SCS5107 Computational Intelligence Unit V

Fuzzy Systems

CLASSICAL PROPOSITION LOGIC

As in our ordinary informal language, ª**sentence**º is used in the logic. Especially, a sentence having only ªtrue (1)º or ªfalse (0)º as its truth value is called ª**proposition**º.

2 + 4 = 7 (false)

For every x, if f(x) = sin x, then f□(x) = cos x. (true) .It rains now. (true or false depending whether it rains or not)

Connectives - combine prepositional variables .

Negation, Conjunction, Disjunction ,Implication

What does it mean if *p* implies *q*? Example: *p* = sky is overcast, *q* = sun not visible, $p \rightarrow q$

p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \longrightarrow q$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	T	T	F	T
F	F	T	F	F	T

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

LOGIC FUNCTIONS

Logic function : a combination of propositional variables by using connectives
Logic formula : Truth values 0 and 1 are logic formulas

- If v is a logic variable, v and v' are a logic formulas
- If a and b represent a logic formulas,

TAUTOLOGY AND INFERENCE RULE

Tautology : A ªtautologyº is a logic formula whose value is always true regardless of its logic variables. A ªcontradictionº is one which is always false.

a	b	$a \to b$	$\overline{(a \to b)}$	$\bar{b}$	$\overline{(a \to b)} \to \bar{b}$
1	1	1	0	0	1
1	0	0	1	1	1
0	1	1	0	0	1
0	0	1	0	1	1

TAUTOLOGY AND INFERENCE RULE

a	b	$(a \to b)$	$(a \wedge (a \to b))$	$(a \wedge (a \to b)) \to b$
1	1	1	1	1
1	0	0	0	1
0	1	1	0	1
0	0	1	0	1

Predicate logic :
"Predicate logic" is a logic which represents a proposition with the predicate and an individual (object)
"Socrates is a man"
"Socrates" – object

"is a man" – predicate
"Two is less than four"
"Two", "four" – objects

"is less than" - predicate

Objects in predicate logic can represented by variables. Then a predicate proposition can be evaluated for truth if an element of a universal set is instantiated to the variable.
"x is a man"
x="Tom", the proposition becomes "Tom is a man"

"x satisfies P" can be denoted P(x)
is_a_man(Tom)

Fuzzy sets

Definition: let X be a non-empty set and be called the universe of discourse. A fuzzy set $A \subset X$ is characterized by the membership function

where $\mu_A(x)$ is a grade (degree) of membership of x in set A.

$$\mu_A : X \rightarrow [0,1]$$

Fuzzy sets

Definition of fuzzy sets:

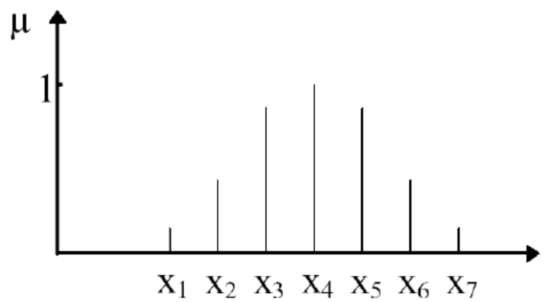
Fuzzy set A can be represented as a set of ordered pairs

$$A = \{(x, \mu_A(x)) | x \in X\}$$

Fuzzy sets

Discrete example:

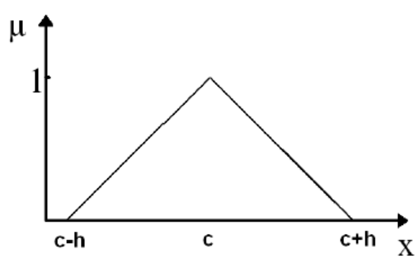
$\mu_A = 0.1/x1 + 0.4/x2 + 0.8/x3 + 1.0/x4 + 0.8/x5 + 0.4/x6 + 0.1/x7$



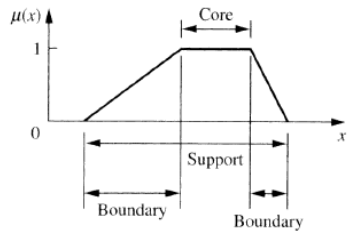
Fuzzy sets

Continuous example:

$$\mu_A(x) = \begin{cases} 1 + \dfrac{x-c}{h}, & x \in [c-h, c] \\ 1 - \dfrac{x-c}{h}, & x \in [c, c+h] \\ 0, & \text{otherwise} \end{cases}$$



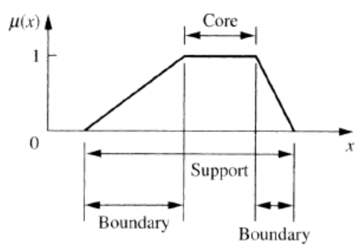
Properties of fuzzy sets



Support : support of a fuzzy set A is a crisp set that contains all elements of A with non-zero membership grade:

$$\text{supp}(A) = \{x \in X | \mu_A(x) > 0\}$$

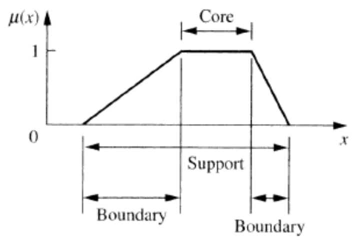
Properties of fuzzy sets



Core: comprises those elements x of the universe such that $\mu_A(x) = 1$.

$$\text{core}(A) = \{x \in X | \mu_A(x) = 1\}$$

Properties of fuzzy sets



Boundary : boundaries comprise those elements x of the universe such that $0< \mu_A (x) <1$

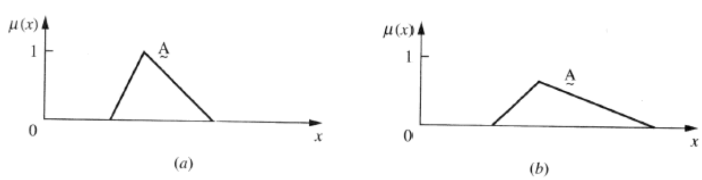
$$\text{bnd}(A) = \{x \in X | 0 < \mu_A(x) < 1\}$$

Properties of fuzzy sets

Height : the height of a fuzzy set A id defined

$$\text{hgt}(A) = \sup_{x \in X} \mu_A(x)$$

Set A is called normal if hgt(A)=1 and subnormal if hgt(A)<1



Fuzzy sets that are normal (a) and subnormal (b).

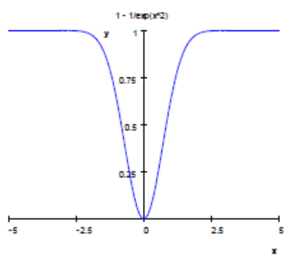
PROPERTIES OF FUZZY SETS

Question?

Is the fuzzy set defined as

$$\mu_A(x) = 1 - 1/e^{-x^2}$$

Normal or subnormal?

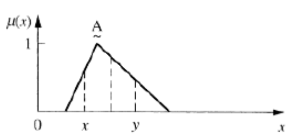


PROPERTIES OF FUZZY SETS

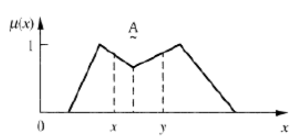
Convex Fuzzy set: a fuzzy set A is convex iff

$$\forall x, y \in X \quad \text{and} \quad \forall \lambda \in [0,1]$$

$$\mu_A(\lambda x + (1-\lambda) y) \geq \min \big(\mu_A(x), \mu_A(y)$$



Convex, normal fuzzy set



Non convex, normal fuzzy set

OPERATIONS ON FUZZY SETS

1. Empty set

$$\mu_{\varnothing} \equiv 0$$

2. Basic set (universe)

$$\mu_X \equiv 1$$

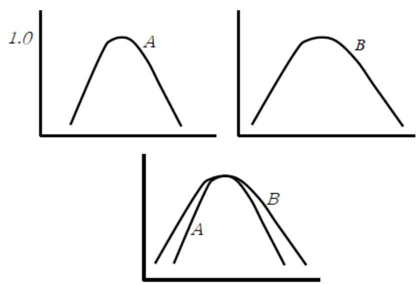
3. Identity

$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x) \quad \forall x \in X$$

OPERATIONS ON FUZZY SETS

4. Subset

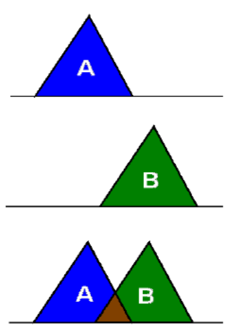
$$A \subset B \Leftrightarrow \mu_A(x) \le \mu_B(x) \quad \forall x \in X$$



OPERATIONS ON FUZZY SETS

5. Union

$$U[\mu_A, \mu_B] = \max[\mu_A, \mu_B],$$
$$\forall x \in X : \mu_{A \cup B}(x) = \max\left(\mu_A(x), \mu_B(x)\right)$$



OPERATIONS ON FUZZY SETS

6. Intersection

Axioms for intersection function

I:[0,1] × [0,1] → [0,1] $\mu_{A\cap B}(x) = I[\mu_A(x), \mu_B(x)]$

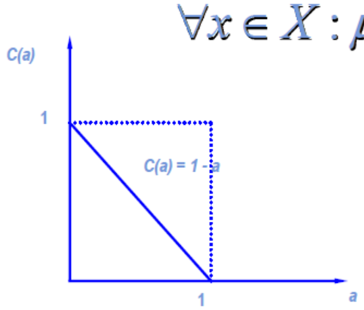
- $I(1, 1) = 1$, $I(1, 0) = 0$, $I(0, 1) = 0$, $I(0, 0) = 0$
- $I(a, b) = I(b, a)$, Commutativity.
- If $a \le a'$ and $b \le b'$, $I(a, b) \le I(a', b')$, monotonicity.
- $I(I(a, b), c) = I(a, I(b, c))$, Associativity.
- I is a continuous function.
- $I(a, a) = a$, idempotency.

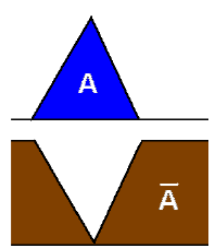
7. Complement

Standard complement function:

$$C(a) = 1 - a$$

$$\forall x \in X : \mu_{\bar{A}}(x) = 1 - \mu_A(x)$$





Properties of fuzzy operations:

(1) Involution $\overline{\overline{A}} = A$

(2) Commutativity $A \cup B = B \cup A$

$A \cap B = B \cap A$

(3) Associativity $(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

(4) Distributivity $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(5) Idempotency $A \cup A = A$

$A \cap A = A$

(6) Absorption $A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

(7) Absorption by X and $\varnothing$ $A \cup X = X$

$A \cap \varnothing = \varnothing$

(8) Identity $A \cup \varnothing = A$

$A \cap X = A$

(9) De Morgan's law $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

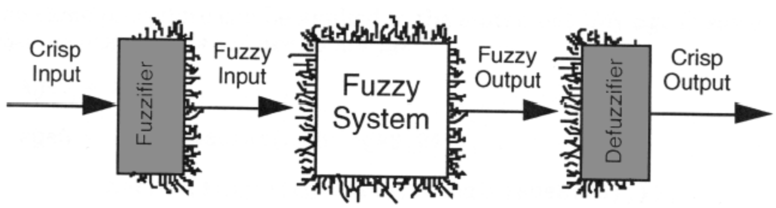
(10) Equivalence formula

$$(\overline{A} \cup B) \cap (A \cup \overline{B}) = (\overline{A} \cap \overline{B}) \cup (A \cap B)$$

(11) Symmetrical difference formula

$$(\overline{A} \cap B) \cup (A \cap \overline{B}) = (\overline{A} \cup \overline{B}) \cap (A \cup B)$$

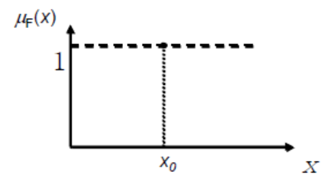
Fuzzy System



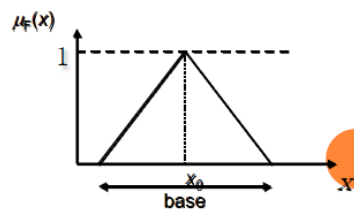
* Fuzzify crisp inputs to get the fuzzy inputs
* Defuzzify the fuzzy outputs to get crisp outputs

FUZZIFICATION

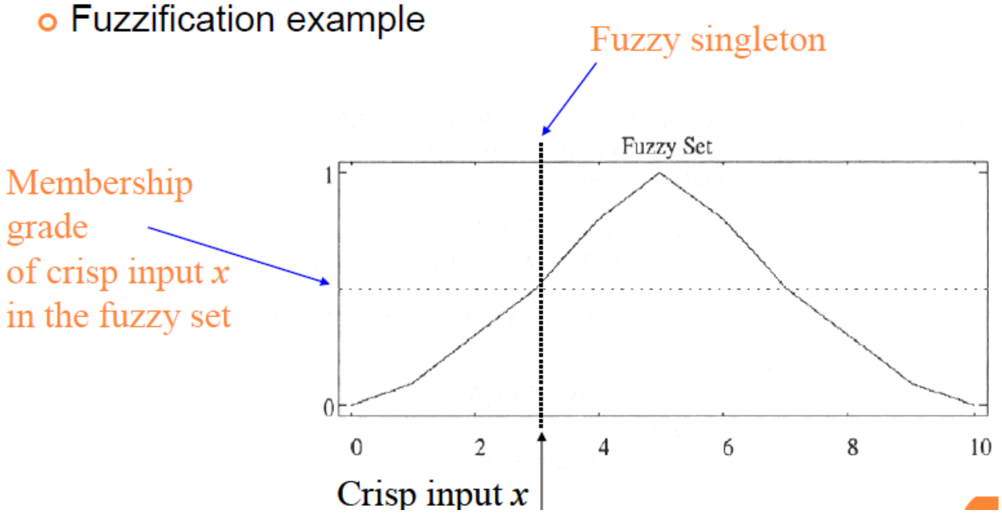
- Process of making a crisp quantity fuzzy
- If it is assumed that input data do not contain noise of vagueness, a fuzzy singleton can be used



- If the data are vague or perturbed by noise, they should be converted into a fuzzy number



- Fuzzification example



Build a Fuzzy Controller

<u>3 Steps</u>

1. Pick the linguistic variables
   - Example: Let temperature (X) be input and motor speed (Y) be output
2. Pick the fuzzy sets
   - Define fuzzy subsets of the X and Y
3. Pick the fuzzy rules
   - Associate output to the input

Example: Build a Fuzzy Controller

Goal: Design a motor speed controller for air conditioner

Step 1: assign input and output variables

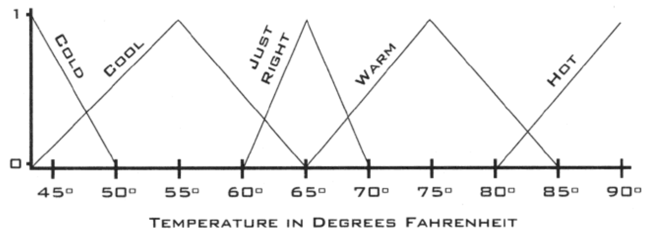
- Let X be the temperature in Fahrenheit
- Let Y be the motor speed of the air conditioner

Step 2: Pick fuzzy sets

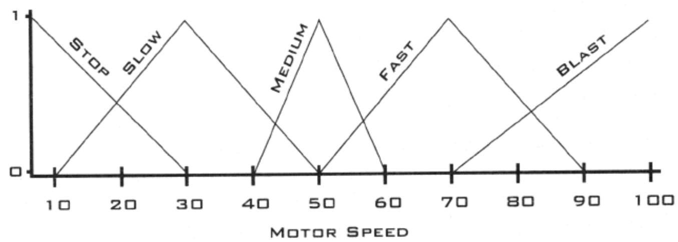
Define linguistic terms of the linguistic variables temperature (X) and motor speed (Y) and associate them with fuzzy sets

- For example, 5 linguistic terms / fuzzy sets on X
  - Cold, Cool, Just Right, Warm, and Hot
- Say 5 linguistic terms / fuzzy sets on Y
  - Stop, Slow, Medium, Fast, and Blast

Input Fuzzy sets



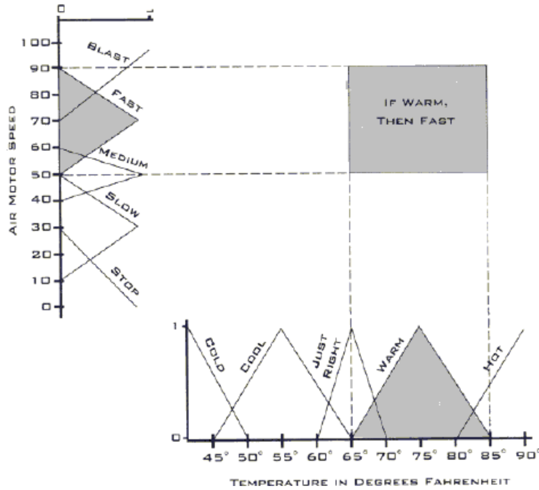
Output Fuzzy sets



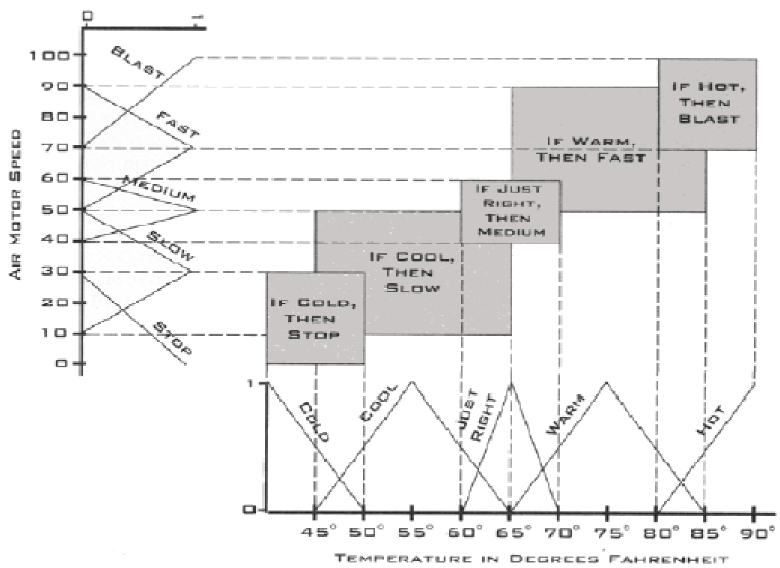
Step 3: Assign a motor speed set to eacl temperature set

- If temperature is ***cold*** then motor speed is ***stop***
- If temperature is ***cool*** then motor speed is ***slow***
- If temperature is ***just right*** then motor speed is ***mediu***
- If temperature is ***warm*** then motor speed is ***fast***
- If temperature is ***hot*** then motor speed is ***blast***

- A Fuzzy Relation expressed by a rule



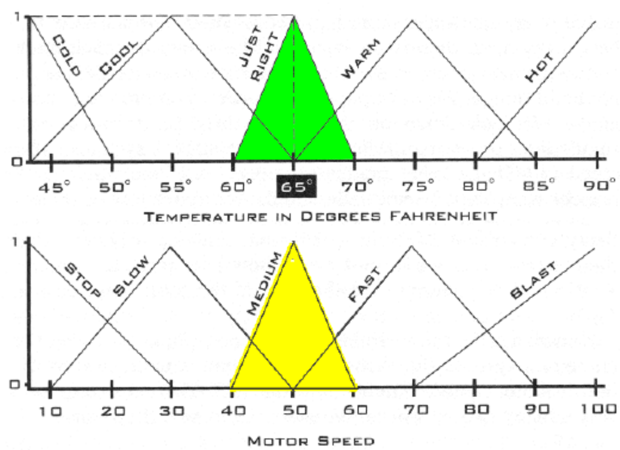
A Fuzzy controller with 5 patches



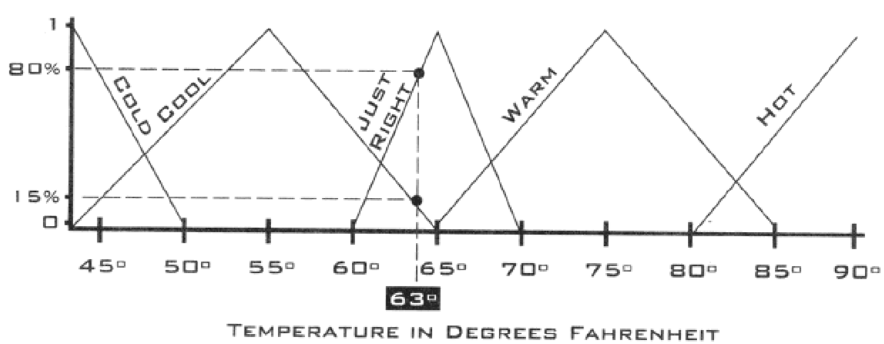
In Mamdani and Larsen model, which rule "fires" or activates at which time?

- They all fire all the time
- They fire in parallel
  - All rules fire to some degree
  - Most fire to zero degree
- The result is a union of fuzzy results from each rule

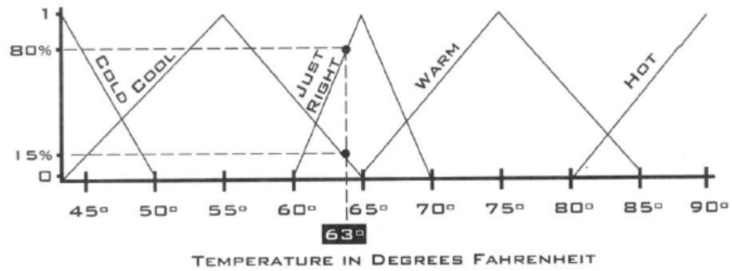
IF TEMPERATURE IS ***JUST RIGHT*** THEN MOTOR SPEED IS ***MEDIUM***

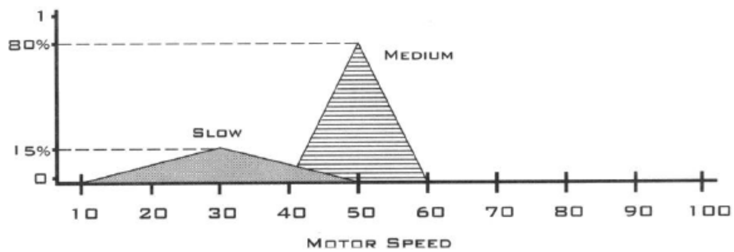


If temperature is ***cool*** then motor speed is ***slow***
If temperature is ***just right*** then motor speed is ***medium***

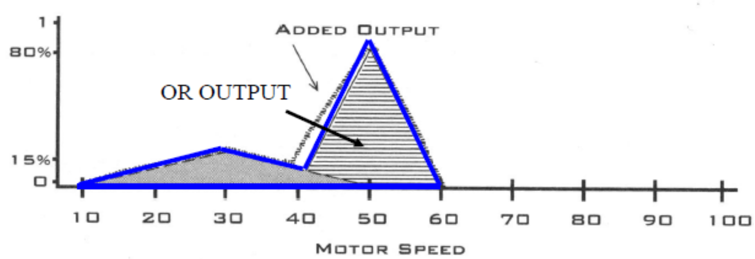


EXAMPLE: T = 63 DEGREE F.

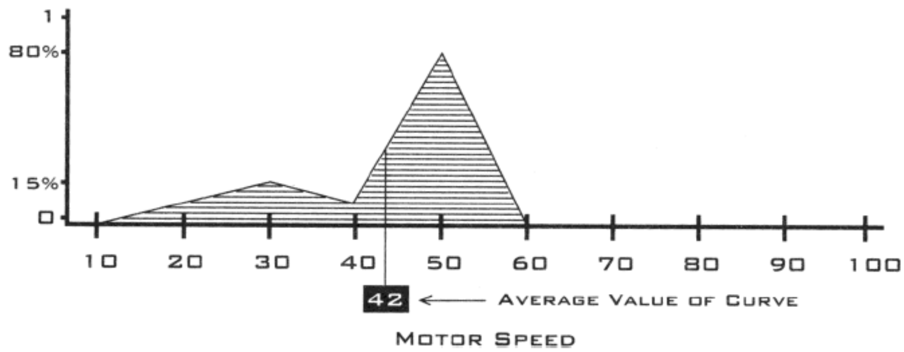




Summed (MAXed) of the partially fired then-part fuzzy sets

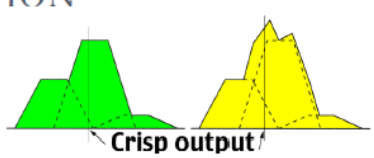


Defuzzify to find the output motor speed



Question: how to convert a fuzzy set into a crisp value?

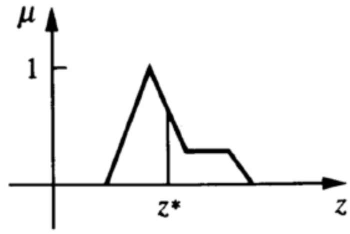
DEFUZZIFICATION



- Converts a fuzzy set into a crisp output.
- Defuzzification is a process to get a non-fuzzy value th best represents the possibility distribution of an inferre fuzzy control action.
- There is no systematic procedure for choosing a good defuzzification strategy.
- Selection of defuzzification procedure depends on the properties of the application.

DEFUZZIFICATION

- Centroid of the Area: the most prevalent and physically appealing of all the defuzzification methods [Sugeno, 1985; Lee, 1990]



- A disadvantage: computationally intensive

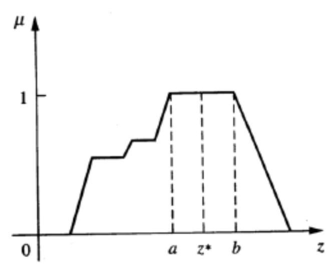
$$z^* = \frac{\int \mu_C(z) \cdot z\, dz}{\int \mu_C(z)\, dz}$$

where $\int$ denotes an algebraic integration.

- Mean of maximum (MOM)

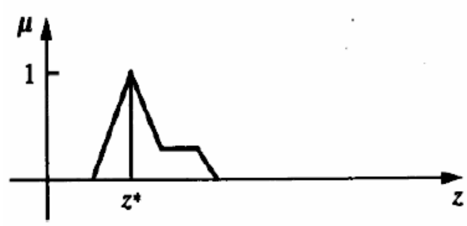
$$z_0 = \sum_{j=1}^{k} \frac{z_j}{k}$$

The defuzzified result represents the mean value of all actions, whose membership functions reach the maximum



z_j: control action whose membership functions reach the max
k: number of such control actions.

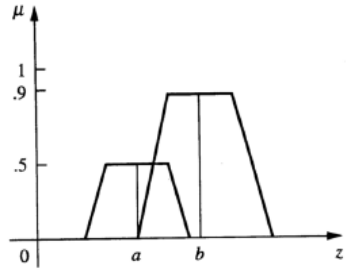
Max-membership principal, also known as height method



$\mu_C(z^*) \geq \mu_C(z)$ for all $z \in Z$

Weighted average method

- Valid for symmetrical output membership functions
- Produces results very close to the COA method
- Less computationally intensive

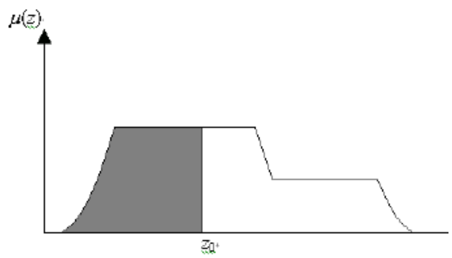


$$z^* = \frac{\sum \mu_C(\bar{z}) \cdot \bar{z}}{\sum \mu_C(\bar{z})} \qquad z^* = \frac{a(.5) + b(.9)}{.5 + .9}$$

Formed by weighting each functions in the output by its respective maximum membership value

Bisector of the Area

- The BOA generates the action (z_0) which partitions the area into two regions with the same area



$$\int_{\alpha}^{z_0} \mu_C(z)\, dz = \int_{z_0}^{\beta} \mu_C(z)\, dz$$

$$\alpha = \min\{z \mid z \in W\}$$

$$\beta = \max\{z \mid z \in W\}$$

First (or last) of maxima

- Determine the smallest value of the domain with maximized membership degree