UNIT 4

DISCRETE PARAMETER MARKOV CHAINS

4.1 Introduction

A random process, $X(t)$, is said to be a Markov process if for any time instants, $t_1 < t_2 < \cdots < t_n < t_{n+1}$, the random process satisfies

$$F_X(X(t_{n+1}) \le x_{n+1} \quad X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_1) = x_1)$$

$$= F_X(X(t_{n+1}) \le x_{n+1} \,|\, X(t_n) = x_n). \qquad (9.1)$$

To understand this definition, we interpret t_n as the present time so that t_{n+1} rep-resents some point in the future and $t_1, t_2, \ldots, t_{n-1}$ represent various points in the past. The Markovian property then states that given the present, the future is independent of the past. Or, in other words, the future of the random process depends only on where it is now and not on how it got there.

EXAMPLE 1: A classical example of a continuous time Markov process is the Poisson counting process. Let $X(t)$ be a Poisson counting process with rate $\boldsymbol{\lambda}$. Then its probability mass function satisfies

$$\Pr(X(t_{n+1}) = x_{n+1} \,|\, X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_1) = x_1)$$

$$= \begin{cases} 0 & x_{n+1} < x_n \\ \dfrac{(\lambda(t_{n+1} - t_n))^{x_{n+1}-x_n}}{(x_{n+1} - x_n)!} e^{-\lambda(t_{n+1}-t_n)} & x_{n+1} \ge x_n \end{cases}.$$

Clearly, this is independent of $\{X(t_{n-1}) = x_{n-1}, \ldots, X(t_1) = x_1\}$. In fact, the Markovian property must be satisfied because of the independent increments assumption of the Poisson process.To start with, we will focus our attention on discrete-valued Markov processes in discrete time, better known as Markov chains. Let $X[k]$ be the value of the process at time instant k. Since the process is discrete-valued, $X[k] \in \{x_1, x_2, x_3, \ldots\}$ and we say that if $X[k] = x_n$, then the process is in state n at time k. A Markov chain is described statistically by its transition probabilities which are defined as follows.

DEFINITION 2: Let $X[k]$ be a Markov chain with states $\{x_1, x_2, x_3, \ldots\}$, then the probability of transitioning from state i to state j in one time instant is

If the Markov chain has a finite number of states, n, then it is convenient to define a *transition probability matrix*,

$$P = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,n} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,n} \\ \\ p_{n,1} & p_{n,2} & \cdots & p_{n,n} \end{bmatrix}. \tag{9.3}$$

One can encounter processes where the transition probabilities vary with time and hence need to be explicitly written as a function of k (e.g., $p_{i,\ j,\ k}$), but we do not consider such processes in this text and henceforth it is assumed that transition probabilities are independent of time.

EXAMPLE 2: Suppose every time a child buys a kid's meal at his favorite fast food restaurant, he receives one of four superhero action figures. Naturally, the child wants to collect all four action figures and so he regularly eats lunch at this restaurant in order to complete the collection. This process can be described by a Markov chain. In this case, let $X[k] \in \{0, 1, 2, 3, 4\}$ be the number of different action figures that the child has collected after purchasing k meals. Assuming each meal contains one of the four superheroes with equal probability and that the action figure in any meal is independent of what is contained in any previous or future meals, then the transition probability matrix easily works.
Initially (before any meals are bought), the process starts in state 0 (the child has no action figures). When the first meal is bought, the Markov chain must move to state 1 since no matter which action figure is contained in the meal, the child will now have one superhero. Hence, $p_{0,\ 1} = 1$ and $p_{\ 0,\ j} = 0$ for all $j\ = 1$. If the child has one distinct action figure, when he buys the next meal he has a 25 percent chance of receiving a duplicate and a 75 percent chance of getting a new action figure. Hence, $p_{1,\ 1} = 1/4$, $p_{1,\ 2} = 3/4$, and $p_{1,\ j} = 0$ for $j = 1, 2$. Similar logic is used to complete the rest of the matrix. The child might be interested in knowing the average number of lunches he needs to buy until his collection is completed. Or, maybe the child has saved up only enough money to buy 10 lunches and wants to know what his chances are of completing the set before running out of money. We will develop the theory needed to answer such questions.
The transition process of a Markov chain can also be illustrated graphically using a state diagram. Such a diagram is illustrated in Figure 9.1 for the Markov chain in Example 2. In the figure, each directed arrow represents a possible transition and the label on each arrow represents the probability of making that transition. Note that for this Markov chain, once we reach state 4, we remain there forever. This type of state is referred to as an *absorbing state*.

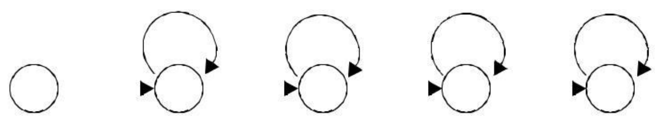


Figure 1 State diagram for the Markov chain of Example 2.

EXAMPLE 3: (The Gambler's Ruin Problem) Suppose a gambler plays a certain game of chance (e.g., blackjack) against the "house." Every time the gambler wins the game, he increases his fortune by one unit (say, a dollar) and every time he loses, his fortune decreases by one unit. Suppose the gambler wins each game with probability p and loses with probability $q = 1 - p$. Let X_n represent the amount of the gambler's fortune after playing the game n times. If the gambler ever reaches the state $X_n = 0$, the gambler is said to be "ruined" (he has lost all of his money). Assuming that the outcome of each game is independent of all others, the sequence x_n , $n = 0, 1, 2, . . .$ forms a Markov chain. The state transition matrix is of the form

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ q & 0 & p & 0 & 0 & \dots \\ 0 & q & 0 & p & 0 & \dots \\ 0 & 0 & q & 0 & p & \dots \\ \dots & & & \dots & & \dots \end{bmatrix}.$$

The state transition diagram for the gambler's ruin problem is shown in Figure 9.2. One might then be interested in determining how long it

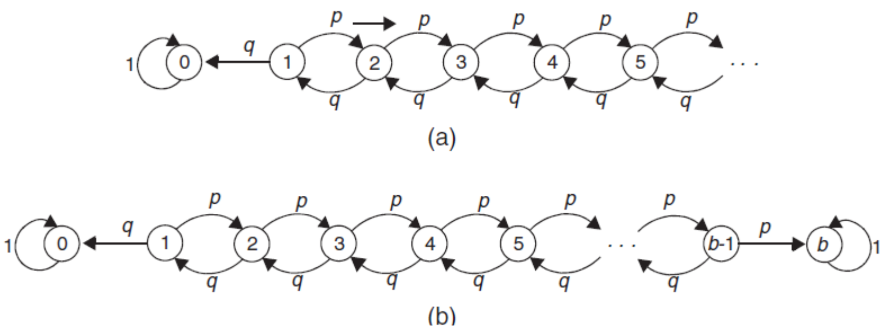


Figure.2 State transition diagram for Example 9 (The Gambler's Ruin Problem), with one absorbing state (a) and with two absorbing states (b).

might take before the gambler is ruined (enters the zero state). Is ruin inevitable for any p, or if the gambler is sufficiently proficient at the game, can he avoid ruin indefinitely? A more realistic alternative to this model is one where the house also has a finite amount of money. Suppose the gambler starts with d dollars and the house has $b - d$ dollars so that between the two competitors there is a total of b dollars in the game. Now if the gambler ever gets to the state 0 he is ruined, while if he gets to the state b he has "broken the bank" (i.e., the house is ruined). Now the Markov chain has two absorbing states as shown in part (b) of the figure. It would seem that sooner or later the gambler must have a run of bad luck sufficient to send him to the 0 state (i.e., ruin) or a run of good luck which will cause him to enter the state b (i.e., break the bank). It would be interesting to find the probabilities of each of these events.

The previous example is one of a class of Markov chains known as *random walks*. Random walks are often used to describe the motion of a particle. Of course, there are many applications that can be described by a random walk that do not involve the movement of a particle, but it is helpful to think of such a particle when describing such a Markov chain. In one dimension, a random walk is a Markov chain whose states are the integers and whose transition probabilities satisfy $p_{i,j} = 0$ for any $j = i - 1, i, i + 1$. In other words, at each time instant, the state of the Markov chain can either increase by one, stay the same, or decrease by one. If $p_{i,i+1} = p_{i,i-1}$, then the random walk is said to be symmetric, whereas if $p_{i,i+1} = p_{i,i-1}$ the random walk is said to have drift. Often the state space of the random walk will be a finite range of integers, n, $n + 1$, $n + 1$, . . . , $m - 1$, m (for $m > n$), in which case the states n and m are said to be boundaries, or barriers. The gambler's ruin problem is an example of a random walk with absorbing boundaries, where $p_{n,n} = p_{m,m} = 1$. Once the particle reaches the boundary, it is absorbed and remains there forever. We could also construct a random walk with reflecting boundaries, in which case $p_{n,n+1} = p_{m,m-1} = 1$. That is, whenever the particle reaches the boundary, it is always reflected back to the adjacent state.

EXAMPLE 4: (A Queueing System) A common example of Markov chains (and Markov processes in general) is that of queueing systems. Consider, for example, a taxi stand at a busy airport. A line of taxis, which for all practical purposes can be taken to be infinitely long, is available to serve travelers. Customers wanting a taxi enter a queue and are given a taxi on a first come, first serve basis. Suppose it takes one unit of time (say, a minute) for the customer at the head of the queue to load himself and his luggage into a taxi. Hence, during each unit of time, one customer in the queue receives service and leaves the queue while some random number of new customers enter the end of the queue. Suppose at each time instant, the number of new customers arriving for service is described by a discrete distribution (p_0, p_1, p_2 , . . .), where p_k is the probability of k new customers. For such a system, the transition probability matrix of the Markov chain would look like

$$P = \begin{bmatrix} p_0 & p_1 & p_2 & p_3 & \cdots \\ p_0 & p_1 & p_2 & p_3 & \cdots \\ 0 & p_0 & p_1 & p_2 & \cdots \\ 0 & 0 & p_0 & p_1 & \cdots \\ \cdots & & \cdots & & \cdots \end{bmatrix}.$$

The manager of the taxi stand might be interested in knowing the probability distribution of the queue length. If customers have to wait too long, they may get dissatisfied and seek other forms of transportation.

4.2 COMPUTATION OF n-STEP TRANSITION PROBABILITIES

Computing n-step transition probabilities

- The probability to move from s to s' in $n \geqslant 0$ steps:

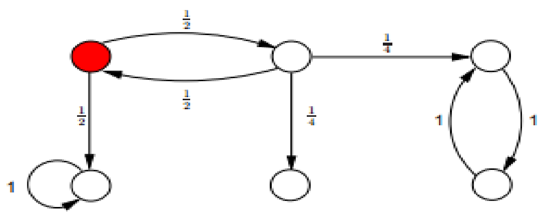
$$p_{s,s'}(n) = \sum_{s''} p_{s,s''}(l) \cdot p_{s'',s'}(n-l) \quad \text{for all } 0 \leqslant l \leqslant n$$

  - this is known as the *Chapman-Kolmogorov* equation

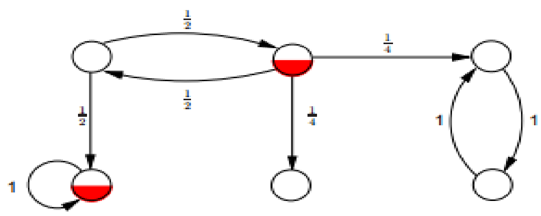
- For $l = 1$ and $n > 0$ we obtain: $p_{s,s'}(n) = \sum_{s''} p_{s,s''}(1) \cdot p_{s'',s'}(n-1)$
  - in matrix-form: $\mathbf{P}^{(n)} = \mathbf{P}^{(1)} \cdot \mathbf{P}^{(n-1)} = \mathbf{P} \cdot \mathbf{P}^{(n-1)}$
  - where $\mathbf{P}^{(n)}$ is the n-step transition probability matrix

- Repeating this scheme: $\mathbf{P}^{(n)} = \mathbf{P} \cdot \mathbf{P}^{(n-1)} = \ldots = \mathbf{P}^{n-1} \cdot \mathbf{P}^{(1)} = \mathbf{P}^{n}$

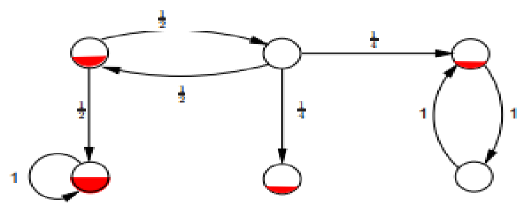
Note: the difference between $\mathbf{P}^{n}$ and $\mathbf{P}^{(n)}$



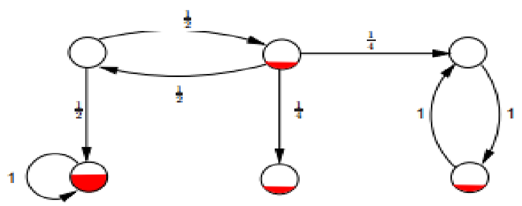
zero-th epoch



first epoch

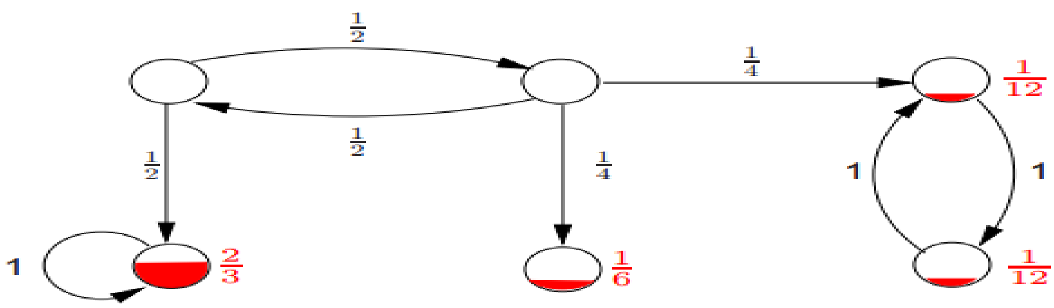


second epoch



third epoch

On the long run



Transient distribution of a DTMC

Probability to be in state s at step n:

$$\begin{aligned} p_s(n) &= \Pr\{\, X(n) = s \,\} \\ &= \sum_{s' \in S} \underbrace{\Pr\{\, X(0) = s' \,\}}_{p_{s'}(0)} \cdot \underbrace{\Pr\{\, X(n) = s \mid X(0) = s' \,\}}_{p_{s',s}(n)} \end{aligned}$$

Using $\underline{p}(n) = (p_{s_0}(n), p_{s_1}(n), \ldots, p_{s_k}(n))$ we obtain in matrix form:

$$\boxed{\underline{p}(n) = \underline{p}(0) \cdot \mathbf{P}^n \quad \text{given} \quad \underline{p}(0)}$$

where $\mathbf{P}^n$ is the n-step transition probability matrix

$\underline{p}(n)$ is called the ***n-step transient-state*** probability vector

4.3 Calculating Transition and State Probabilities in Markov Chains

The state transition probability matrix of a Markov chain gives the probabilities of transitioning from one state to another in a single time unit. It will be useful to extend this concept to longer time intervals.

DEFINITION .3: The *n*-step transition probability for a Markov chain is

(9.4)

Also, define an *n*-step transition probability matrix $\boldsymbol{P}^{(n)}$ whose elements are the *n*-step transition probabilities just described in Equation 9.4.

Given the one-step transition probabilities, it is straightforward to calculate higher order transition probabilities using the following result.

THEOREM .1: (Chapman-Kolmogorov Equation)

$$p_{i,j}^{(n)} = \sum_k p_{i,k}^{(m)} p_{k,j}^{(n-m)}, \quad \text{for any } m = 0, 1, 2, \ldots, n \qquad (9.5)$$

PROOF: First, condition on the event that in the process of transitioning from state *i* to state *j*, the Markov chain passes through state *k* at some intermediate point in time. Then, using the principle of total probability,

$$\Pr(X_{l+n}=j \mid X_l=i)=\sum_k \Pr(X_{l+n}=j \mid X_l=i,\ X_{l+m}=k)\Pr(X_{l+m}=k \mid X_k=i). \qquad (9.6)$$

Using the Markov property, the expression reduces to the desired form:

$$\Pr(X_{l+n}=j \mid X_l=i)=\sum_k \Pr(X_{l+n}=j \mid X_{l+m}=k)\Pr(X_{l+m}=k \mid X_k=i). \qquad (9.7)$$

■

This result can be written in a more compact form using transition probability matrices. It is easily seen that the Chapman-Kolmogorov equations can be written in terms of the n-step transition probability matrices as

$$\boldsymbol{P}(n) = \boldsymbol{P}(m)\boldsymbol{P}(n-m).$$

Then, starting with the fact that $\boldsymbol{P}^{(1)} = \boldsymbol{P}$, it follows that $\boldsymbol{P}^{(2)} = \boldsymbol{P}^{(1)}\boldsymbol{P}^{(1)} = \boldsymbol{P}^2$, and using induction, it is established that

$$\boldsymbol{P}^{(n)} = \boldsymbol{P}^n .$$

Hence, we can find the n-step transition probability matrix through matrix multiplication. If n is large, it may be more convenient to compute $\boldsymbol{P}^n$ via eigen-decomposition. The matrix $\boldsymbol{P}$ can be expanded as $\boldsymbol{P} = \boldsymbol{U}\ \boldsymbol{U}^{-1}$, where is the diagonal matrix of eigenvalues and $\boldsymbol{U}$ is the matrix whose columns are the corresponding eigenvectors. Then

$$\boldsymbol{P}^n = \boldsymbol{U}\quad{}^n\ \boldsymbol{U}^{-1}.$$

Another quantity of interest is the probability distribution of the Markov chain at some time instant k. If the initial probability distribution of the Markov chain is known, then the distribution at some later point in time can easily be found. Let $\boldsymbol{\pi}_j(k) = \Pr(X_k = j)$ and $\boldsymbol{\pi}(k)$ be the row vector whose jth element is $\boldsymbol{\pi}_j(k)$. Then

$$\pi_j(k) = \Pr(X_k = j) = \sum_i \Pr(X_k = j \mid X_0 = i)\Pr(X_0 = i) = \sum_i p_{i,j}^{(k)}\pi_i(0),$$

or in vector form,

$$\pi(k) = \pi(0)\boldsymbol{P}^k.$$

4.4 *Classification of states*

Irreducible Chain:
If for every *i, j* ,we can find some n such that p_{ij}>0,then every state can be reached from every other state, and the Markov chain is said to be irreducible. Otherwise the chain is non-irreducible or reducible.
Return state:
State I of a Markov chain is called a return state if $p_{i,j}$>0 for some n>1.
Periodic state:
Period d_i of a return state *i* is the greatest common divisor of all m such that p_{ij}>0.State i is periodic with period d_i,if d_i>1 and aperiodic if d_i=1.
Recurrent state:
If f_{ii}=1 the return to state i is certain and the state i is said to be persistent or recurrent. Otherwise transient.
Ergotic state:
A non-null persistent and aperiodic state are called ergotic.

4.5 *The M/M/1 Queue*

In this section, we investigate Markov processes where the time variable is continuous. In particular, most of our attention will be devoted to the so-called birth-death processes which are a generalization of the Poisson counting process studied in the previous chapter. To start with, consider a random process $X(t)$ whose state space is either finite or countable infinite so that we can represent the states of the process by the set of integers, $X(t) \in \{. . . , -3, -2, -1, 0, 1, 2, 3, . . .\}$. Any process of this sort that is a Markov process has the interesting property that the time between any change of states is an exponential random variable. To see this, define T_i to be the time between the *i*th and the $(i + 1)$th change of state and let $h_i(t)$ be the complement to its CDF, $h_i(t) = \Pr(T_i > t)$. Then, for $t > 0$, $s > 0$,

$$h_i(t+s) = \Pr(T_i > t+s) = \Pr(T_i > t+s, T_i > s) = \Pr(T_i > t+s \mid T_i > s)\Pr(T_i > s). \tag{9.31}$$

Due to the Markovian nature of the process, $\Pr(T_i > t+s \mid T_i > s) = \Pr(T_i > t)$, and hence the previous equation simplifies to

$$h_i(t+s) = h_i(t)h_i(s). \tag{9.32}$$

Furthermore, for this function to be a valid probability, the constant $\boldsymbol{\rho}_i$ must not be negative. From this, the PDF of the time between change of states is easily found to be $f_{T_i}(t) = \boldsymbol{\rho}_i e^{-\rho_i t} u(t)$.
As with discrete time Markov chains, the continuous time Markov process can be described by its transition probabilities.

THEOREM 4: For a Markov birth-death process with birth rate $\boldsymbol{\lambda}_n$, $n = 0, 1, 2, . . .$, and death rate $\boldsymbol{\mu}_n$, $n = 1, 2, 3, . . .$, the steady state distribution is given by

$$\pi_k = \lim_{t\to\infty} p_{i,k}(t) = \frac{\prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}}{1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}}.$$

If the series in the denominator diverges, then $\boldsymbol{\pi}_k = 0$ for any finite k. This indicates that a steady state distribution does not exist. Likewise, if the series converges, the $\boldsymbol{\pi}_k$ will be nonzero, resulting in a well-behaved steady state distribution.

EXAMPLE 5: (The M/M/1 Queue) In this example, we consider the birth-death process with constant birth rate and constant death rate. In particular, we take

$\boldsymbol{\lambda}_n = \boldsymbol{\lambda}$, $n = 0, 1, 2, \ldots$ and $\boldsymbol{\mu}_0 = 0$, $\boldsymbol{\mu}_n = \boldsymbol{\mu}$, $n = 1, 2, 3, \ldots$.

This model is commonly used in the study of queueing systems and, in that context, is referred to as the M/M/1 queue. In this nomenclature, the first "M" refers to the arrival process as being Markovian, the second "M" refers to the departure process as being Markovian, and the "1" is the number of servers. So this is a single server queue, where the interarrival time of new customers is an exponential random variable with mean $1/\boldsymbol{\lambda}$ and the service time for each customer is exponential with mean $1/\boldsymbol{\mu}$. For the M/M/1 queueing system, $\boldsymbol{\lambda}_{i-1}/\boldsymbol{\mu}_i = \boldsymbol{\lambda}/\boldsymbol{\mu}$ for all i so that

$$1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i} = \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j = \frac{1}{1-\lambda/\mu} \quad \text{for } \lambda < \mu.$$

The resulting steady state distribution of the queue size is then

$$\pi_k = \frac{(\lambda/\mu)^k}{\frac{1}{1-\lambda/\mu}} = (1-\lambda/\mu)(\lambda/\mu)^k, \quad k = 0, 1, 2, \ldots, \quad \text{for } \lambda < \mu.$$

Hence, if the arrival rate is less than the departure rate, the queue size will have a steady state. It makes sense that if the arrival rate is greater than the departure rate, then the queue size will tend to grow without bound.

4.6 M/G/1 queue

The number of customers in the system, N (t), does not now constitute a Markov process.

The probability per time unit for a transition from the state {N = n} to the state {N = n − 1}, i.e. for a departure of a customer, depends also on the time the customer in service has already spent in the server;

– this information is not contained in the variable N (t)

– only in the case of an exponential service time the amount of service already received does not have any bearing (memoryless property)

In spite of this, the mean queue length, waiting time, and sojourn time of the M/G/1 queue can be found. The results (the Pollaczek-Khinchin formulae) will be derived in the following. It turns out that even the distributions of these quantities can be found. A derivation based on considering an embedded Markov chain will be presented after the mean formulae.

4.7 Pollaczek-Khinchin mean formula

We start with the derivation of the expectation of the waiting time W . W is the time the customer has to wait for the service (time in the "waiting room", i.e. in the actual queue).

$$\mathrm{E}[W] = \underbrace{\underbrace{\mathrm{E}[N_q]}_{\text{number of waiting customers}} \cdot \underbrace{\mathrm{E}[S]}_{\text{mean service time}}}_{\text{mean time needed to serve the customers ahead in the queue}} + \underbrace{\mathrm{E}[R]}_{\text{unfinished work in the server}} \qquad (R = \text{residual service time})$$

- R is the remaining service time of the customer in the server (unfinished work expressed as the time needed to discharge the work).
  If the server is idle (i.e. the system is empty), then R = 0.
- In order to calculate the mean waiting time of an arriving customer one needs the expec-tation of N_q (number of waiting customers) at the instant of arrival.
- Due to the PASTA property of Poison process, the distributions seen by the arriving
- customer are the same as those at an arbitrary instant.

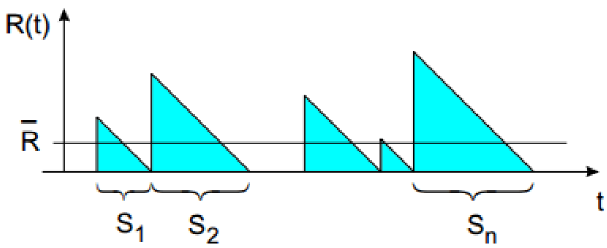
The key observation is that by Little's result the mean queue length E[N$_q$] can be expressed in terms of the waiting time (by considering the waiting room as a black box)

$$\mathrm{E}[N_q] = \lambda \mathrm{E}[W] \quad \Rightarrow \quad \boxed{\mathrm{E}[W] = \frac{\mathrm{E}[R]}{1-\rho}}$$

It remains to determine E

$\rho = \lambda \mathrm{E}[S]$

The residual service time can be deduced by using similar graphical argument as was used in explaining the hitchhiker's paradox. The graph represents now the evolution of the unfinished work in the server, R(t), as a function of time



$$\mathrm{E}[R] = \frac{1}{t}\int_0^t R(t')dt' = \frac{1}{t}\sum_{i=1}^{n}\frac{1}{2}S_i^2 = \underbrace{\frac{n}{t}}_{\to\lambda} \cdot \underbrace{\frac{1}{n} \cdot \sum_{i=1}^{n}\frac{1}{2}S_i^2}_{\to\frac{1}{2}\mathrm{E}[S^2]}$$

$$\boxed{\mathrm{E}[W] = \frac{\lambda \mathrm{E}[S^2]}{2(1-\rho)}}$$

Pollaczek-Khinchin mean formula for the waiting time

From the mean waiting time one immediately gets the mean sojourn time

$$\mathrm{E}[T] = \underbrace{\mathrm{E}[S]}_{\text{the customer's own service time}} + \mathrm{E}[W]$$

<u>*Mean waiting and sojourn times*</u>

$$\boxed{\begin{cases} \mathrm{E}[W] &= \dfrac{\lambda \mathrm{E}[S^2]}{2(1-\rho)} &= \dfrac{1+C_v^2}{2} \cdot \dfrac{\rho}{1-\rho} \cdot \mathrm{E}[S] \\ \mathrm{E}[T] &= \mathrm{E}[S] + \dfrac{\lambda \mathrm{E}[S^2]}{2(1-\rho)} &= \left(1 + \dfrac{1+C_v^2}{2} \cdot \dfrac{\rho}{1-\rho}\right) \cdot \mathrm{E}[S] \end{cases}}$$

<u>Squared coefficient of variation C_v^2</u>

$$\begin{aligned} C_v^2 &= \mathrm{V}[S]/\mathrm{E}[S]^2 \\ \mathrm{E}[S^2] &= \mathrm{V}[S] + \mathrm{E}[S]^2 \\ &= (1 + C_v^2) \cdot \mathrm{E}[S]^2 \end{aligned}$$

Mean number of waiting customers and customers in system

$$\boxed{\begin{cases} \mathrm{E}[N_q] &= \lambda \mathrm{E}[W] &= \dfrac{\lambda^2 \mathrm{E}[S^2]}{2(1-\rho)} &= \dfrac{1+C_v^2}{2} \cdot \dfrac{\rho^2}{1-\rho} \\ \mathrm{E}[N] &= \lambda \mathrm{E}[T] &= \lambda \mathrm{E}[S] + \dfrac{\lambda^2 \mathrm{E}[S^2]}{2(1-\rho)} &= \rho + \dfrac{1+C_v^2}{2} \cdot \dfrac{\rho^2}{1-\rho} \end{cases}}$$

Remarks on the PK mean formulae

- Mean values depend only on the expectation E[S] and variance V[S] of the service time distribution but not on higher moments.
- Mean values increase linearly with the variance.
- Randomness, ‘disarray’, leads to an increased waiting time and queue length.
- The formulae are similar to those of the M/M/1 queue; the only difference is the extra factor $(1 + C_v^2)/2$.

$M/M/1$ queue

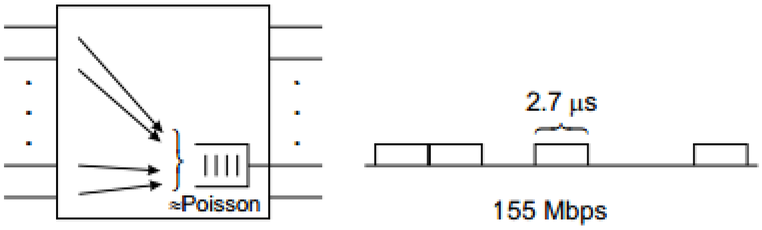
In the case of the exponential distribution one has

$$\mathrm{V}[S] = \mathrm{E}[S]^2 \quad \Rightarrow \quad C_v^2 = 1$$

$$\boxed{\begin{cases} \mathrm{E}[N] &= \rho + \dfrac{\rho^2}{1-\rho} &= \dfrac{\rho}{1-\rho} \\ \mathrm{E}[T] &= \left(1 + \dfrac{\rho}{1-\rho}\right) \cdot \mathrm{E}[S] &= \dfrac{1}{1-\rho} \cdot \mathrm{E}[S] \end{cases}}$$

The familiar formulae for the $M/M/1$ queue

Example. The output buffer of an ATM multiplexer can be modelled as an M/D/1 queue. Constant service time means now that an ATM cell has a fixed size (53 octets) and its transmission time to the link is constant. If the link speed is 155 Mbit/s, then the transmission time is S = 53 · 8/155 µs = 2.7µs. What is the mean number of cells in the buffer (including the cell being transmitted) ant the mean sojourn time of the cell in the buffer when the average information rate on the link is 124 Mbit/s? The load (utilization) of the link is ρ = 124/155 = 0.8



Then

$$\begin{cases} \mathrm{E}[N] &= 0.8 + \frac{1}{2} \cdot \frac{0.8^2}{1-0.8} = 2.4 \\ \mathrm{E}[T] &= \left(1 + \frac{1}{2} \cdot \frac{0.8}{1-0.8}\right) 2.7\,\mu\mathrm{s} = 8.1\,\mu\mathrm{s} \end{cases}$$

Problems

1. A housewife buys 3 kinds of cereals A,B and C. She nssive weever buys the same cereal in successive weeks.If she buys cereal A,the next week she buys B.However if she buys B or C ,the next week she is 3 times as likely to buy A as the other cereal. In the long run ,how often does she buy each of the three cereals?

   P=[0 1 0
   3/4 0 1/4
   3 /4 1/4 0].

   Let Π =[Π_1,Π_2] be the steady state distribution of the Markov chain.Then ΠP= Π

   =[Π_1,Π_2 Π_3] [0 1 0
   3/4 0 1/4
   3 /4 1/4 0] = (Π_1,Π_2 Π_3).

   Solving
   Π_1=15/35
   Π_2 =16/35
   Π_3=4/35.
   Prob of buying A=15/35
   Prob of buying B=16/35
   Prob of buying C=4/35

2. Consider the Markov chain with three states, S={1,2,3}S={1,2,3}, that has the following transition matrix

$$P=\begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

If we know P(X1=1)=P(X1=2)=1/4 find $P(X_1=3, X_2=2, X_3=1)$.

$P(X_1=3)=1-P(X1=1)-P(X_1=2)=1-1/4-1/4=1/2.$

$P(X_1=3, X_2=2, X_3=1)=P(X_1=3)\cdot p_{32}\cdot p_{21}=1/2\cdot 1/2\cdot 1/3=1/12.$

3. Three boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix.

The transition matrix is given by

$$P=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

States of Xn depends only on states of Xn-1,but not on states of Xn-2,Xn-3,…. Or earlier states. Therefore Xn is markov chain.