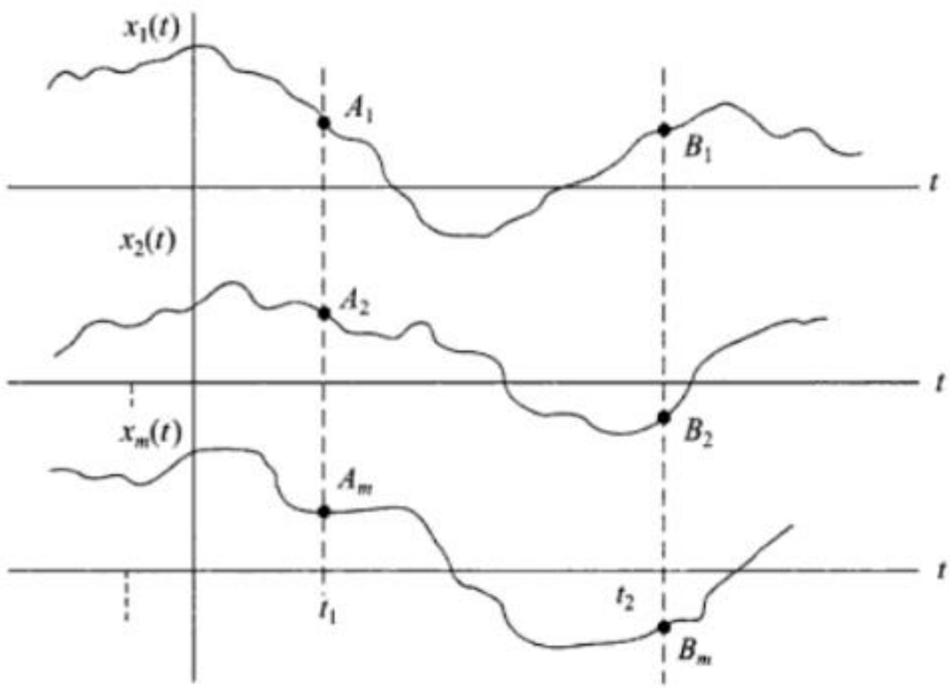
UNIT - II
RANDOM SIGNAL THEORY

RANDOM PROCESS:

- Let us consider an experiment of measuring the temperature of a room.
- Let there be a collection of thermometers. Each thermometer reading is a random variable which can take on any value from the sample space S.
- Also, at different times the reading of thermometers may be different.
- Thus the room temperature is function of a both the sample space and the time.
- In this example, we have extended the concept of random satiable by taking into consideration the time dimension.
- Here we assign a time function x(t, a) to every outcome S.
- There will be a family of all such functions. This family of fimetions S(x.5) is known as random process or stochastic process.
- A random process X(xS)represents an ensemble or a set or a family of time functions where t and S are variables. Is place of '(tS) and A(1.5), the short notations x(t) and X(t) are often used.



- The absve figure shows a few members of the ensemble. [fx,(1) is the reading of first thermometer, st(t)] is the reading of second thermometer and so on.]
- Each member is also known as sample function or ensemble member or realization of the process. A random process represents single time functions when t is variable and s is fixed. $x1(t)$ and $x2(t)$ are the examples of single time functions.
- To determine the statistics of the mom temperature, say mean value, we may follow one of the following two procedures.

- I. We may fix t to some value, say t.
- The result is a random variable
    - $X(t1, S) = X(t1)$— [A1 A2 ... Am].
- The mean value of $X(t1)$ $E[X(t1)]$, can now be calculated. It is known as ensemble average. It may be noted that ensemble average is a function of time. There is an ensemble average corresponding to each time.
- Thus at time 12, we have
    - $X(t2, s)= X(t2)=$ [B1 B2 ... Bm]
- The ensemble average corresponding to time t, = $E[X(t2)]$, can also be found out. Similarly, ensemble average corresponding to any time can be found out.
- We may consider a sample function, say si(t) over the entire time scale. Then the mean value of x1 (t) is defined as

$$< x_1(t) > = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_1(t) dt$$

The expected value of all mean values is known as time average and is given as,

$$< x(t) > = E [< x(t) >]$$

A random process for which the mean values of all sample functions are the same is known as the regular random process.

In this case,

$$< x1(t) > = < x2(t) > = \ldots\ldots\ldots\ldots < x(t) > = < x(a) >$$

For some process, ensembles average is independent to time (i.e)

$$E(x(t1)) = E[x(t2)] = \ldots\ldots\ldots = E[x(t)]$$

Such process are known as stationary process in restricted sense.

STATIONARY PROCESS:

- Such processes are known as stationary processes in restricted sense. (Here it is restricted to mean.

ERGODIC PROCESS:

- When an ensemble average is equal to the time average, then the process is known as ergodic process in restricted sense.
- When all statistical ensemble properties are equal to statistical time properties, then the process is known as ergodic process in strict sense. When we say ergodic process. then it is meant that the process ergodic in strict sense.
- It may be rioted that ergodic process is a subset of a stationary process, i.e. if a process is ergodic, then it is also stationary, but the vice versa is not necessarily true.

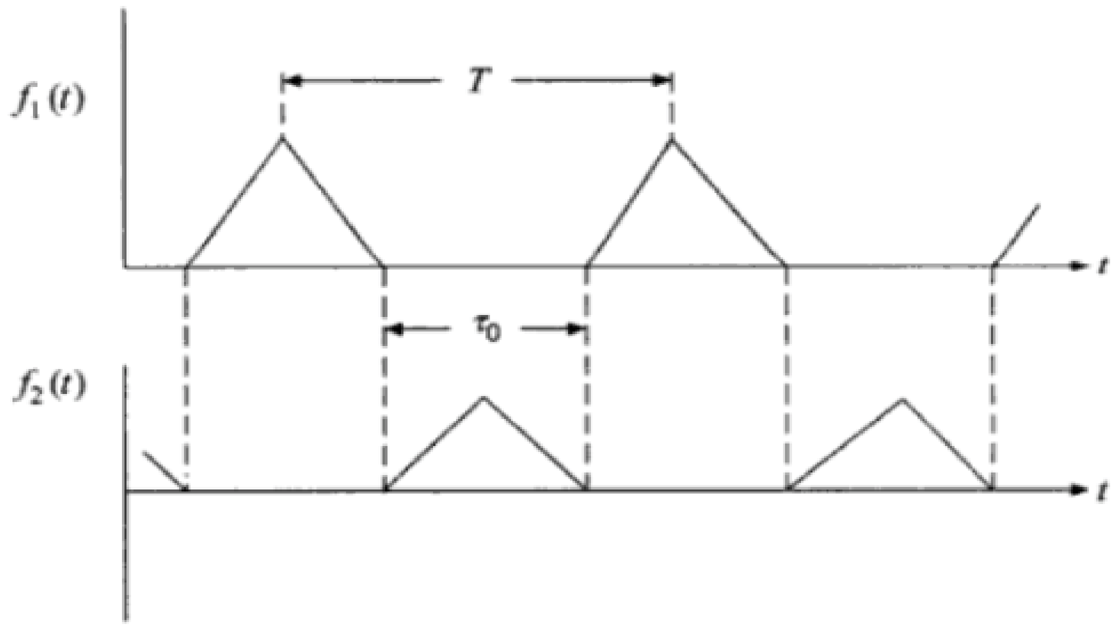
SEARCHING PARAMETER: (T - TOW)

- The time delay introduced in the expression of correlation is known as searching or scanning parameter.
- The time t is a dummy variable and the correlation R1,s (t-tow) a function of the delayed parameter.

UN CORRELATED (incoherent) SIGNALS:

- Functions for which , (r) is zero for all values of r are called iincorrelated or incoherent functions. Correlation is also known as coherence.

During the process of scanning, it is essential to specify which function is being shifted. In general, Ru(r), which is obtained by shifting /2W in one direction is not the same as Ru(T), which is obtained by shiftingfl(r) in the same direction. It can be seen that , for real functions F1(t) and f2(t),



be seen that , for real functions F1(t) and f2(t),

Where R1, 2(z) id defined as.

$$R_{1,2}(-\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_1(t) f_2(t-\tau)\,dt$$

- Thus the cross correlation function is non- commulative.
- This is true as the shifting of one function in one direction is equivalent to shifting the other in opposite direction

$$R_{2,1}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_1(t+\tau) f_2(t)\,dt = R_{1,2}(-\tau)$$

Correlation:

* The Correlation, or cross-Correlation between two two waveforms is the measure of similarity between one waveform, and time delayed version of the other waveform.

* Consider two general Complex function $f_1(t)$ & $f_2(t)$ which may or may not be periodic, and not restricted, to finite interval.

* The cross-correlation, or simply Correlation $R_{1,2}(\tau)$ between two function is defind as follows:

$$R_{1,2}(\tau) = \lim_{T \to \infty} \int_{-T/2}^{T/2} f_1(t)\, f_2^*(t+\tau)\, dt$$

* This represent the Shift of function $f_2(t)$ by an amount $(-\tau)$. A similar effect can be obtained by shifting $f_1(t)$ by an

amount (f τ). Hence, Correlation may also be defined in an equivalent way, as

$$R_{1,2}(\tau) = \lim_{T\to\infty} \int_{-T/2}^{T/2} f_1(t-\tau)\, f_2^*(t)\, dt$$

* Let us define the Correlation for two cases

(i) energy (non-periodic) function)

(ii) power (periodic) function

* In the definition of correlation, limits of integration may be taken as infinite for energy signals,

$$R_{1,2}(\tau) = \int_{-\infty}^{\infty} f_1(t)\, f_2^*(t+\tau)\, dt = \int_{-\infty}^{\infty} f_1(t-\tau)\, f_2^*(t)\, dt.$$

* For power signals of period T_0 may not converge. Hence average Correlation over a period T_0 is defind as,

$$R_{1,2}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_1(t)\, f_2^*(t+\tau)\, dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_1(t-\tau)\, f_2^*(t)\, dt$$

Auto-Correlation:

* Auto correlation is a special form of cross-correlation. It is defind as the correlation of a function with itself.

* If $f_1(t) = f_2(t) = f(t)$. Then the expression for auto correlation as given below;

$$R(\tau) = \lim_{T \to \alpha} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, f^*(t \mp \tau)\, dt$$

which is equivalent to

$$R(\tau) = \lim_{T \to \alpha} \frac{1}{T} \int_{-T/2}^{T/2} f(t-\tau)\, f^*(t)\, dt$$

* It is obvious that auto correlation function is the measure of similarity of a function with its delayed replica. An analogy case may be stated as "Comparison of your present photograph and the photograph taken five years back".

Properties of Auto-Correlation

(a) The auto correlation for $\tau = 0$ is average power P of the signal, ie

$$R(0) = \lim_{T \to \infty} \int_{-T/2}^{T/2} f(t) f^*(t)\, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2\, dt = P$$

For energy signal, R(0) is the energy of the signal $R(0) = \int_{-\infty}^{\infty} |f(t)|^2 dt = E$

(b) The auto correlation function exhibits Conjugate symmetry. ie

$$R(\tau) = R^*(-\tau)$$

For real function,

$$R(\tau) = R(-\tau)$$

* In other words, the real part of $R(\tau)$ is an even function of τ and the imaginary part is an odd function of τ.

* This property follows directly from the definition of autocorrelation function. Recall that shifting the function towards right or left is equivalent when integration extends from $-\infty$ to ∞

ie $R(\tau) = R(-\tau)$

(c) The maximum value of autocorrelation function $R(\tau)$ occurs at origin,

$R(0) \geq R(\tau)$ for all τ.

That is, $R(0)$ is the maximum value of $R(\tau)$.

Power Spectral Density

If $\{x(t)\}$ is a stationary process [either in the strict sense or wide sense] with auto correlation function $R_{xx}(\tau)$ then the Fourier Transform of $R_{xx}(\tau)$ is called the power sepectral density function of $\{x(t)\}$ and is denoted by $S_{xx}(\omega)$ (or) $S(\omega)$

$$\text{(ie)} \quad S_{xx}(\omega) = \int_{-\alpha}^{\alpha} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

If 'ω' is replaced by $2\pi f$. Where 'f' is the frequency variable, then the power spectral density function will be a function of f denoted by $S_{xx}(f)$

$$S_{xx}(f) = \int_{-\alpha}^{\alpha} R_{xx}(\tau)\, e^{-j2\pi f\tau}\, d\tau.$$

* Given the power spectral density function $S_{xx}(\omega)$, the autocorrelation function $R_{xx}(\tau)$ is given by the Inverse Fourier Transform of $S_{xx}(\omega)$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$$

(or) $$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(f)\, e^{i2\pi f\tau}\, df.$$

Properties of Power Spectral Density

Property (1):

The value of the spectral density function at zero frequency is equal to the total area under the graph of the autocorrelation function.

ie $$S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, d\tau$$

Property (2):

The spectral density function of a real random process is an even function of frequency,

(ie) $S_{XX}(-\omega) = S_{XX}(\omega)$; if $\{X(t)\}$ is real.

Property (3):

The spectral density and the auto correlation function of a real wide sense stationary process form a Fourier cosine transform pair.

$$S_{XX}(\omega) = 2 \cdot F_c\{R_{XX}(\tau)\}$$

Property (4):

The mean square value of a wide sense stationary process is equal to the total area under the graph of the spectral density

$$E[X^2(t)] = R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(f)\,df.$$

Property (5):

WIENER - KHINCHINE THEOREM

If $X_T(\omega)$ is the Fourier Transform of the trancated random process defind as,

$$X_T(t) = \begin{cases} X(t) & ; -T \le t \le T \\ 0 & ; \text{otherwise} \end{cases}$$

Where $\{X(t)\}$ is a real WSS process with power spectral function $S_{XX}(\omega)$ then

$$S_{XX}(\omega) = \lim_{T \to \infty} \left[\frac{1}{2T} E\left\{ |X_T(\omega)|^2 \right\} \right]$$

Proof:

Given that,

$$X_T(\omega) = F\{X_T(t)\}$$

$$= \int_{-\infty}^{\infty} X_T(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T}^{T} X_T(t)\, e^{-j\omega t}\, dt.$$

Since $\{X(t)\}$ is real, we have

$$|X_T(\omega)|^2 = X_T(\omega) \cdot \overline{X_T(\omega)}$$

$$= X_T(\omega) \cdot X_T(-\omega)$$

$$= \int_{-T}^{T} X(t_1)\, e^{-j\omega t_1}\, dt_1 \int_{-T}^{T} X(t_2)\, e^{j\omega t_2}\, dt_2$$

$$|X_T(\omega)|^2 = \int_{-T}^{T}\int_{-T}^{T} x(t_1)\, x(t_2)\, e^{-j\omega(t_1-t_2)}\, dt_1\, dt_2$$

$$\therefore E\{|X_T(\omega)|^2\} = \int_{-T}^{T}\int_{-T}^{T} E\{x(t_1)\, x(t_2)\}\, e^{-j\omega(t_1-t_2)}\, dt_1\, dt_2$$

We know that the autocorrelation function,

$$R_{XX}(t_1, t_2) = E\{x(t_1)x(t_2)\} = \text{a function of } (t_1-t_2)$$

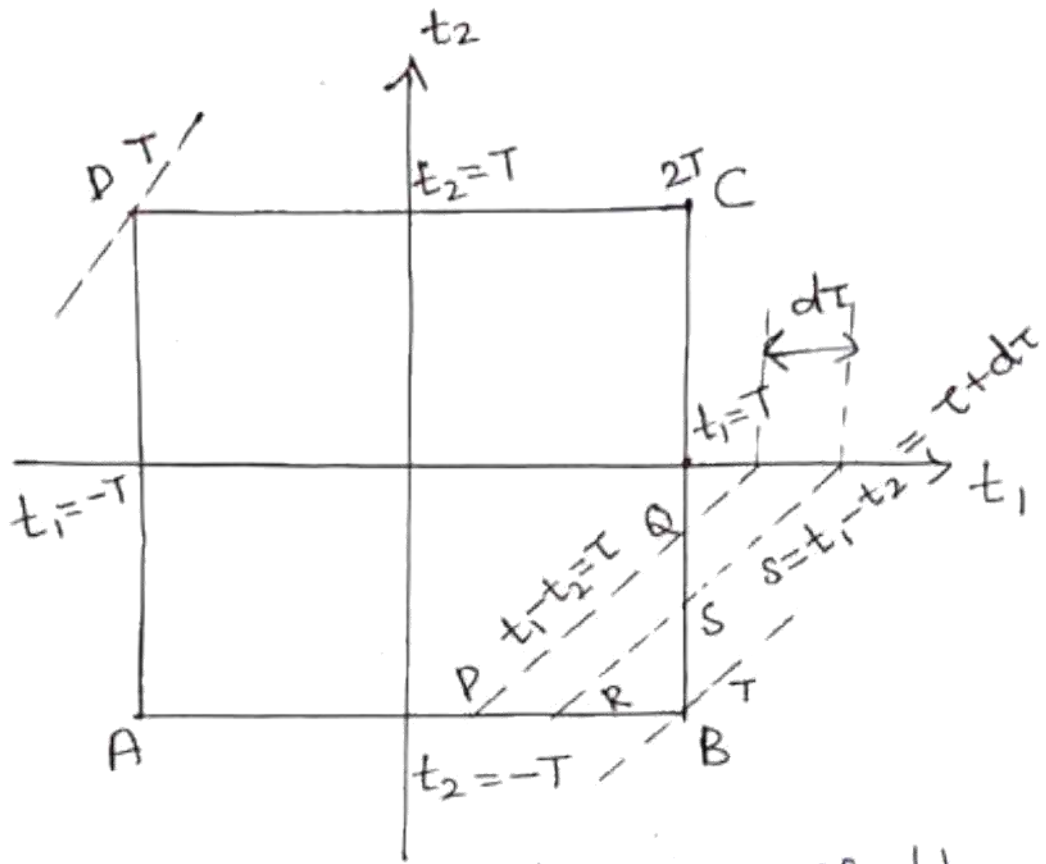
Hence $E|X_T(\omega)|^2 = \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1-t_2) \cdot e^{-j\omega(t_1-t_2)}\, dt_1\, dt_2$

$$= \int_{-T}^{T}\int_{-T}^{T} \phi(t_1-t_2) \cdot dt_1 \cdot dt_2.$$

Where $\phi(t_1 - t_2) = R_{XX}(t_1-t_2)\, e^{-j\omega(t_1-t_2)}$

The above double integral is evaluated as follows:

We will evaluate the double integral over the area of the square ABCD bounded by $t_1 = -T, +T$ and $t_2 = T, -T$ as shown in the figure.



Let us divide the area of the Square in the number of strips like PQRS. PQ is given by $t_1 - t_2 = \tau$ and RS is given by $t_1 - t_2 = \tau + d\tau$

* When PQRS is at the initial position D, $t_1 - t_2 = -2T$. When PQRS is at final position B, $t_1 - t_2 = 2T$.

* Hence when τ varies from $-2T$ to $2T$, the area ABCD is Covered.

Now $dt_1 \cdot dt_2$ = elemental area of the $t_1 t_2$ plane.

= area of PQRS

Area of PQRS = Area of triangle PBQ − Area of triangle RBS

$$= \frac{1}{2}(2T-\tau)^2 - \frac{1}{2}(2T-\tau-d\tau)^2$$

$$= \frac{1}{2}\left[4T^2+\tau^2-4T\tau-(4T^2+\tau^2+d\tau^2-4Td\tau+2\tau d\tau-4T\tau)\right]$$

$$= \frac{1}{2}\left[4T\,d\tau - 2\tau d\tau\right] \quad \text{omitting } d\tau^2$$

$$= (2T-\tau)\,d\tau$$

Note that in the above equation we have taken $BQ = PB = T-(t-T)$

$$= \begin{cases} 2T-\tau & ; \text{if } \tau \geq 0 \\ 2T+\tau & ; \text{if } \tau < 0 \end{cases}$$

For all τ, we have

Area of PQRS $= dt_1 \cdot dt_2 = (2T-|\tau|)\,d\tau$

$$E\left\{|X_T(\omega)|^2\right\} = \int_{-2T}^{2T} \phi(\tau)\left(2T - |\tau|\right) d\tau$$

$$\frac{1}{2T} E\left\{|X_T(\omega)|^2\right\} = \int_{-2T}^{2T} \phi(\tau) \left\{1 - \frac{|\tau|}{2T}\right\} d\tau$$

Taking limit $T \to \alpha$, we have

$$\lim_{T\to\alpha} \frac{1}{2T} E\left\{|X_T(\omega)|^2\right\} = \lim_{T\to\alpha} \int_{-2T}^{2T} \phi(\tau)\, d\tau$$

$$- \lim_{T\to\alpha} \frac{1}{2T} \int_{-2T}^{2T} |\tau| \phi(\tau)\, d\tau$$

$$= \int_{-\alpha}^{\alpha} \phi(\tau)\, d\tau$$ [assuming that $\int_{-\alpha}^{\alpha} |\tau| \phi(\tau)\, d\tau$ is Bounded]

$$= \int_{-\alpha}^{\alpha} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$\because \phi(\tau) = R_{xx}(\tau)\, e^{-j\omega\tau}$

$$\lim_{T\to\alpha} \frac{1}{2T}\left[E\left\{|X_T(\omega)|^2\right\}\right] = S_{xx}(\omega)$$

Hence the proof.

Relation between PSD of input and Output random process:

In the Consideration of the transmission of stationary random waveforms through fixed linear system, a basic tool is the relationship of the output power spectral density to the input power spectral density, given as

$$S_y(f) = |H(f)|^2 S_x(f)$$

The autocorrelation function of the output is the inverse Fourier transform

$$R_y(\tau) = F^{-1}[S_y(f)]$$

$$= \frac{1}{2\pi} \int_{-\alpha}^{\alpha} |H(f)|^2 S_x(f) e^{j2\pi f\tau} df$$

where

$H(f) \longrightarrow$ system transfer function

$S_X(f) \rightarrow$ power spectral density of the input $x(t)$

$S_Y(f) \rightarrow$ power spectral density of the output $y(t)$

$R_Y(\tau) \rightarrow$ AutoCorrelation function of the O/p

We know that,

The cross-correlation function between input and output $R_{xy}(\tau)$ defined as,

$$R_{xy}(\tau) = E\left[x(t)\; y(t+\tau)\right]$$

Using the superposition integral we have,

$$y(t) = \int_{-\alpha}^{\alpha} h(u)\, x(t-u)\, du.$$

Where $h(u)$ is the systems impulse response. The above eqn relates each sample function of the i/p & o/p process

$$R_{xy}(\tau) = E\left[x(t) \int_{-\alpha}^{\alpha} h(u)\, x(t+\tau-u)\, du\right] \text{— (1)}$$

* Since the integral does not depend on t, we can take x(t) inside and interchange the operations of expectation and convolution.

The above eqn can be written as,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} h(u)\, E\left[x(t)\, x(t+\tau-u)\right] du \quad \text{—} \; ②$$

By definition of the autocorrelation function of x(t)

$$E\left\{x(t)\, x(t+\tau-u)\right\} = R_x(\tau-u)$$

Therefore eqn ② can be written as,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} h(u)\, R_x(\tau-u)\, du.$$

$$R_{xy}(\tau) = h(\tau) * R_x(\tau) \quad \text{———} \; ③$$

ie, the cross-correlation function i/p with o/p is the autocorrelation function of the i/p convolved with the impulse response.

* Since eqn ③ is a Convolution, the Fourier transform of $R_{xy}(\tau)$, the cross power spectral density of $x(t)$ with $y(t)$ is,

$$S_{xy}(f) = H(f)\, S_x(f) \quad \text{——} \quad (4)$$

we know that

$$H^*(f) = H(-f)$$

$$S_x^*(f) = S_x(f)$$

$$S_{yx}(f) = H(-f)\, S_x(f)$$

$$= H^*(f)\, S_x(f) \quad \text{——} \quad (5)$$

where the order of subscript is importan

Taking the inverse Fourier transform of eqn ⑤ with aid of the Convolution theorem and again using the time-reversal theorem, we obtain

$$R_{yx}(\tau) = h(-\tau) * R_x(\tau) \quad \text{——} \quad (6)$$

* By definition, $R_{xy}(\tau)$ can be written as,

$$R_{xy}(\tau) = E\left\{x(t)\,[h(t) * x(t+\tau)]\right\}$$

$$R_{xy}(\tau) = E\{x(t) \cdot y(t+\tau)\} \quad \text{—— (7)}$$

where $[h(t) * x(t+\tau)] = y(t+\tau)$

* Combining this with (3) we have

$$E[x(t) \cdot [h(t) * x(t+\tau)]] = h(\tau) * R_x(\tau)$$

$$= h(\tau) * E[x(t) \cdot x(t+\tau)] \quad \text{—— (8)}$$

eqn (6) becomes,

$$R_{xy}(\tau) = E\{[h(t) * x(t)]\, x(t+\tau)\}$$

$$= h(-\tau) * R_x(\tau)$$

$$= h(-\tau) * E[x(t) \cdot x(t+\tau)] \quad \text{—— (9)}$$

* Thus, bringing the convolution operation outside the expectation gives a convolution of $h(\tau)$ with the autocorrelation function if $h(t) * x(t+\tau)$ is inside the expectation or a convolution of $h(-\tau)$ with the auto-correlation function if $h(t) * x(t)$ is inside the expectation.

* These Results are Combined to obtain the auto correlation function of the output of a linear system in terms of the i/p auto correlation function as follows

$$R_y(\tau) = E\left[y(t) \cdot y(t+\tau)\right]$$

$$= E\left\{y(t) \cdot \left[h(t) * x(t+\tau)\right]\right\} \quad \text{(10)}$$

which follows because $y(t+\tau) = h(t) * x(t+\tau)$ using eqn (8), with $x(t)$ replaced by $y(t)$, we obtain

$$R_y(\tau) = h(\tau) * E\left[y(t) \cdot x(t+\tau)\right]$$

$$= h(\tau) * R_{yx}(\tau)$$

$$= h(\tau) * \left\{h(-\tau) * R_x(\tau)\right\} \quad \text{(11)}$$

where the last line follows by substituting from (6) written in terms of integrals eqn (11) is

$$R_y(\tau) = \int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} h(u) \cdot h(v) R_x(\tau+u-v)\, du \cdot dv \quad \text{(12)}$$

* The Fourier transform of (11) is the output power spectral density & is easily obtained as follows.

$$S_y(f) = F\left[R_y(\tau)\right]$$

$$= F\left[h(\tau) * R_{yx}(\tau)\right]$$

$$= H(f)\, S_{yx}(f)$$

$$S_y(t) = |H(f)|^2\, S_x(f)$$

where eqn (5) has been substitued to obtain the last line.