UNIT – IV ANALYSIS OF DISCRETE TIME SIGNALS

Spectrum of D.T signals, Discrete Time Fourier Transform (DTFT), Discrete Fourier Transform (DFT) – Basic principles of z-transform – z-transform definition – Region of convergence – Properties of ROC – Properties of z-transform – Poles and Zeros – Inverse z-transform using Contour integration – Residue Theorem, Power Series expansion and Partial fraction expansion.

Discrete Time Fourier Transform (DTFT)

The discrete-time Fourier transform (DTFT) of a real, discrete-time signal x[n] is a complex-valued function defined by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

for any (integer) value of n.

Inverse Discrete Time Fourier Transform (IDTFT)

The function X ($e^{j\omega}$) or X (ω) is called the Discrete-Time Fourier Transform(DTFT) of the discrete-time signal x(n). The inverse DTFT is defined by the following integral:

$$x(n) = \frac{1}{2}\pi \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Properties of DTFT

Property	Periodic signal	Fourier Series Coefficients
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k \cdot e^{-jk\left(\frac{2\pi}{N}\right) n_0}$
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Frequency Shifting	$e^{jMw_0 n} x[n]$	a_{k-M}
First Difference	$x[n] - x[n-1]$	$\left(1 - e^{-jk\left(2\pi/N\right)}\right) a_k$
Conjugate Symmetry for Real Signals	x[n] real	$a_k = a^*_{-k}$
Real & Even Signals	x[n] real and even	a_k real and even
Real & Odd signals	x[n] real and odd	a_k purely imaginary and odd
Even-Odd Decomposition Of Real Signals	$x_e[n] = Ev\{x[n]\} \quad [x[n] real]$ $x_o[n] = Od\{x[n]\} \quad [x[n] real]$	$\text{Re}\{a_k\}$ $j\,\text{Im}\{a_k\}$
Parseval's Relation	$\frac{1}{N} \sum_{n=\langle N \rangle} \lvert x[n] \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$	

1. Find the DTFT of an impulse function which occurs at time zero.

$$x[n] = \delta[n]$$

$$X\left(e^{jw}\right) = \sum \delta[n] e^{-jwn} = 1$$

$$\delta[n] \overset{F}{\Leftrightarrow} 1$$

$$\delta[n-1] \overset{F}{\Leftrightarrow} (1) \cdot e^{-jw(1)}$$

Discrete Fourier Transform

The DFT is used to convert a finite discrete time sequence x (n) to an N point frequency domain sequence denoted by X (K). The N point DFT of a finite duration sequence x (n) is defined as

N-1
X (K) = ∑ x (n) e-j2ռnk/Nfor K=0, 1, 2, …….N-1
n=0

The discrete Fourier transform (DFT) is the Fourier transform for finite-length sequences because, unlike the (discrete-space) Fourier transform, the DFT has a discrete argument and can be stored in a finite number of infinite word-length locations. Yet, it turns out that the DFT can be used to exactly implement convolution for finite-size arrays

Inverse Discrete Fourier Transform

n=0

The IDFT is used to convert the N point frequency domain sequence X (K) to an N point time sequence. The IDFT of the sequence X (K) of length N is defined as

N-1
x (n) =1/N ∑X (K) $e^{+j2ռnk/N}$ for n=0, 1,2,……N-1
K=0

Properties of DFT

1. Periodicity: X (K+N) =X (K) for all K.
2. Linearity: DFT[a1 x1 (n)+a2 x2(n)]=a1 X1 (K)+a2 X2 (K)
3. DFT of time reversed sequence: DFT[x(N-n)]=X(N-K)
4. Circular convolution :DFT[x1(n)*x2(n)]=X1(K) X2(K)
5. Shifting: If DFT {x (n)} =X (K), then DFT{x (n-no)} =X (K) e $^{-j2ռno k/N}$
6. Symmetry property

   Re[X(N-k)]=ReX(k)

   This implies that amplitude has symmetry

   Im[X(N-k)]= - Im[X(k)]

   This implies that the phase spectrum is antisymmetric.
7. If x[n] is an even function $x_e[n]$ then

$$F\left[x_e[n]\right] = X_e(k) = \sum_{n=0}^{N-1} x_e[n]\cos(k\Omega nT)$$

This implies that the transform is also even

8. If x[n] is odd function $x_o[n]$ than

$$F[x_o[n]] = X_o(k) = -j\sum_{n=0}^{N-1} x_o[n]\sin(k\Omega nT)$$

This implies that the transform is purely imaginary and odd

9. Parseval's Theorem

The normalized energy in the signal is given by either of the following expressions

$$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N}\sum_{k=0}^{N-1}|X(k)|^2$$

10. Delta Function

$$F[\delta(nT)] = 1$$

11. Unit step function

$$F[u[n]] = \frac{1}{1-e^{-jw}} + \sum_{k=-\infty}^{\infty}\pi\delta(w+2\pi k)$$

$$F\left[e^{jw_0 n}\right] = \sum_{k=-\infty}^{\infty} 2\pi\delta(w - w_0 + 2\pi k)$$

12. Fourier transform of a CT complex exponential is interpreted as an impulse at $w=w_0$. For discrete-time we expect something similar but difference is that DTFT is periodic in w with period 2π. This says that FT of x[n] should have impulses at w_0, $w_0 \pm 2\pi$, $w_0 \pm 4\pi$ etc.

$$\alpha^n u[n] \quad (|n| < 1) \quad \overset{F}{\Leftrightarrow} \quad \frac{1}{1-\alpha e^{-jw}}$$

13. Linear cross-correlation of two data sequences or series may be computed using DFTs. The linear cross correlation of two finite-length sequences $x_1[n]$ and $x_2[n]$ each of length N is defined to be:

$$r_{x_1x_2}(j) = \frac{1}{N}\sum_{n=-\infty}^{\infty} x_1(n)x_2(n+j) \qquad ,-\infty \le j \le \infty$$

Circular correlation of finite length periodic sequences $x_{1p}[n]$ and $x_{2p}[n]$ is described as:

$$r_{cx_1x_2}(j) = \frac{1}{N}\sum_{n=0}^{N-1} x_{1p}(n)x_{2p}(n+j) \qquad , j = 0,.......,(N-1)$$

This circular correlation can be evaluated using DFTs as shown below:

$$r_{cx_1x_2}(j) = F^{-1}\left[X_1^*(k)X_2(k)\right]$$

The circular correlation can be converted into a linear correlation by using augmenting zeros. If the sequences are $x_1[n]$ of length N_1 and $x_2[n]$ of length N_2, then their linear correlation will be of length N1+N2-1.

To achieve this $x_1[n]$ is replaced by $x_{1a}[n]$ which consists of $x_1[n]$ with (N_2-1) zeros added and $x_2[n]$ is augmented by (N_1-1) zeros to become $x_{2a}[n]$.

$$\Rightarrow \qquad r_{x_1x_2}(j) = F^{-1}\left[X_{1a}^*(k)X_{2a}(k)\right]$$

1. Find the DFT of the following signal x(n)=δ(n)

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N} \text{ for } K=0, 1, 2\ldots,N-1$$

$$X(K) = \sum_{n=0}^{N-1} \delta(n) e^{-j2\pi nk/N} \text{ for } K=0, 1, 2,\ldots N-1$$

$$X(K) = 1$$

2. Consider a length-N sequence defined for n = 0,1,2,……,(N-1) where

$$x[n] = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$ Find the DFT of the given sequence.

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1,2,\ldots\ldots,(N-1)$$

The N-point DFT is equal to $= 1$

Basic Principles of Z Transform:

The z-transform is useful for the manipulation of discrete data sequences and has acquired a new significance in the formulation and analysis of discrete-time systems. It is used extensively today in the areas of applied mathematics, digital signal processing, control theory, population science, economics.

These discrete models are solved with difference equations in a manner that is analogous to solving continuous models with differential equations. The role played by the z-transform in the solution of difference equations corresponds to that played by the Laplace transforms in the solution of differential equations.

Types of Z Transform

Unilateral Z-transform

Alternatively, in cases where x[n] is defined only for n ≥ 0, the single-sided or unilateral Z-transform is defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

In signal processing, this definition can be used to evaluate the Z-transform of the unit impulse response of a discrete-time causal system

Bilateral Z-transform

The bilateral or two-sided Z-transform of a discrete-time signal x[n] is the formal power series X(Z) defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where n is an integer and z is, in general, a complex number:

$z = A e^{j\phi} = A(\cos\phi + j\sin\phi)$

where A is the magnitude of z, j is the imaginary unit, and ϕ is the complex argument (also referred to as angle or phase) in radians.

Inverse Z Transform

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

1. **The z-transform of the sequence** $x_n = \cos(an)$ **find its Z transform**

$$\mathcal{Z}[x_n] = \mathcal{Z}[\cos(a\,n)] = \mathcal{Z}\left[\frac{1}{2}e^{i a n} + \frac{1}{2}e^{-i a n}\right]$$

$$= \frac{1}{2}\mathcal{Z}\left[e^{i a n}\right] + \frac{1}{2}\mathcal{Z}\left[e^{-i a n}\right] = \frac{1}{2}\frac{z}{z - e^{i a}} + \frac{1}{2}\frac{z}{z - e^{-i a}}$$

$$= \frac{1}{2}\left(\frac{z\left(z - e^{-i a}\right)}{(z - e^{i a})(z - e^{-i a})} + \frac{z\left(z - e^{i a}\right)}{(z - e^{i a})(z - e^{-i a})}\right)$$

$$= \frac{z\left(2z - e^{i a} - e^{-i a}\right)}{2(z - e^{i a})(z - e^{-i a})} = \frac{z\left(z - \frac{e^{i a}+e^{-i a}}{2}\right)}{(z - e^{i a})(z - e^{-i a})}$$

$$= \frac{z(z - \cos(a))}{(z - e^{i a})(z - e^{-i a})} = \frac{z(z - \cos(a))}{z^2 - 2z\cos(a) + 1}$$

2. Find the z-transform of the unit pulse or impulse sequence $x_n = \delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$.

$$X(z) = \mathcal{Z}[x_n] = \sum_{n=0}^{\infty} x_n z^{-n} = 1 + \sum_{n=1}^{\infty} 0\, z^{-n} = 1$$.

3. The z-transform of the unit-step sequence $x_n = u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$ is $X(z) = \frac{z}{z-1}$
.

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} (z^{-1})^n$$

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

4. The z-transform of the sequence $x_n = b^n$ is $X(z) = \frac{z}{z - b}$.

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} b^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{b^n}{z^n}\right) = \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^n$$

$$= \frac{1}{1 - \frac{b}{z}} = \frac{z}{z - b}$$

5. The z-transform of the exponential sequence $x_n = e^{an}$ is $X(z) = \frac{z}{z - e^a}$.

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} e^{an} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{an}}{z^n}\right) = \sum_{n=0}^{\infty} \left(\frac{e^a}{z}\right)^n$$

$$= \frac{1}{1 - \frac{e^a}{z}} = \frac{z}{z - e^a}$$

6. Prove $Z[\{a^n\}] = \frac{z}{z-a}$

<u>Proof</u>

$$Z[\{a^n\}] = \sum_{0}^{\infty} a^n z^{-n} = \sum_{0}^{\infty} \left[\frac{a}{z}\right]^n = 1 + \left[\frac{a}{z}\right] + \left[\frac{a}{z}\right]^2 + \left[\frac{a}{z}\right]^3 + \left[\frac{a}{z}\right]^4 + \ldots\ldots\ldots\ldots$$

$$= \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}, \quad \left|\frac{a}{z}\right| < 1 \quad \text{i.e; } |z| > |a|$$

	Sequence	**z - transform**
1	$\delta[n]$	1
2	$u[n]$	$\frac{z}{z - 1}$
3	b^n	$\frac{z}{z - b}$
4	$b^{n-1} u[n-1]$	$\frac{1}{z - b}$
5	e^{an}	$\frac{z}{z - e^a}$
6	n	$\frac{z}{(z - 1)^2}$
7	n^2	

		$\frac{z(z+1)}{(z-1)^3}$
8	$b^n n$	$\frac{bz}{(z-b)^2}$
9	$e^{an} n$	$\frac{z e^a}{(z-e^a)^2}$
10	$\sin(an)$	$\frac{\sin(a)\, z}{z^2 - 2\cos(a)\, z + 1}$
11	$b^n \sin(an)$	$\frac{\sin(a)\, b z}{z^2 - 2\cos(a)\, b z + b^2}$
12	$\cos(an)$	$\frac{z(z - \cos(a))}{z^2 - 2\cos(a)\, z + 1}$
13	$b^n \cos(an)$	$\frac{z(z - b\cos(a))}{z^2 - 2\cos(a)\, b z + b^2}$

Properties of Z-Transform

Z-Transform has the following properties:

1. Linearity Property

If $x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then linearity property states that

$$a\,x(n) + b\,y(n) \stackrel{\text{Z.T}}{\longleftrightarrow} a\,X(Z) + b\,Y(Z)$$

2. Time Shifting Property

   If $x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then Time shifting property states that

$$x(n-m) \stackrel{\text{Z.T}}{\longleftrightarrow} z^{-m} X(Z)$$

3. Multiplication by Exponential Sequence Property

   If $x(n) \stackrel{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then multiplication by an exponential sequence property states that

$$a^n \, . \, x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z/a)$$

4. Time Reversal Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then time reversal property states that

$$x(-n) \overset{\text{Z.T}}{\longleftrightarrow} X(1/Z)$$

5. Differentiation in Z-Domain OR Multiplication by n Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

Then multiplication by n or differentiation in z-domain property states that

$$n^k x(n) \overset{\text{Z.T}}{\longleftrightarrow} [-1]^k z^k \frac{d^k X(Z)}{dZ^K}$$

6. Convolution Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \overset{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then convolution property states that

$$x(n) * y(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z)$$

7. Correlation Property

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \overset{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then correlation property states that

$$x(n) \otimes y(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z^{-1})$$

8. Initial Value and Final Value Theorems

Initial value and final value theorems of z-transform are defined for causal signal.

Initial Value Theorem

For a causal signal x(n), the initial value theorem states that

$$x(0) = \lim_{z \to \infty} X(z)$$

This is used to find the initial value of the signal without taking inverse z-transform

Final Value Theorem

For a causal signal x(n), the final value theorem states that

$$x(\infty) = \lim_{z \to 1} [z-1] X(z)$$

This is used to find the final value of the signal without taking inverse z-transform.

Region of Convergence (ROC) of Z-Transform

The range of variation of z for which z-transform converges is called region of convergence of z-transform.

Properties of ROC of Z-Transforms

- ROC of z-transform is indicated with circle in z-plane.
- ROC does not contain any poles.
- If x(n) is a finite duration causal sequence or right sided sequence, then the ROC is entire z-plane except at z = 0.
- If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROC is entire z-plane except at z = ∞.
- If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. |z| > a.
- If x(n) is a infinite duration anti-causal sequence, ROC is interior of the circle with radius a. i.e. |z| < a.
- If x(n) is a finite duration two sided sequence, then the ROC is entire z-plane except at z = 0 & z = ∞.

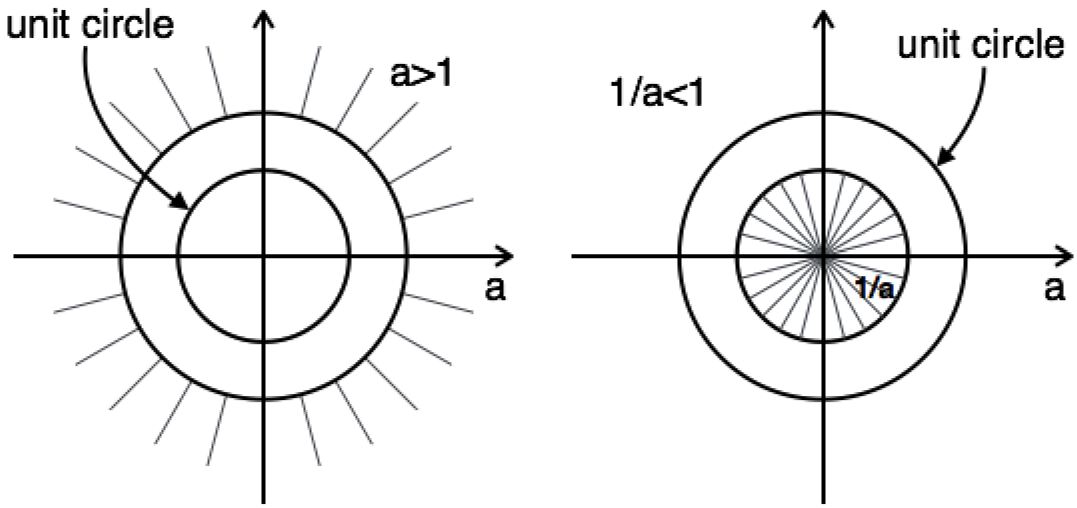
The concept of ROC can be explained by the following example:

Example 1: Find z-transform and ROC of

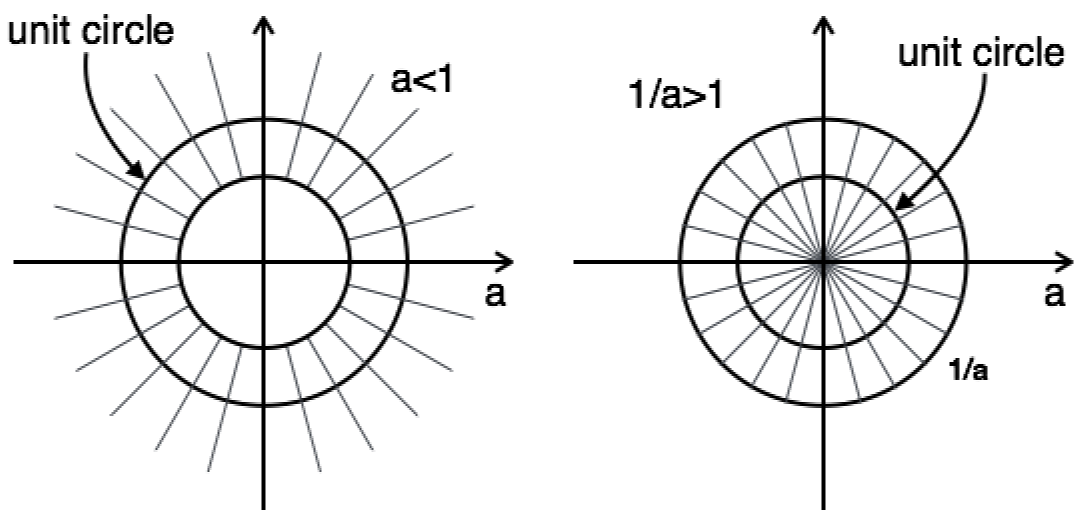
$$a^n u[n] + a^{-} nu[-n-1]$$

$$Z.T[a^n u[n]] + Z.T[a^{-n} u[-n-1]] = \frac{Z}{Z-a} + \frac{Z}{Z-\frac{1}{a}}$$

The plot of ROC has two conditions as a > 1 and a < 1, as the value of ‘a’ is not known.



In this case, there is no combination ROC.



Here, the combination of ROC is from

$$a < |z| < \frac{1}{a}$$

Hence for this problem, z-transform is possible when a < 1.

$$\mathcal{Z}[x[n]] = \frac{1}{1-az^{-1}} - \frac{1}{1-a^{-1}z^{-1}} = \frac{a^2-1}{a}\frac{z}{(z-a)(z-1/a)}$$

1. The Z transform of a right sided signal $x[n] = a^n u[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

For this summation to converge, i.e., for $X(z)$ to exist, it is necessary to have $|az^{-1}| < 1$, i.e., the ROC is $|z| > |a|$. As a special case when $\underline{a = 1}$, $x[n] = u[n]$ and we have

$$\mathcal{Z}[u[n]] = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

2. The Z-transform of a left sided signal $x[n] = -a^n u[-n-1]$ is:

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

For the summation above to converge, it is required that $|a^{-1}z| < 1$, i.e., the ROC is $|z| < |a|$. Comparing the two examples above we see that two different signals can have identical z-transform, but with different ROCs.

3. Find the inverse of the given z-transform $X(z) = 4z^2 + 2 + 3z^{-1}$. Comparing this with the definition of z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[-2]z^2 + x[-1]z^1 + x[0] + x[1]z^{-1} + x[2]z^{-2}$$

we get $x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$

In general, we can use the time shifting property

$$\mathcal{Z}[\delta[n+n_0]] = z^{n_0}$$

to inverse transform the $X(z)$ given above to $x[n]$ directly.

Zeros and Poles of Z-Transform

All z-transforms in the above examples are rational, i.e., they can be written as a ratio of polynomials of variable z in the general form

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{M} b_k z^k}{\sum_{k=0}^{N} a_k z^k} = \frac{b_M}{a_N} \frac{\prod_{k=1}^{M}(z - z_{z_k})}{\prod_{k=1}^{N}(z - z_{p_k})}$$

where $N(z)$ is the numerator polynomial of order M with roots $z_{z_k}, (k = 1, 2, \cdots, M)$, and $D(z)$ is the denominator polynomial of order N with roots $z_{p_k}, (k = 1, 2, \cdots, N)$.

In general, we assume the order of the numerator polynomial is lower than that of the denominator polynomial, i.e., $M < N$. If this is not the case, we can always expand $X(z)$ into multiple terms so that $M < N$ is true for each of terms.

The zeros and poles of a rational $X(z) = N(z)/D(z)$ are defined as:

- **Zero**: Each of the roots of the numerator polynomial z_z for which $X(z)\Big|_{z=z_z} = X(z_z) = 0$ is a zero of $X(z)$.

If the order of $D(z)$ exceeds that of $N(z)$ (i.e., $N > M$), then $X(\infty) = 0$, i.e., there is a zero at infinity:

$$\left.\frac{b_1 z + b_0}{a_2 z^2 + a_1 z + a_0}\right|_{z \to \infty} = 0$$

- **Pole**: Each of the roots of the denominator polynomial z_p for which $\left. X(z) \right|_{z=z_p} = X(z_p) = \infty$ is a pole of $X(z)$.

If the order of $N(z)$ exceeds that of $D(z)$ (i.e., $M > N$), then $X(\infty) = \infty$, i.e, there is a pole at infinity:

$$\left.\frac{b_2 z^2 + b_1 z + b_0}{a_1 z + a_0}\right|_{z \to \infty} \to \infty$$

Most essential behavior properties of an LTI system can be obtained graphically from the ROC and the zeros and poles of its transfer function $H(z)$ on the z-plane

Inverse Z Transform using Contour Integration Method

1.
$$F(z) = \frac{z\left(z^2 - 2z - 1\right)}{\left(z^2 + 1\right)^2}$$

$$F(z) = 2z \frac{(z-1)^2 - (z-1)(z-2) - (z-2)}{(z-1)(z-2)^2}$$

$$F(z) = 2z \frac{z^2 - 2z + 1 - z^2 + 3z - 2 - z + 2}{(z-1)(z-2)^2}$$

$$F(z) = 2z \frac{1}{(z-1)(z-2)^2}$$

2. Evaluate the inverse z transform of $X(z)=\frac{1}{1-az^{-1}}, |z|>|a|$ using the complex inversion integral.

Long Division Method

The *z*-transform is a power series expansion,

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)z^{-n} = \cdots + x(-2)z^{2} + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

where the sequence values *x*(*n*)are the coefficients of z^{-n} in the expansion. Therefore, if we can find the power series expansion for *X*(*z*), the sequence values *x*(*n*)may be found by simply picking off the coefficients of z^{-n}.

1. Sometimes the inverse transform of a given $X(z)$ can be obtained by long division.

$$X(z)=\frac{1}{1-az^{-1}}$$

By a long division, we get

$$1 \div (1-az^{-1}) = 1 + az^{-1} + a^2z^{-2} + \cdots$$

which converges if the ROC is $|z|>|a|$, i.e., $|az^{-1}|<1$ and we get

$$x[n]=a^n u[n]$$

. Alternatively, the long division can also be carried out as:

$$1 \div (-az^{-1}+1) = -a^{-1}z - a^{-2}z^2 - \cdots$$

which converges if the ROC is $|z|<|a|$, i.e., $|a^{-1}z|<1$ and we get

$$x[n]=-a^n u[-1-n]$$

2. To understand how an inverse Z Transform can be obtained by long division, consider the function

$$F(z) = \frac{z}{z - 0.5}$$

If we perform long division

$$\begin{array}{r|l} & 1 + 0.5z^{-1} + 0.25z^{-2} + \cdots \\ \hline z - 0.5 & z \\ & z - 0.5 \\ \hline & 0.5 \\ & 0.5 - 0.25z^{-1} \\ \hline & 0.25z^{-1} \\ & 0.25z^{-1} - 0.125z^{-2} \\ \hline & \vdots \end{array}$$

we can see that

$$F(z) = 1 + 0.5z^{-1} + 0.25z^{-2} + \cdots.$$

So the sequence f[k] is given by

$$f = \{1, 0.5, 0.25, \cdots\}$$

Upon inspection

$$f[k] = 0.5^k$$

3. Find the Inverse Z Transform using Long Division Method

$$F(z) = \frac{2z^2 + z}{z^2 - 1.5z + 0.5}$$

$$\begin{array}{r|l} & 2 + 4z^{-1} + 5z^{-2} + \cdots \\ \hline z^2 - 1.5z + 0.5 & 2z^2 + z \\ & 2z^2 - 3z + 1 \\ \hline & 4z - 1 \\ & 4z - 6 + 2z^{-1} \\ \hline & 5 - 2z^{-1} \\ & 5 - 7.5z^{-1} + 2.5 \\ \hline & \vdots \end{array}$$

$$F(z) = 2 + 4z^{-1} + 5z^{-2} + \cdots$$

and the sequence f[k] is given by $f = \{2, 4, 5, \cdots\}$

4. E(z) = $\dfrac{0.5}{(z-1)(z-0.6)}$

$$
\begin{array}{r|l}
 & 0.5z^{-2} + 0.8z^{-3} + 0.98z^{-4} + \ldots \\
\hline
z^2 - 1.6z + 0.6 & 0.5 \\
 & \underline{0.5 - 0.8z^{-1} + 0.3z^{-2}} \\
 & 0.8z^{-1} - 0.3z^{-2} \\
 & \underline{0.8z^{-1} - 1.28z^{-2} + 0.48z^{-3}} \\
 & 0.98z^{-2} - 0.48z^{-3}
\end{array}
$$

e(0) = 0, e(1) = 0, e(2) = 0.5, …

Inverse Z Transform using Residue Method:

Find the solution using the formula

$$y[n] = Z^{-1}[Y(z)] = \sum_{i=1}^{k} \mathrm{Res}[Y(z)\, z^{n-1}, z_i]$$

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z) = Y(z)\, z^{n-1}$.

Partial fraction method

Inverse Z Transform by Partial Fraction Expansion

This technique uses Partial Fraction Expansion to split up a complicated fraction into forms that are in the Z Transform table. As an example consider the function

$$F(z) = \frac{2z^2 + z}{z^2 - 1.5z + 0.5}$$

For reasons that will become obvious soon, we rewrite the fraction before expanding it by dividing the left side of the equation by "z."

$$\frac{F(z)}{z} = \frac{2z + 1}{z^2 - 1.5z + 0.5}$$

Now we can perform a partial fraction expansion

$$\frac{F(z)}{z} = \frac{2z+1}{z^2 - 1.5z + 0.5}$$
$$= \frac{2z+1}{(z-1)(z-0.5)}$$
$$= \frac{A}{z-1} + \frac{B}{z-0.5}$$
$$= \frac{6}{z-1} + \frac{-4}{z-0.5}$$

These fractions are not in our table of Z Transforms. However if we bring the "z" from the denominator of the left side of the equation into the numerator of the right side, we get forms that are in the table of Z Transforms; this is why we performed the first step of dividing the equation by "z"

$$F(z) = 6\frac{z}{z-1} - 4\frac{z}{z-0.5}$$

So

$$f[k] = 6u[k] - 4 \cdot 0.5^k$$

or

$$f = \{2, 4, 5, 5.5, \cdots\}$$

PART - A

1. Distinguish between DFT and DTFT?
2. State and prove Parseval's relation for DFT.
3. Find the DFT of the sequence {0,1,0,1}
4. Define DFT and IDFT
5. Find IDFT of X(k) = {1, 0, 1, 0}.
6. Define Z transform
7. Mention the types of Z transform
8. Find the Z transform of u(n)
9. Define ROC
10. Mention the properties of ROC

PART – B

1. Explain the Properties of Z Transform
2. Explain the properties of DFT
3. Mention the Properties of DTFT
4. Find Inverse Z Transform for the following function using partial fraction method

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} \qquad \text{ROC}: |z| > \frac{1}{2}$$

5. Find Inverse Z Transform for the following function using Long division method

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{\left(1 + z^{-1}\right)^2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)} \qquad |z| > 1$$

6. Find Inverse Z Transform for the following function using power series method

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$