UNIT – V

MODELING AND SIGNAL DETECTION IN NOISE

* SOURCES OF NOISE :=

⇒ There are various sources of random noise. They are broadly classified as:

(a) External Noise and

(b) Internal Noise.

⇒ The external noise is created outside the circuit, and includes:

(i) Erratic Natural Disturbances: This type of noise does not occur regularly. It is caused by lightning, electrical storms, and intergalatic or other atmospheric disturbances. This noise is unpredictable in nature, and is known as atmospheric or static noise. Besides this, extraterristral noise is also created by erratic natural disturbances. The atmospheric noise is less severe above 30 MHz.

(ii) Man-Made noise: This noise is because of the undesired pick-ups from electrical applicanes, such as motors, switch gears, automobile and aircraft ignitions etc. This type of noise is under human control, and can be eliminated by removing the source of the noise. This noise is effective in frequency range of 1MHz - 500 MHz.

⇒ The internal noise is created by the active and passive components present within the communication circuit itself. This type of noise is also known as fluctuation noise. It is caused by spontaneous fluctuations in the physical system. Examples of such fluctuations are:

(a) Thermal motion of the free electrons inside a resistor, known as "Brownian motion", which is random in nature.

(b) The random emission of electrons in vaccum tubes
(c) The random diffusion of electrons and holes in a semiconductor.

=) The two important types of fluctuation noise are (i) shot noise and (ii) thermal noise.

* THERMAL NOISE :-

=> This type of noise occurs due to random motion of free electrons inside a resistor. Every electron has got its thermal energy, due to which it is in motion.

=) Its motion is zigzag and random because during its movement it collides with lattice structure. The net effect of the motion of all electrons constitutes an electric current flowing through resistor.

=) The direction of current flow is random and has a zero mean value. Such a noise is referred to either thermal, agitation, white or Johnson's noise.

=) In thermodynamics, we know that kinetic energy of a particle is expressed by its, temperature so that the temperature of the body is the statistical rms value of the velocity of the particles in the body.

=) It is thus apparent that the noise power generated by a resistor is proportional to its absolute temperature in addition being proportional to the B.W over which the noise is to be measured.

=) Thus,

$$P \propto TB = kTB \longrightarrow (1)$$

where, k = Boltzmann's constant

$T = 273 + ^\circ c$

B = Bandwidth of operation

P = Maximum noise power output of a resistor.

from eq(2)

$$V_n^2 = 4RP_n = 4RKT\,\delta f$$

$$V_n = \sqrt{4RP_n} = \sqrt{4RKT\delta f} \longrightarrow (3).$$

=> Note that the generated noise voltage is quite independent of the frequency at which it is measured.

=> This is due to the fact that it is random and therefore evenly distributed on the average.

=> It may be noted that the thermal noise contains all frequencies in equal amount. For this reason, it is also called "white noise agitation" or "Johnson's noise."

* NOISE TEMPERATURE :=

=> The available noise power is directly proportional to temperature and it is independent of value of resistance. This power specified in terms of temperature is called as " Noise temperature".

=> The available thermal noise power from eq, we have

$$P_n = KTB \longrightarrow (1)$$

which basically depends upon the temperature and bandwidth. A thermal resistor therefore delivers a maximum of KT watts per unit bandwidth, regardless of the value of R.

=> clearly temperature is the fundamental parameter of thermal noise. But there are other white noise sources which are non-thermal in the sense that the noise power is not related to the physical temperature; still we can talk of the noise temperature T_n of any white noise source, thermal or non-thermal by definition we have

$$T_n = \frac{P_n}{KB}. \longrightarrow (2).$$

where P_n = maximum noise power the source can deliver.

⇒ Since, the noise voltage is caused by the random movement of electrons within the resistors, which in turn constitutes a current.

⇒ At any instant of time there are more electrons arriving at one particular end than at other because of their random motion. Over a period of time, the imbalance will be retreated.

⇒ As the rate of arrival of electrons at either end of the resistor varies randomly, so does the potential difference between the two ends, thus there will be a random voltage across the resistor.

⇒ We are only considering rms values of noise while referring to all the formulas and not its instantaneous value.

⇒ Using eq(1), the equivalent circuit of a resistor as a noise generator may be drawn as shown in Fig(1) and from this the resistor's equivalent noise voltage V_n may be calculated.

⇒ Assume that R_1 is noiseless and is receiving the maximum noise power generated by R. Here we can take the help of maximum power transfer theorem that maximum power as transmitted when $R = R_L$.

$$P_n = \frac{V_i^2}{R_L} = \frac{V_i^2}{R} = \frac{(V_n/2)^2}{R} = \frac{V_n^2}{4R} \longrightarrow (2)$$

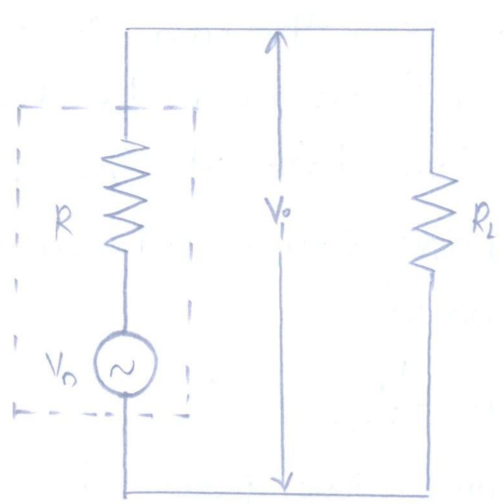


Fig. 4.1 Resistor as a noise generator

K = Boltzman's constant

B = Bandwidth.

⇒ Another advantage of the use of noise temperature for low noise levels is that it shows a greater variation for any given noise level change than does the noise figure, therefore changes are easier to grasp in their true sense.

⇒ For thermal sources T_n is physical temperature, where as for non-thermal sources it is measure of the available noise power.

⇒ This means that a non-thermal white noise source having equivalent internal resistance R_n and noise temperature T_n can be replaced by a resistor $R = R_n$ at physical temperature $T = T_n$.

$$F = 1 + \frac{R'_{eq}}{R_a} = 1 + \frac{K T_{eq} B R'_{eq}}{K T_0 \cdot B \cdot R_a}$$

$$F = 1 + \frac{T_{eq}}{T_0}. \longrightarrow (3)$$

T_{eq} = Equivalent noise temperature of amplifier.

⇒ Note that here F cannot be expressed in decibels. Also T_{eq} may be effected by the actual value of ambient temperature of receiver.

⇒ Here we are again stressing upon the fact that equivalent noise temperature is just a fictitious entity.

$$T_0 F = T_0 + T_{eq}$$

$$\boxed{T_{eq} = T_0 (F-1)} \longrightarrow (4).$$

* NOISE BANDWIDTH :-

⇒ The noise bandwidth is an important parameter for specifying the noise power at the output of a bandpass linear system.

⇒ Consider a linear bandpass system shown in fig (a). The square of transfer function, i.e $|H(\omega)|^2$ is plotted in fig (b).

⇒ Only the positive half of the plot $|H(\omega)|^2$ is shown assuming that the negative half is symmetrical about vertical axis.

⇒ The noise power (m.s. value) at the output of the system is given as:

$$P_o = \overline{V_{no}^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ni}(\omega)\cdot|H(\omega)|^2 d\omega.$$

$$= \frac{1}{\pi}\int_{0}^{\infty} S_{ni}(\omega)\,|H(\omega)|^2 d\omega.$$

⇒ For all practical purposes, the input noise power density is assumed to be constant with frequency.

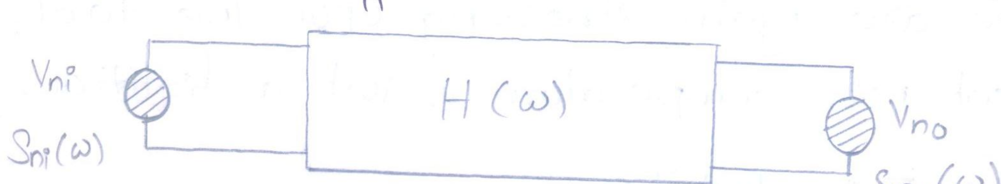
⇒ Let us take this constant value as c. Thus

$$S_{ni}(\omega) = c.$$

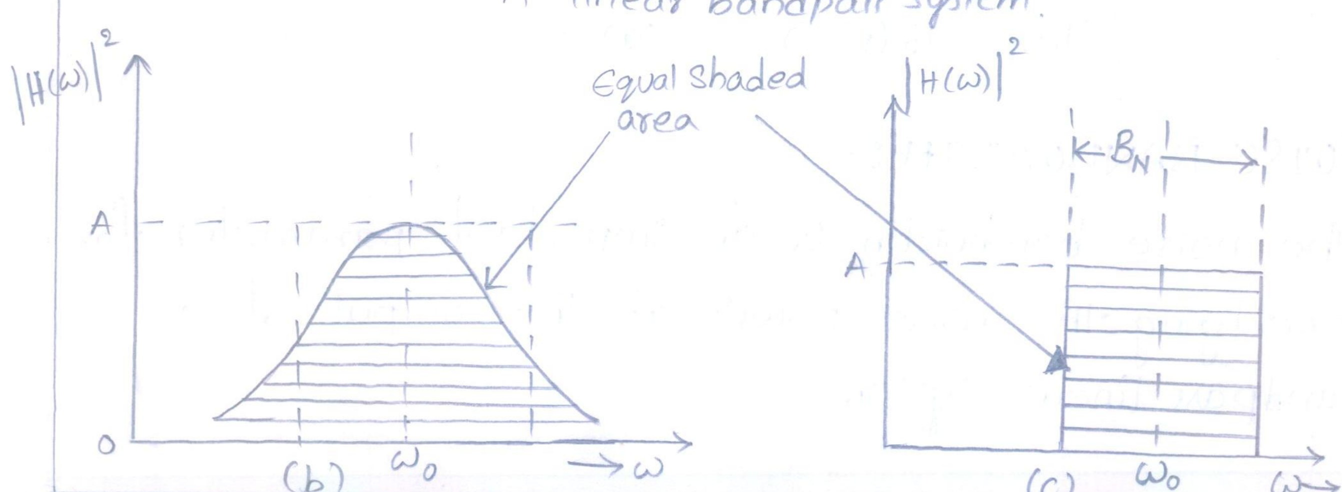
⇒ Then, the output noise power is given by

$$P_o = \frac{c}{\pi}\int_{0}^{\infty}|H(\omega)|^2 d\omega.$$

⇒ The integral in the right hand side is the area under the curve square of transfer function, i.e plot of $|H(\omega)|^2$ show in fig (b). We can say that power P_o is proportional to the area under the $|H(\omega)|^2$ curve i.e $P_o = \frac{c}{\pi} \times$ area under the curve $|H(\omega)|^2$.



(a)
A linear bandpass system.



(b) $|H(\omega)|^2$ of actual system. b

(c) $|H(\omega)|^2$ of ideal system.

⇒ Let us consider an ideal bandpass system with rectangular characteristic of $|H(\omega)|^2$ such that the area under this curve is same as that of an actual system, and the height of the ideal curve is equal to A, which is the maximum value of $|H(\omega)|^2$ in actual system.

⇒ Then bandwidth of the ideal system is called as "equivalent noise bandwidth" denoted by B_N.

⇒ The characteristic of the ideal system is shown in fig (c). The area under this ideal rectangular characteristic specifying the power of the signal is given by

$$A \times B_N$$

where,

$$A = |H(\omega_0)|^2$$

Equating the areas of actual and ideal systems,

$$A \times B_N = \int_0^\infty |H(\omega)|^2 d\omega .$$

which gives the expression for noise bandwidth B_N as

$$B_N = \frac{1}{A} \int_0^\infty |H(\omega)|^2 d\omega$$

$$B_N = \frac{\text{Area under the } |H(\omega)|^2 \text{ curve of actual system}}{\text{max value of the } |H(\omega)|^2 \text{ curve of actual system}}$$

⇒ The noise power, in terms of B_N is obtained as follows:

$$P_0 = \overline{V_{no}^2} = \frac{C}{\pi} \times [\text{area under the curve } |H(\omega)|^2 \text{ of actual system}].$$

$$= \frac{C}{\pi} \times [\text{area under the curve } |H(\omega)|^2 \text{ of ideal system}]$$

$$= \frac{C}{\pi} \times [AB_N]$$

Hence,

$$P_0 = \overline{V_{no}^2} = \frac{CAB_N}{\pi}.$$

⇒ Here, C is a constant equal to the power density spectrum of noise voltage at the input of the system.

⇒ "Equivalent noise bandwidth" is the bandwidth of that ideal bandpass system which produces the same noise power as the actual system.

* NARROW BAND NOISE :-

=> In communication systems, message signals intermixed with noise are usually passed through bandpass filters. The bandpass filters have narrow bandwidths in the sense that the bandwidth is small as compared to centre frequency.

=> The wideband noise accompained with the desired signal is also passed through this bandpass filter.

=> Hence, in general, we have to deal with bandpass noise for evaluating the noise performance of a communication system.

=> Consider a narrow band Gaussian noise $n(t)$ with the spectral density function $S_n(\omega)$ shown in fig (a). The spectrum is centered about ω_c and has a bandwidth of $2\omega_m$ radians.

=> The power density curve may be approximated by a set of delta functions as shown in fig (a). This is due to the fact that an element area given by $S_n(\omega_c+k_1\Delta\omega)$.

=> $\Delta\omega$ becomes a delta function under the limit $\Delta\omega \to 0$. The pair of delta functions corresponding to this element area will be located at $\omega = \pm(\omega_c - k_1\Delta\omega)$ and have a strength equal to the area $S_n(\omega_c + k_1\Delta\omega)\cdot\Delta\omega$.

=> The corresponding pair of delta function is denoted by $\{S_n(\omega_c + k_1\Delta\omega)\Delta\omega\}\,\delta[\omega \pm (\omega_c - k_1\Delta\omega)]$

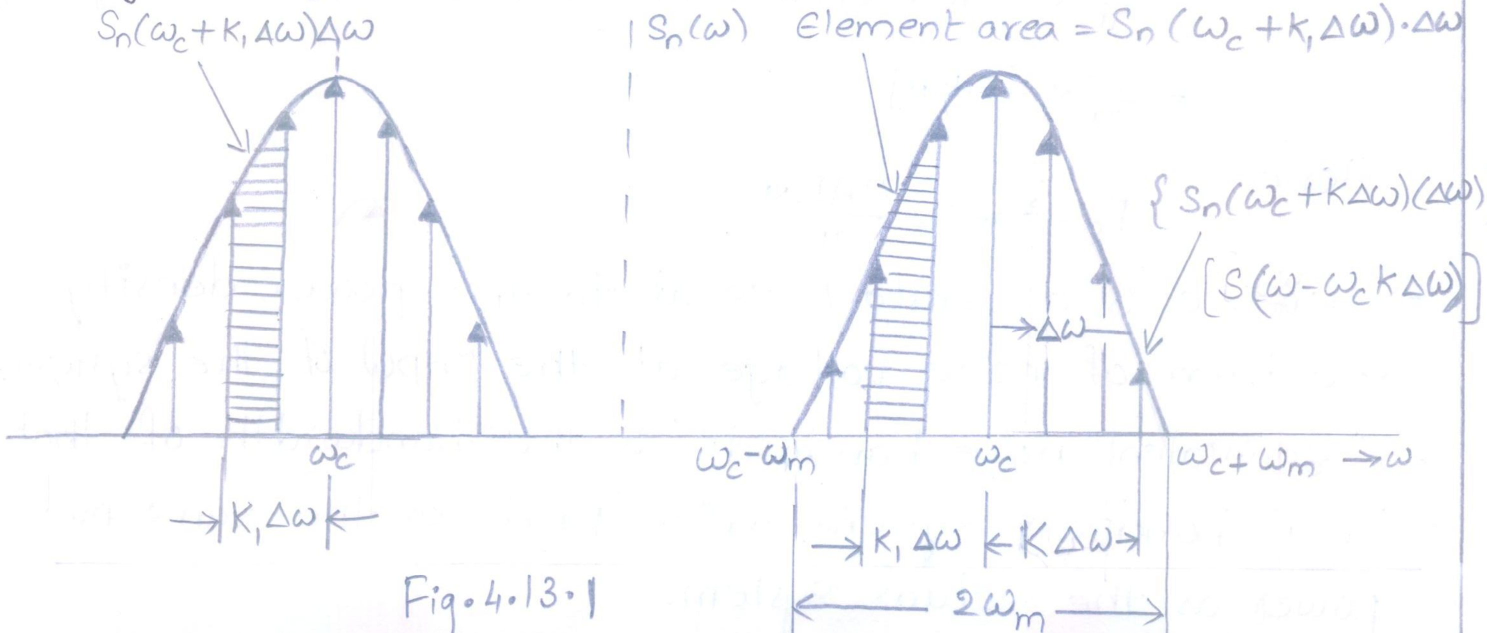


Fig. 4.13.1

⇒ Similarly, a pair of delta functions located at $\pm(\omega_c + k\Delta\omega)$ is given by $\{S_n(\omega_c + k\Delta\omega)\cdot\Delta\omega\}\,\delta[\omega \pm (\omega_c - k\Delta\omega)]$

⇒ The characteristic curve may be approximated by summation of similar delta functions located at a uniform spacing of $\Delta\omega$.

⇒ The entire curve can be represented in terms of delta function as follows:

$$S_n(\omega) = \lim_{\Delta\omega \to 0} \sum_k \{S_n(\omega_c + k\Delta\omega)\Delta\omega\}\left[\delta(\omega - \omega_c - k\Delta\omega) + \delta(\omega + \omega_c + k\Delta\omega)\right] \quad \to (1)$$

⇒ The spectrum of $\cos\omega_c t$ consists of pair of impulses located at $\pm\omega_c$ and it can be represented in time domain as the sum of cosine functions given below:

$$n(t) = \lim_{\Delta\omega \to 0} \sum_k A_k \cos[(\omega_c + k\Delta\omega)t + \theta_k] \longrightarrow (2)$$

where A_k is defined by

$$A_k = \sqrt{4 S_n(\omega_c + k\Delta\omega)\Delta\omega} \longrightarrow (3)$$

and θ_k belongs to a set of values of an independent random variable $\{\theta_k\}$ each with a uniform distribution defined by

$$p(\theta_k) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta_k \le 2\pi \\ 0, & \text{elsewhere} \end{cases} \longrightarrow (4)$$

⇒ As $\Delta\omega$ goes to zero, the number of sine functions in eq(2) tends to infinity.

⇒ According to central limit theorem, it can be defined by a Gaussian process. Equation (2) is a mathematical description of a sine wave model of narrow band noise $n(t)$.

⇒ This noise is Gaussian distributed with a zero mean and a variance defined by

$$\sigma^2 = \sum_k \frac{A_k^2}{2} \longrightarrow (5)$$

* QUADRATURE REPRESENTATION OF NARROW-BAND NOISE :-

⇒ The expansion of the sinusoidal function in eq(2) yields:

$$\cos[(\omega_c + k\Delta\omega)t + \theta_k] = \cos\omega_c t \cdot \cos(k\Delta\omega t + \theta_k) - \sin\omega_c t \cdot \sin(k\Delta\omega t + \theta_k)$$

substituting this expanded form in eq(2) we get

$$n(t) = n_c(t) \cdot \cos\omega_c t - n_s(t) \cdot \sin\omega_c t \longrightarrow (6)$$

where

$$n_c(t) = \lim_{\Delta\omega \to 0} \sum_k A_k \cdot \cos(k\Delta\omega t + \theta_k) \longrightarrow (7)$$

and

$$n_s(t) = \lim_{\Delta\omega \to 0} \sum_k A_k \cdot \sin(k\Delta\omega t + \theta_k) \longrightarrow (8).$$

⇒ It is obvious that term $n_c(t)$ is in phase with the carrier wave $\cos\omega_c(t)$; whereas $n_s(t)$ is $90°$ out of phase with the carrier wave term.

⇒ Hence, $n_c(t)$ is referred to as in-phase component and $n_s(t)$ as quadrature component of the narrow-band noise $n(t)$.

* NOISE FIGURE :-

⇒ The amount of noise power contributed by a two-port network is also characterized by its noise figure.

⇒ It is defined as the ratio of the total noise power spectral density S_{no} at the output of the two-port network to the noise spectral density S'_{no} at the output assuming the network to be entirely noiseless.

⇒ This ratio is the same as the ratio of corresponding available power densities. Thus the noise figure F of a two-port network is

$$F = \frac{(S_{no})_a}{(S'_{no})_a} \longrightarrow (1).$$

⇒ When a two-port network is assumed to be noiseless, the power density at the output is solely due to the noise source at the input. Hence, alternately we can define,

$$\boxed{F = \frac{\text{Power density of the total noise at the output of n/w}}{\text{power density at the output due to the source}}}$$

⇒ The total noise power density at the output S_{no} is equal to the sum of the power density solely due to the input source S'_{no} and the power density contributed by network itself denoted by S''_{no}

$$S_{no} = S'_{no} + S''_{no}$$

and noise figure in eq(1) may be written as

$$F = \frac{S'_{no} + S''_{no}}{S'_{no}} = 1 + \frac{S''_{no}}{S'_{no}}.$$

* NOISE-FIGURE OF CASCADED AMPLIFIERS (OR) FRISS FORMULA :=

⇒ A cascade of two stages is shown in fig. The available gain and the noise-figure of the first stage is G_{a1} and F_1, respectively.

⇒ Similarly, the second stage has G_{a2} and F_2 as the available gain and the noise-figure respectively.

⇒ let N_i be the noise power generated by resistor R at the input of the first stage.

⇒ The noise power at the final output due to N_i is given by $\boxed{N_{o1} = G_{a1} G_{a2} N_i}$

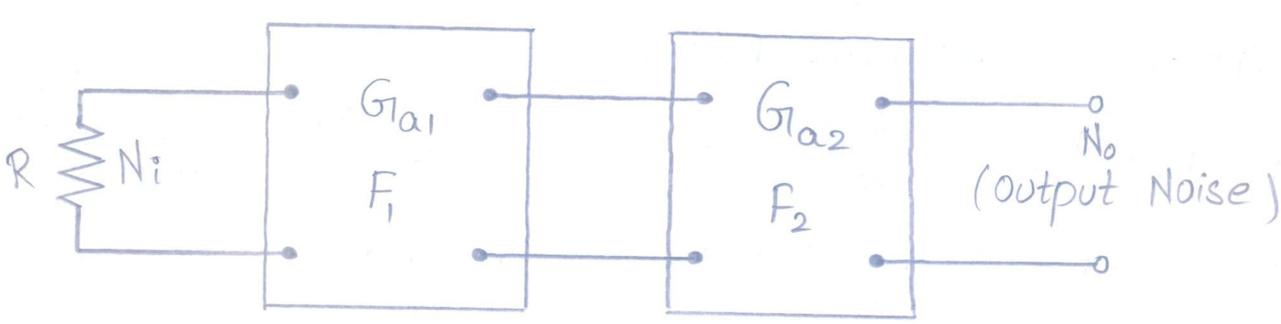


Fig. 4.10.1 Cascaded Two-port Network

⇒ The first stage introduces its own internal Noise. The contribution of this noise at the output of first stage is equal to $G_{a1}(F_1-1)N_i$.

⇒ This noise is amplified by second stage and appears at its output as given by

$$N_{o2} = G_{a1}G_{a2}(F_1-1)N_i$$

Similarly, the noise contributed by the second stage to the final output noise power can be evaluated to be

$$N_{o3} = G_{a2}(F_2-1)N_i$$

⇒ The total noise power at the output thus is,

$$N_o = N_{o1} + N_{o2} + N_{o3}.$$

$$= G_{a1}G_{a2}N_i + G_{a1}G_{a2}(F_1-1)N_i + G_{a2}(F_2-1)N_i. \quad \rightarrow (1)$$

Eq (1) can be rearranged as

$$\frac{1}{G_{a1}G_{a2}} \cdot \frac{N_o}{N_i} = 1 + F_1 - 1 + \frac{F_2-1}{G_{a1}} = F_1 + \frac{F_2-1}{G_{a1}} \quad \longrightarrow (2)$$

⇒ The overall available gain G_a of the cascaded stages is given by $\boxed{G_a = G_{a1}G_{a2}}$

Substituting this value in eq(2) we get

$$\frac{1}{G_a} \cdot \frac{N_o}{N_i} = F_1 + \frac{F_2-1}{G_{a1}}$$

⇒ The left hand side of the equation represents the overall noise-figure F of the cascaded amplifier. Hence, the overall noise-figure F of the two cascaded amplifier is

$$F = F_1 + \frac{F_2-1}{G_{a1}} \quad \longrightarrow (3).$$

⇒ The expression can be extended to a multistage amplifier as furnished below:

$$F = F_1 + \frac{F_2-1}{G_{a1}} + \frac{F_3-1}{G_{a1} \cdot G_{a2}} + \cdots\cdots + \frac{F_n}{G_{a1} \cdot G_{a2} \cdots G_{a(n-1)}} \rightarrow (4)$$

⇒ This is known as the Friss formula.

* EQUIVALENT NOISE-TEMPERATURURE OF CASCADED STAGES :-

⇒ Let us assume that the first and second stages of above fig are characterized by equivalent noise temperatures T_{e1} and T_{e2} respectively. we have

$$F-1=\frac{T_e}{T}$$

where T_e is the overall noise temperature of the two cascaded stages. substituting noise-figures in terms of the corresponding equivalent noise temperature in eq(3), we get

$$1+\frac{T_e}{T}=1+\frac{T_{e1}}{T}+\frac{T_{e2}}{G_{a1}T}.$$

$$T_e=T_{e1}+\frac{T_{e2}}{G_{a1}}$$

⇒ The expression can be extended to multistage amplifiers. The equivalent noise temperature of a multistage network is, thus, given as

$$T_e=T_{e1}+\frac{T_{e2}}{G_{a1}}+\frac{T_{e3}}{G_{a1}\cdot G_{a2}}+\dots+\frac{T_{en}}{G_{a1}G_{a2}\cdots G_{a(n-1)}}.$$

* BAYES CRITERIAN :-

⇒ To illustrate the use of minimum average cost optimization criteria to find optimum receiver structures, we will first consider detection.

⇒ For example, suppose we are faced with a situation in which the presence or absence of a constant signal of value k to be detected in presence of an additive Gaussian Noise component N.

⇒ Thus we may hypothesize two situations for the the observed data z.

Hypothesis 1 (H_1) : $Z=N$ (Noise alone) $P(H_1 \text{ true})$
$=P_0$.

Hypothesis 2 (H_2) : $z = K+N$ (Signal plus noise) $p(H_2 \text{ true}) = 1-P_0$.

=) Assuming the noise to have zero mean & variance σ_n^2 we may write down the pdf's of z given hypothese H_1 and H_2 respectively. under hypothesis H_1, z is Gaussian with mean zero and variance σ_n^2.

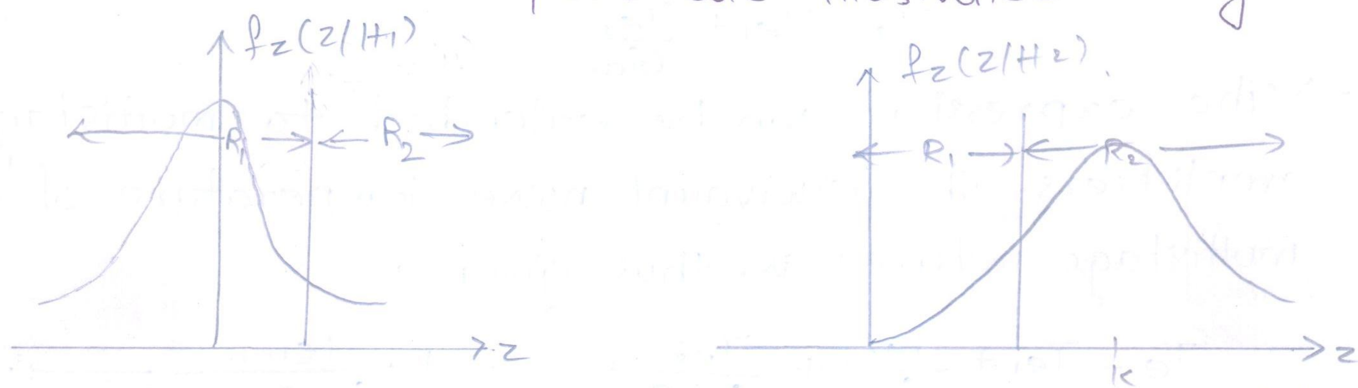
Thus

$$f_z(z/H_1) = \frac{e^{-z^2/2\sigma_n^2}}{\sqrt{2\pi}\sigma_n} \longrightarrow (1)$$

Under hypothesis H_2, since the mean is k,

$$f_z(z/H_2) = \frac{e^{-(z-k)^2/2\sigma_n^2}}{\sqrt{2\pi\sigma_n^2}} \longrightarrow (2)$$

=) These conditional pdf's are illustrated in figure



=) we note in this example that the observed data, can range over the real line $-\infty < z < \infty$

=) Our objective is to parition this one dimensional observation space into two regions R_1 and R_2 such that if z falls into R_1, we decide hypothesis H_1 is true, while if z is in R_2, we decide hypothesis H_2 is true.

=) Taking a general approach to the prob we note that four apriori costs are required. Since there are four types of decisions that can make. These costs are

$C_{11} \rightarrow$ the cost of deciding in favour of H_1 when H_1 is actually true.

$C_{12} \rightarrow$ the cost of deciding in favour of H_1, when H_2 is actually true.

$C_{21} \rightarrow$ the cost of deciding in favour of H_2, when H_1 is actually true.

$C_{22} \rightarrow$ the cost of deciding in favour of H_2, when H_2 is actually true.

$\Rightarrow$ Given that H_1 was actually true, the conditional average cost of making a decision, $c(D/H_1)$ is

$$c(D/H_1) = C_{11} \cdot p(\text{decide } H_1/H_1 \text{ true}) + C_{21} \cdot p[\text{decide } H_2/H_1 \text{ true}] \rightarrow (3)$$

$\Rightarrow$ In terms of conditional pdf of z given H_1, we may write

$$p[\text{decide } H_1/H_1 \text{ true}] = \int_{R_1} f_z(z/H_1)\,dz \longrightarrow (4)$$

and $$p[\text{decide } H_2/H_1 \text{ true}] = \int_{R_2} f_z(z/H_1)\,dz \longrightarrow (5)$$

$\Rightarrow$ we note that z must lie in either R_1 or R_2, since we are forced to make a decision.

Thus,

$$p[\text{decide } H_1/H_1 \text{ true}] + p[\text{decide } H_2/H_1 \text{ true}] = 1 \rightarrow (6)$$

$\Rightarrow$ conditional pdf $[f_z(z/H_1)]$ can be expressed as

$$\int_{R_2} f_z(z/H_1)\,dz = 1 - \int_{R_1} f_z(z/H_1)\,dz \rightarrow (7)$$

$\Rightarrow$ Thus combining eq (3) & (7), the conditional average cost given H_1, $c(D/H_1)$ becomes

$$c(D/H_1) = C_{11} \int_{R_1} f_z(z/H_1)\,dz + C_{21}\left[1 - \int_{R_1} f_z(z/H_1)\,dz\right] \rightarrow (8).$$

$\Rightarrow$ In a similar manner, the average cost of making a decision given that H_2 is true, $c(D/H_2)$ can be written as,

$$c(D/H_2) = C_{12}\, P[\text{decide } H_1 / H_2 \text{ true}] + C_{22}\, P[\text{decide } H_2 / H_2 \text{ true}]$$

$$= C_{12} \int_{R_1} f_z(z/H_2)\,dz + C_{22} \int_{R_2} f_z(z/H_2)\,dz.$$

$$= C_{12} \int_{R_1} f_z(z/H_2)\,dz + C_{22}\left[1 - \int_{R_1} f_z(z/H_2)\,dz\right] \rightarrow (9)$$

=> To find the average cost without regard to which hypothesis is actually true, we must average eq(8) and eq(9), with respect to the prior probabilities of hypothesis H_1 & H_2.

$$P_0 = P[H_1 \text{ true}] \text{ and } q_0 = 1 - P_0 = P[H_2 \text{ true}].$$

=> The average cost of making a decision is then,

$$C(D) = P_0 \cdot C(D/H_1) + q_0 C(D/H_2) \longrightarrow (10).$$

Substituting eq(8) & eq(9) in eq(10) and collecting terms, we obtain

$$C(D) = P_0\left\{ C_{11} \int_{R_1} f_z(z/H_1)\,dz + C_{21}\left[1 - \int_{R_1} f_z(z/H_1)\,dz\right]\right\} + q_0\left\{ C_{12} \int_{R_1} f_z(z/H_2)\,dz + C_{22}\left[1 - \int_{R_1} f_z(z/H_2)\,dz\right]\right\}. \rightarrow (11)$$

$$C(D) = [P_0 C_{21} + q_0 C_{22}] + \int_{R_1} q_0 (C_{12} - C_{22}) f_z(z/H_2)\,dz - \int_{R_1} P_0 (C_{21} - C_{11}) f_z(z/H_1)\,dz \longrightarrow (12).$$

=> The first term in brackets represents a fixed cost once P_0, q_0, C_{21} and C_{22} are specified. The value of the integral is determined by those values that are assigned to R_1.

⇒ Since wrong decisions should be more costly than right decisions, it is reasonable to assume than $C_{12} > C_{22}$ and $C_{21} > C_{11}$.

⇒ Thus the two bracketed terms within the integral are positive because q_0, P_0, $f_z(\cancel{z}/H_2)$ and $f_z(z/H_1)$ are probabilities.

⇒ Values of z that give a larger value for the first bracketed term than for the second should be assigned to R_2. In this manner $C(D)$ will be minimized. Mathematically the preceding discussion can be summarized by the pair of inequalities.

$$q_0(C_{12} - C_{22}) \cdot f_z(z/H_2) \underset{H_1}{\overset{H_2}{\gtrless}} P_0(C_{21} - C_{11}) \cdot f_z(z/H_1)$$

(or)

$$\frac{f_z(z/H_2)}{f_z(z/H_1)} \underset{H_1}{\overset{H_2}{\gtrless}} \frac{P_0(C_{21} - C_{11})}{q_0(C_{12} - C_{22})} \longrightarrow (13).$$

which are interpreted as follows. If an observed value of z results in the left hand ratio of pdf's being greater than the right hand ratio of constants, choose H_2, if not choose H_1.

⇒ The left-hand side of eq(13) is denoted by Δz,

$$\Delta(z) \triangleq \frac{f_z(z/H_2)}{f_z(z/H_1)}$$ is called likelihood ratio.

⇒ The right hand side of eq(13) is

$$\eta \triangleq \frac{P_0(C_{21} - C_{11})}{q_0(C_{12} - C_{22})}$$

is called the threshold of the test.

⇒ Thus, Bayes criterion of minimum average cost has resulted in a test of the likelihood ratio, which is a random variable, against the threshold value η.

Bayes Criterion Minimum probability of Error Dectector:

$$C(D) = [P_0 C_{21} + q_0 C_{22}] + \int_{R_1} \{ q_0 (C_{12} - C_{22}) f_z(Z/H_2) - P_0 (C_{21} - C_{11}) f_z(Z/H_1) \} dz.$$

⇒ It follows that if $C_{11} = C_{22} = 0$ (zero cost for making right decision) and $C_{12} = C_{21} = 1$ (equal cost for making either type of wrong decision), then the risk reduces to,

$$C(D) = P_0 [1 - \int_{R_1} f_z(Z/H_1) dz] + q_0 \int_{R_1} f_z(Z/H_2) dz.$$

$$= P_0 \int_{R_2} f_z(Z/H_1) dz + q_0 \int_{R_1} f_z(Z/H_2) dz.$$

$$= P_0 \cdot P_F + q_0 \cdot P_M \longrightarrow (14).$$

where eq(14) is the probability of making a wrong decision, averaged over hypotheses, which is the same as the probability of error used as the optimization criterion. Thus Bayes receives with thus special cost assignment are minimum - probability of error - receiver.

$$\frac{f_z(Z/H_2)}{f_z(Z/H_1)} \underset{H_1}{\overset{H_2}{\gtrless}} \frac{P_0 (C_{21} - C_{11})}{q_0 (C_{12} - C_{22})}.$$

Maximum a posteriori (MAP) Detector :=

Let $C_{11} = C_{22} = 0$ & $C_{21} = C_{12}$ in eq (13), we can rearrange the equation in the form

$$\frac{f_z(z/H_2) \cdot p(H_2)}{f_z(z)} \underset{H_1}{\overset{H_2}{\gtrless}} \frac{f_z(z/H_1) \cdot p(H_1)}{f_z(z)} \longrightarrow (15)$$

where the definitions of P_0 and q_0 have been Substituted, both sides of eq (13), have been multiplied by $p(H_2)$ and both sides have been divided by,

$$f_z(z) \triangleq f_z(z/H_1) \cdot p(H_1) + f_z(z/H_2) \cdot p(H_2) \longrightarrow (16)$$

=) Using Bayes rule, the above eq becomes,

$$p(H_2/z) \underset{H_1}{\overset{H_2}{\gtrless}} p(H_1/z) \rightarrow (17) \; [C_{11} = C_{22} = 0 \; \& \; C_{12} = C_{12}].$$

Eq (17) states that the most probable hypothesis, given a particular observation z is to be chosen in order to minimize the risk, which for the special cost assignment assumed is equal to the probability of error.

=) The probabilities $p(H_1/z)$ and $p(H_2/z)$ are called a posteriori probabilities, for they gave the probability of a particular hypothesis after the observation of z, in contrast to $p(H_1)$ and $p(H_2)$ which gave us the probabilities of the same events before observation of z.

=) Because the hypothesis corresponding to the maximum a posteriori probability choosen, such detectors are referred to as maximum a posteriori (MAP) detectors. Minimum probability of error detectors and MAP detectors are therefore equivalent.

Minimax Detectors :=

The minimax decision rule corresponding to the Bayes decision rule, where the a posteriori probabilities have been choosen to make the Bayes risk a maximum.

* NEYMANN PEARSON CRITERION TEST :=

In Radar detection,

1. P_F - False Arm probability.
2. P_D - Detection probability.
3. P_M - Missing probability.

Size of the test :

The value of α is chosen as the value of P_F.

$$\alpha \rightarrow \text{size of the test.}$$

power of the test :

The value of β is chosen as the maximum attained value of P_D.

$\Rightarrow$ let us now obtain the minimizing P_M subject to constraint.

$$P_F \rightarrow \alpha \longrightarrow (1).$$

we construct the objective function

$$\boxed{J = P_m + \lambda (P_F - \alpha)} \longrightarrow (2).$$

where $\lambda \geq 0$ is lagrangian multiplier.

making use of the equation from probability of error,

$$P_F = P(D_1/H_0) = \int_{y_1} P(y/H_0)\,dy \rightarrow (3).$$

$$P_M = P(D_0/H_1) = \int_{y_0} P(y/H_1)\,dy \rightarrow (4)$$

Sub (3), (4) in eq (2).

$$J = \int_{y_0}^{y_1} P(y/H_1)\,dy + \lambda\left(\int_{y_1} P(y/H_0)\,dy - \alpha\right)$$

H_0 is chosen if $P(H_1/y) < P(H_0/y)$.

H_1 is chosen if $P(H_1/y) > P(H_0/y)$.

Generally,

Decision rule is written as

$$\frac{P(H_1/y)}{P(H_0/y)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

⟹ Thus using decision rule the likelihood ratio minimize the value of P_M.

The decision made becomes

$$V(y) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda.$$

$$V(y) = \frac{P(y/H_1)}{P(y/H_0)}$$ is the likelihood Ratio.

⟹ The threshold of the test γ is equal to the lagrangian multiplier λ and is chosen to satisfy the constraints ~~of~~ on P_F.

Therefore we have

$$\alpha = P_F = \int_{y_1} p(y/H_0)\,dy = \int_{n}^{\alpha} r\left(\frac{v(y)}{H_0}\right) dy.$$

$\Rightarrow$ Thus using Neymann - pearson test the P_F is reduced to its minimum value α we can attain as much maximum value as to the detection probability, P_D.