Correlation & Spectral Density

Auto Correlation

$$R_{xx}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

Properties of Auto Correlation

1. $$R_{xx}(0) = E[X^2(t)]$$

$$E[X(t).X(t+\tau)] = R_{xx}(\tau)$$

$$\tau = 0$$

$$E[X(t)X(t)] = E[X^2(t)$$

2. $$|R_{xx}(\tau)| = R_{xx}(0)$$

By def $$R_{xx}(\tau) = E[X(t)\, X(t+\tau)]$$

By Cauchy- Schwartz equality

$$(E(XY))^2 \leq E[X^2] \cdot E[Y^2]$$

Let $X = X(t)$; $Y = X(t+\tau)$

$$\therefore E[X(t) \cdot X(t+\tau)]^2 \leq E[X^2(t) \cdot E[X^2(t+\tau)]$$

$$\Rightarrow (R_{xx}(\tau))^2 \leq E[X^2(t) \cdot E[X^2(t+\tau)] \quad (1)$$

Since $E[X(t)]$ & $Var\, X(t)$ are constants for a stationary process, we get

$$E[X^2(t)] = E[X^2(t+\tau)]$$

Hence by (1) we have

$$(R_{xx}(\tau))^2 \leq [E[x^2(t)]]^2 \quad (2)$$

By Property 1, we have

$$E[x^2(t)] = R_{xx}(0) \quad (3)$$

Subs (3) & (2) we get ~~$E(x^2(t)) = R_{xx}(0)$~~

$$(R_{xx}(\tau))^2 \leq (R_{xx}(0))^2$$

Taking square root on both sides, we get

$$|R_{xx}(\tau)| \leq (R_{xx}(0))$$

$(\because R_{xx}(0) = E[x^2(t)]$ is +ve$)$

Hence ~~by~~ property 2 is proved

Property 3

$$R_{xx}(-\tau) = R_{xx}(\tau)$$

i.e $R_{xx}(\tau)$ is an even funct

Proof:

$$R_{xx}(\tau) = E[X(t) \cdot X(t+\tau)]$$

$$R_{xx}(-\tau) = E[X(t) \cdot X(t-\tau)]$$

Let $t - \tau = a$ Then

$$R_{xx}(-\tau) = E[X(a+\tau) \cdot X(a)] = R_{xx}(\tau)$$

4. If $X(t)$ is a Stationary process & it has no periodic component, Then

$$\lim_{|\tau| \to \infty} R_{xx}(\tau) = \bar{X}^2 \quad \text{where } \bar{X} = E(X(t))$$

5. If $X(t)$ is ergodic, zero mean & has no periodic component, Then

$$\lim_{|\tau| \to \infty} R_{xx}(\tau) = 0$$

Cross Correlation Function

$$R_{XY}(t_1, t_2) = E[X(t_1) \cdot Y(t_2)]$$

Cross covariance of $X(t)$ & $Y(t)$ is

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1)\,\mu_Y(t_2)$$

Two process are J. Stationary in wide sense if

① $E[X(t)] = E(Y(t)) = \mu$ is constant

2) $R_{XY}(t_1, t_2)$ is a function of time difference

Properities of Cross Correlation Function

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

$$R_{XY}(\tau) = E[X(t)\,Y(t+\tau)]$$

$$R_{YX}(\tau) = E[Y(t)\cdot X(t+\tau)]$$

Replace $\tau = -\tau$

$$R_{YX}(-\tau) = E[Y(t)\,X(t-\tau)]$$

Put $t - \tau = a$

$$= E[Y(a+\tau)\,X(a)] = E[X(a)\,Y(a+\tau)]$$
$$= R_{XY}(\tau)$$

2). If $X(t)$ and $Y(t)$ are two WSS process, Then

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)\,R_{YY}(0)}$$

i.e the max. of $R_{XY}(\tau)$ can occur anywhere, but can't exceed $\sqrt{R_{XX}(0) \cdot R_{YY}(0)}$.

Proof: By Cauchy - Schwartz inequality, we have

$$E(XY)^2 \leq E[X^2] \cdot E[Y^2]$$

$$E[X(t)\,Y(t+\tau)]^2 \leq E[X^2(t)] \cdot E[Y^2(t+\tau)]$$

$$(R_{XY}(\tau))^2 \leq R_{XX}(0) \cdot R_{YY}(0)$$

thus (By prop. 1 of ACF)

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0) \cdot R_{YY}(0)} \quad -(A)$$

3. If $X(t)$ & $Y(t)$ are two WSS processes, then

$$|R_{XY}(\tau)| \leq \frac{1}{2}(R_{XX}(0) + R_{YY}(0))$$

By property 1

$$R_{XX}(0) = E(X^2(t))$$

$$R_{YY}(0) = E(Y^2(t))$$

G.M of 2 ~~+ve~~ no. does not exceed their A.M

$$\sqrt{R_{XX}(0) \cdot R_{YY}(0)} \leq \frac{1}{2}(R_{XX}(0) + R_{YY}(0))$$

From Prop. 2 A

$$|R_{XY}(\tau)| \leq \frac{1}{2}(R_{XX}(0) + R_{YY}(0))$$

4. If the processes $X(t)$ & $Y(t)$ are orthogonal then $R_{XY}(\tau) = 0$

5. If $X(t)$ & $Y(t)$ are indep., then the auto covariance is zero.

(4) (3)

Q.2 $X(t)$ and $Y(t)$ are zero mean & stochastically independent R.P having A.C.F $R_{XX}(\tau) = e^{-|\tau|}$ & $R_{YY}(\tau) = \cos(2\pi\tau)$ resp.

(1) Find the A.C.F of $W(t) = X(t) + Y(t)$

(2) Find the A.C.F of $Z(t) = X(t) - Y(t)$

(3) Find the cross correlation function of $W(t)$ & $Z(t)$

Sol: $E[X(t)] = 0 = E[Y(t)]$

(1) ⇒ $R_{WW}(\tau) = E[W(t)\, W(t+\tau)]$

$= E[[X(t)+Y(t)] \cdot [X(t+\tau) + Y(t+\tau)]]$

$\cancel{= E[e^{-|\tau|} + (\cos 2\pi\tau)]\{e^{-|t+\tau|} + \cos(2\pi(t+\tau))\}}$

$= E[X(t)\, X(t+\tau) + X(t)\, Y(t+\tau) + Y(t)\, X(t+\tau) + Y(t)\, Y(t+\tau)]$

$= E[R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)]$

[$X(t)$ & $Y(t)$ are I zero mean $R_{XY}(\tau) = 0 = R_{YX}(\tau)$]

$= E[e^{-|\tau|} + \cos(2\pi\tau)]$

$= e^{-|\tau|} + \cos(2\pi\tau)$

2 ⇒ $R_{ZZ}(\tau) = E[Z(t)\, Z(t+\tau)]$

$= E[(X(t) - Y(t))(X(t+\tau) - Y(t+\tau))]$

$= E[X(t) \cdot X(t+\tau) - X(t)\, Y(t+\tau)$

$- Y(t)\, X(t+\tau) + Y(t)\, Y(t+\tau)]$

$$= R_{XX}(\tau) - R_{YX}(\tau) - R_{XY}(\tau) + R_{YY}(\tau)$$

$$= e^{-|\tau|} + \cos(2\pi\tau)$$

(iii)

$$R_{WZ}(t) = E[W(t) \cdot Z(t+\tau)]$$

$$= E[(X(t) + Y(t))(X(t+\tau) - Y(t+\tau))]$$

$$= E[X(t) \cdot X(t+\tau) - X(t) Y(t+\tau) + Y(t) X(t+\tau) - Y(t) Y(t+\tau)]$$

$$= E[R_{XX}(\tau) - R_{XY}(\tau) + R_{YX}(\tau) - R_{YY}(\tau)]$$

$$= e^{-|\tau|} - \cos(2\pi\tau)$$

③ 4

Q.2. Consider the R.P $X(t) = 3\cos(\omega t+\theta)$ & $Y(t) = 2\cos(\omega t+\phi)$, where $\phi = \theta - \frac{\pi}{2}$ & θ is uniformly distributed R.V over $(0, 2\pi)$. Verify that $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0) \cdot R_{YY}(0)}$

$$R_{XX}(\tau) = E[X(t) \cdot X(t+\tau)]$$

$$= E[3\cos(\omega t+\theta) \cdot 3\cos(\omega(t+\tau)+\theta)]$$

$$= \frac{9}{2} E[2\cos(\omega t+\theta)\cos(\omega t+\omega\tau+\theta)]$$

$\Rightarrow$ $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$$= \frac{9}{2} E[\cos(\omega t+\theta+\omega t+\omega\tau+\theta) + \cos(\cancel{\omega t}+\cancel{\theta} - \cancel{\omega t} - \omega\tau - \cancel{\theta})]$$

$$= \frac{9}{2} E[\cos(2\omega t+2\theta+\omega\tau) + \cos(-\omega\tau)]$$

$\Rightarrow$ $\cos(-\theta) = \cos\theta$

$$= \frac{9}{2} E[\cos(2\omega t+2\theta+\omega\tau) + \cos(\omega\tau)]$$

$$= \frac{9}{2}\{E(\cos(2\omega t+2\theta+\omega\tau)) + E[\cos(\omega\tau)]\} \quad \text{—①}$$

$$E[\cos(2\omega t+2\theta+\omega\tau)] = \int_0^{2\pi} \cos(2\omega t+2\theta+\omega\tau) \cdot f(\theta)$$

$$f(\theta) = \begin{cases} \frac{1}{2\pi} & 0<\theta<2\pi \\ 0 & \text{otherwise} \end{cases}$$ $\rightarrow$ Since θ is U.D over $(0, 2\pi)$

$$\Rightarrow \frac{1}{2\pi}\left[\frac{\sin(2\omega t + 2\theta + \omega\tau)}{2}\right]_0^{2\pi}$$

$$\frac{1}{4\pi}\left[\sin((2\omega t + \omega\tau) + 4\pi) - \sin(2\omega t + \omega\tau)\right]$$

$$= \frac{1}{4\pi}\left[\sin(2\omega t + \omega\tau) - \sin(2\omega t + \omega\tau)\right]$$

$$= 0$$

Sub in Eq (1) we get

$$R_{xx}(\tau) = \frac{9}{2}\left[E(\cos\omega\tau)\right] = \frac{9}{2}\cos\omega\tau$$

$$R_{xx}(0) = \frac{9}{2}$$

$$Y(t) = 2\cos(\omega t + \phi)$$

$$= 2\cos\left(\omega t + \theta - \frac{\pi}{2}\right)$$

$$= 2\cos\left\{-\left[\frac{\pi}{2} - (\omega t + \theta)\right]\right\}$$

$$= \cancel{2\cos}\left[2\sin(\omega t + \theta)\right]$$

In similar way, we can prove that,

$$R_{YY}(\tau) = 2\cos\omega\tau$$

$$R_{YY}(0) = 2$$

Now

$$R_{XY}(t, t+\tau) = E[X(t) \cdot Y(t+\tau)]$$

$$= E\left[3\cos(\omega t + \theta) \cdot 2\sin(\omega(t+\tau) + \theta)\right]$$

$$2\cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$3E\left[\sin(\omega t+\theta + \underset{+\omega\tau}{\omega t+\theta}) - \sin(\omega t+\theta - \underset{-\omega\tau}{\omega t-\theta})\right]$$

$$= 3E\left[\sin(\omega t+\theta+\underset{+\omega\tau}{\omega t+\theta}) - \sin(-\omega\tau)\right]$$

$$= 3E\left[\sin(2\omega t+2\theta+\omega\tau)\right] + 3E\left[\sin\omega\tau\right]$$

$$= 3E\left[\sin(2\omega t+\omega\tau+2\theta)\right] + 3\sin\omega\tau \quad — (2)$$

$$E\left[\sin(2\omega t+\omega\tau+2\theta)\right] = \frac{1}{2\pi}\int_0^{2\pi} \sin(2\omega t+\omega\tau+2\theta)\,d\theta$$

$$\Rightarrow \frac{-1}{2\pi}\left[\frac{\cos(2\omega t+\omega\tau+2\theta)}{2}\right]_0^{2\pi}$$

$$= -\frac{1}{4\pi}\left[\cos(2\omega t+\omega\tau+4\pi) - \cos(2\omega t+\omega\tau)\right]$$

$$-\frac{1}{4\pi}\left[\cos(2\omega t+\omega\tau) - \cos(2\omega t+\omega\tau)\right]$$

$$= 0$$

Eq (2) $\Rightarrow$ $R_{XY}(\tau) = 3\sin\omega\tau$

$X(t)$ & $Y(t)$ are jointly WSS

We have

$$R_{xy}(\tau) = 3 \sin \omega \tau$$

$$|R_{xy}(\tau)| = |3 \sin \omega \tau|$$

$$= 3 |\sin \omega \tau| \leq 3$$

$$R_{xy}(\tau) \leq \sqrt{R_{xx}(0) R_{yy}(0)}$$

Power Spectral Densities

Prob: The A.C.F for a Stationary process is given by $R_{xx}(\tau) = 9 + 2e^{-|\tau|}$. Find the Mean Value of the R.V $Y = \int_0^2 x(t)\,dt$ & Var. of $x(t)$.

Sol:

$$E[Y(t)] = \int_0^2 E[x(t)]\,dt$$

$$E[x(t)] = \mu_x^2 = \lim_{\tau \to \infty} 9 + 2e^{-|\tau|} = 9 = 3$$

$$E[Y(t)] = \int_0^2 3\,dt \Rightarrow 6$$

$$Var(x(t)) = E[x^2(t)] - (E[x(t)])^2$$

$$E[x^2(t)] = R_{xx}(0) = 9 + 2 = 11$$

$$Var = 11 - 9 = 2$$

Let $X(t)$ & $Y(t)$ be defined by $X(t) = A\cos\omega t + B\sin\omega t$ & $Y(t) = B\cos\omega t + A\sin\omega t$ where ω is constant & A & B are I.R.V both having zero mean & variance σ^2. Find the Cross-correlation of $X(t)$ & $Y(t)$. Are $X(t)$ & $Y(t)$ jointly WSS process? (3 unit)

Sol:

$$E[A] = E[B] = 0, \quad E[A^2] = \sigma^2, \quad E[B^2] = \sigma^2$$

$$R_{XY}(\tau) = E[X(t)\,Y(t+\tau)]$$

$$= E[[A\cos\omega t + B\sin\omega t][B\cos(\omega t+\omega\tau) + A\sin(\omega t+\omega\tau)]]$$

$$= E[AB\cos\omega t\cos(\omega t+\omega\tau) + B^2\sin\omega t\cos(\omega t+\omega\tau) + A^2\cos\omega t\sin(\omega t+\omega\tau) + BA\sin\omega t\sin(\omega t+\omega\tau)]$$

$$= \cos\omega t\cos(\omega t+\omega\tau)\,E[AB] + \sigma^2\sin\omega t\cos(\omega t+\omega\tau) + \sigma^2[\cos\omega t\sin(\omega t+\omega\tau)] + E[BA]\sin\omega t\sin(\omega t+\omega\tau)$$

$$E(AB) = E(A)\cdot E(B)$$

$$\sigma^2[\sin\omega t\cos(\omega t+\omega\tau) + \cos\omega t\sin(\omega t+\omega\tau)]$$

$$\sigma^2\sin(2\omega t+\omega\tau)$$

$\Rightarrow$ A function of time difference

(a) If $X(t)$ & $Y(t)$ are independent WSS Processes with zero means, find the A.C.F of

① $Z(t) = a + bX(t) + cY(t)$

② $Z(t) = aX(t)\,Y(t)$

Sol:

$$R_{\cancel{XX}}(\tau) = E[Z(t)\,Z(t+\tau)]$$

$$= E[[a + bX(t) + cY(t)][a + bX(t+\tau) + cY(t+\tau)]]$$

$$= E[a^2 + ab\,X(t+\tau) + ac\,Y(t+\tau) + ba\,X(t) + b^2 X(t)\,X(t+\tau) + bc\,X(t)\,Y(t+\tau) + ca\,Y(t) + cb\,Y(t)\,X(t+\tau) + c^2\,Y(t)\,Y(t+\tau)]$$

$$= E[a^2] + ab\,E[X(t+\tau)] + ac\,E[Y(t+\tau)] + ba\,E[X(t)] + b^2 E[X(t)\cdot X(t+\tau)] + bc\,E[X(t)]\cdot E[Y(t+\tau)] + ca\,E[Y(t)] + cb\,E[Y(t)\,X(t+\tau)] + c^2 E[Y(t)\,Y(t+\tau)]$$

$$= a^2 + b^2 R_{XX}(\tau) + c^2 R_{YY}(\tau)$$

Q. 3. If $\{X(t)\}$ is a R.P $\bar{c}$ mean 3 & Autocorrelation $R_{xx}(\tau) = 9+4e^{-0.2|\tau|}$. Determine the mean, variance & covariance of R.Y $Y = X(5)$ & $Z = X(8)$.

<u>Sol</u>:

$E[X(t)] = 3$

$E[Y] = E[X(5)] = 3$

$Var(Y) = E[Y^2] - (E[Y])^2$

$E[Y^2] = \cancel{R_{YY}(0)} = E[X^2(5)]$

$E[X^2(t)] = R_{XX}(0) = 9+4 = 13$

$E[X^2(5)] = 13$

$Var(Y) = 13 - 9 = 4$

$\parallel^{ly}$ $Var(Z) = 4$

Cov. $= R(t_1, t_2) - E[X(t_1)\, X(t_2)]$

$= R(5, 8) - E[X(5)\, X(8)]$

$= 9+4e^{-0.2\times 3} - E[5]\cdot E[8]$

$= 9+4e^{-0.6} - 9$

$= 4e^{-0.6}$ //.

Power Spectral Density & Its properties

If $\{x(t)\}$ is a S.P with the ACF $R_{xx}(\tau)$, then the F.T of $R_{xx}(\tau)$ is called the PSD or simply S.P of $x(t)$ & is given by

$$S_{xx}(\omega) = S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau$$

Note Given the PSD the A.C.F $R(\tau)$ is given by

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\cdot e^{i\omega\tau}\, d\omega$$

Properties

(1) $S_{xx}(\omega) \geq 0$

2) $S_{xx}(\omega) = S_{xx}(-\omega)$

3) $S_{xx}(0) = \int_{-\infty}^{\infty} R(\tau)\, d\tau$

4) $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega =$ Time average of $E[x^2(t)]$

Q.1 An A.CF $R(\tau)$ is given by $C\cdot e^{-a|\tau|}$; $C>0$; $a>0$ obtain the S.D of $x(t)$

Sol: $R(\tau) = Ce^{-a|\tau|}$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} Ce^{-a|\tau|}\, e^{-i\omega\tau}\, d\tau$$

$$= c\int_{-\alpha}^{\alpha} e^{-\alpha|\tau|}\left[\cos\omega\tau - i\sin\omega\tau\right] d\tau$$

$$\Rightarrow c\int_{-\alpha}^{\alpha} e^{-\alpha|\tau|}\cos\omega\tau\, d\tau - ci\int_{-\alpha}^{\alpha} e^{-\alpha|\tau|}\sin\omega\tau\, d\tau$$

[odd funct.]

$$= 2c\int_{0}^{\alpha} e^{-\alpha\tau}\cos\omega\tau\, d\tau$$

$$= 2c\left[\frac{e^{-\alpha\tau}}{\alpha^2+\omega^2}\left[\alpha\cos\omega\tau + \omega\sin\omega\tau\right]\right]_0^{\alpha}$$

$$= \frac{2c\alpha}{\alpha^2+\omega^2}$$

Q-2 Find the PSD of WSS process with ACF $R(\tau) = e^{-\alpha\tau^2}$

Sol: $$S_{xx}(\omega) = \int_{-\alpha}^{\alpha} R_{xx}(\tau)\, e^{-i\omega\tau} d\tau$$

$$= \int_{-\alpha}^{\alpha} e^{-\alpha\tau^2} e^{-i\omega\tau} d\tau$$

$$= \int_{-\alpha}^{\alpha} e^{-\alpha\left[\tau^2 + \frac{i\omega\tau}{\alpha}\right]} d\tau$$

$$= \int_{-\alpha}^{\alpha} e^{-\alpha\left[\tau^2 + \frac{i\omega\tau}{\alpha} + \left(\frac{i\omega}{2\alpha}\right)^2 - \left(\frac{i\omega}{2\alpha}\right)^2\right]} d\tau$$

$$= \int_{-\alpha}^{\alpha} e^{-\alpha\left[\left[\left(\tau + \frac{i\omega}{2\alpha}\right)^2\right] + \frac{\omega^2}{4\alpha}\right]} d\tau$$

$$e^{-\frac{\alpha\omega^2}{4\alpha^2}} \int_{-\alpha}^{b} e^{-\alpha\left(\tau + \frac{i\omega}{2\alpha}\right)^2} d\tau \Rightarrow 2e^{-\frac{\alpha\omega^2}{4\alpha^2}} \int_{0}^{\alpha} e^{-\alpha\left(\tau + \frac{i\omega}{2\alpha}\right)^2} \cdot d\tau$$

Put $y = \tau + \frac{i\omega}{2\alpha}$ | $dy = d\tau$

$$= 2e^{-\frac{\alpha\omega^2}{4\alpha^2}} \int_{0}^{\alpha} e^{-\alpha y^2} dy$$

Put $\alpha y^2 = t$

$\alpha 2y\, dy = dt$

$$2e^{-\alpha\omega^2/4\alpha^2} \int_{0}^{\alpha} e^{-t} \frac{dt}{\frac{2\alpha\sqrt{t}}{\sqrt{\alpha}}}$$

$$= e^{-\alpha\omega^2/4\alpha^2} \int_{0}^{\alpha} e^{-t} \frac{dt}{\sqrt{\alpha}\sqrt{t}}$$

$$e^{-\frac{\alpha\omega^2}{4\alpha^2}} \frac{1}{\sqrt{\alpha}} \int_{0}^{\alpha} e^{-t} t^{-1/2} dt$$

$$e^{-\omega^2/4\alpha} \frac{\sqrt{\pi}}{\sqrt{\alpha}}$$

Q.3 If the PSD of a WSS process is given by $S_{xx}(\omega) = \begin{cases} b/a\,(a-|\omega|) & |\omega| \le a \\ 0 & |\omega| > a \end{cases}$

Find its ACF

Sol:

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-a}^{a} \frac{b}{a}(a-|\omega|)\,(\cos\omega\tau + i \sin\omega\tau)\, d\omega$$

$$= \frac{1}{2\pi}\frac{b}{a}\int_{-a}^{a}(a-|\omega|)\cos\omega\tau\, d\omega + \underbrace{\frac{i}{2\pi}\frac{b}{a}\int_{-a}^{a}(a-|\omega|)\sin\omega\tau\, d\omega}_{\text{odd funct.}}$$

$$= \frac{1}{2\pi}\,2\,\frac{b}{a}\int_{0}^{a}(a-\omega)\cos\omega\tau\, d\omega$$

$$= \frac{b}{a\pi}\left[(a-\omega)\left(\frac{\sin\omega\tau}{\tau}\right) - (-1)\left(-\frac{\cos\omega\tau}{\tau^2}\right)\right]_0^a$$

$$= \frac{b}{a\pi}\left[-\frac{\cos a\tau}{\tau^2} - \left(-\frac{1}{\tau^2}\right)\right]$$

$$= \frac{b}{\pi a}\left[\frac{1-\cos a\tau}{\tau^2}\right]$$

Q.4. Given the PSD of a continuous process as $S_{xx}(\omega) = \dfrac{\omega^2+9}{\omega^4+5\omega^2+4}$. Find the Mean Square Value of the process.

<u>Sol</u>. $$S_{xx}(\omega) = \frac{\omega^2+9}{\omega^2+4\omega^2+\omega^2+9}$$

$$= \frac{\omega^2+9}{(\omega^2+1)\;(\omega^2+4)}$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\omega\tau}\, d\omega$$

Mean Square Value is $E[X^2(t)] = R_{xx}(0)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega$$

$$\frac{\cancel{2}}{\cancel{2}\pi}\int_{0}^{\infty} S_{xx}(\omega)\, d\omega \qquad [S_{xx}(\omega) \text{ is an even func}^n]$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \frac{\omega^2+9}{(\omega^2+1)\,(\omega^2+4)}\, d\omega$$

To evaluate the integral we use Partial Fraction

$$\frac{\omega^2+9}{(\omega^2+1)(\omega^2+4)} = \frac{A}{\omega^2+1} + \frac{B}{\omega^2+4}$$

$$\omega^2+9 = A(\omega^2+4) + B(\omega^2+1)$$

By solving $A = 8/3$, $B = -5/3$

$$= \frac{1}{\pi}\left[\int_0^\infty \frac{8/3}{\omega^2+1}\,d\omega + \int_0^\infty \frac{(-5/3)}{\omega^2+4}\,d\omega\right]$$

$$= \frac{1}{\pi}\left[\frac{8}{3}\left[\tan^{-1}\omega\right]_0^\infty - \frac{5}{3}\,\frac{1}{2}\left[\tan^{-1}\frac{\omega}{2}\right]_0^\infty\right]$$

$$= \frac{1}{\pi}\left[\frac{8}{3}\left[\frac{\pi}{2}\right] - \frac{5}{6}\,\frac{\pi}{2}\right]$$

$$\frac{1}{2}\left[\frac{16-5}{6}\right] = 11/12$$

Q.4 The A.C.F of the Poisson increment process is given by $R(\tau) = \begin{cases} \lambda^2 & \text{for } |\tau| > \epsilon \\ \lambda^2 + \frac{\lambda}{\epsilon}\left(1 - \frac{|\tau|}{\epsilon}\right) & \text{for } |\tau| \le \epsilon \end{cases}$

Prove that its spectral density is given by

$$S(\omega) = 2\pi\lambda^2\delta(\omega) + \frac{4\lambda\sin^2\left(\frac{\epsilon\omega}{2}\right)}{\epsilon^2\omega^2}$$

Sol:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{-\epsilon} \lambda^2 e^{-i\omega\tau}\,d\tau + \int_{-\epsilon}^{\epsilon} \lambda^2 + \frac{\lambda}{\epsilon}\left(1-\frac{|\tau|}{\epsilon}\right) e^{-i\omega\tau}\,d\tau$$

$$+ \int_{\epsilon}^{\infty} \lambda^2 e^{-i\omega\tau}\,d\tau$$

$$\Rightarrow \int_{-\infty}^{\epsilon} \lambda^2 e^{-i\omega\tau}\,d\tau + \int_{-\epsilon}^{\epsilon} \lambda^2 e^{-i\omega\tau}\,d\tau + \int_{-\epsilon}^{\epsilon} \frac{\lambda}{\epsilon}\left(1-\frac{|\tau|}{\epsilon}\right) e^{-i\omega\tau}\,d\tau$$

$$+ \int_{\epsilon}^{\infty} \lambda^2 e^{-i\omega\tau}\,d\tau$$

$$F(d^2) + \int_{-\epsilon}^{\epsilon} \frac{d}{\epsilon}\left(1-\frac{|\tau|}{\epsilon}\right)\left[\cos\omega\tau + i\sin\omega\tau\right] d\tau$$

$$= F(d^2) + \int_{-\epsilon}^{\epsilon} \frac{d}{\epsilon}\left(1-\frac{|\tau|}{\epsilon}\right)\cos\omega\tau + \underbrace{i\int_{-\epsilon}^{\epsilon}\frac{d}{\epsilon}\left(1-\frac{|\tau|}{\epsilon}\right)\sin\omega\tau\, d\tau}_{\substack{\text{odd func} \\ =0}}$$

$$= F(d^2) + 2\int_{0}^{\epsilon} \frac{d}{\epsilon}\left(1-\frac{\tau}{\epsilon}\right)\cos\omega\tau\, d\tau$$

$$\Rightarrow F(d^2) + \frac{2d}{\epsilon}\left[\left(1-\frac{\tau}{\epsilon}\right)\left(\frac{\sin\omega\tau}{\omega}\right) - \left(-\frac{1}{\epsilon}\right)\left(-\frac{\cos\omega\tau}{\omega^2}\right)\right]_0^{\epsilon}$$

$$\Rightarrow F(d^2) + \frac{2d}{\epsilon}\left[-\frac{\cos\omega\epsilon}{\epsilon\omega^2} - \left(-\frac{1}{\omega^2\epsilon}\right)\right]$$

$$\Rightarrow F(d^2) + \frac{2d}{\epsilon^2\omega^2}\left[1-\cos\omega\epsilon\right]$$

$$F(d^2) + \frac{4d}{\epsilon^2\omega^2}\left[\sin^2\left(\frac{\epsilon\omega}{2}\right)\right]$$

By dirac delta function

$$\int_{-\infty}^{\infty} d^2 e^{-i\omega\tau}\, d\tau = 2\pi d^2 \delta(\omega)$$

Hence $2\pi d^2\delta(\omega) + \frac{4d}{\epsilon^2\omega^2}\sin^2\left(\frac{\epsilon\omega}{2}\right)\Big]$

Prob: The P.S of WSS process $\{X(t)\}$ is given by $S_{xx}(\omega) = \frac{1}{(1+\omega^2)^2}$. Find the A.C.F & hence deduce the Average Power.

Sol: Given $S_{xx}(\omega) = \frac{1}{(1+\omega^2)^2}$

$$= \left[\frac{1}{(1+i\omega)(1-i\omega)}\right]^2$$

$$= \left[\frac{1}{2}\frac{(1+i\omega)+(1-i\omega)}{(1+i\omega)(1-i\omega)}\right]^2$$

$$= \frac{1}{4}\left[\frac{1}{(1-i\omega)} + \frac{1}{(1+i\omega)}\right]^2$$

$$= \frac{1}{4}\left[\frac{1}{(1-i\omega)^2} + \frac{1}{(1+i\omega)^2} + \frac{2}{(1-i\omega)(1+i\omega)}\right]$$

$$F^{-1}\left[\frac{1}{(\alpha+i\omega)^2}\right] = U(\tau)\tau e^{-\alpha\tau}$$ where $U(\tau)$ is a unit step function

$$F^{-1}\left[\frac{1}{(\alpha-i\omega)^2}\right] = U(\tau)\tau e^{\alpha\tau}$$

$$F^{-1}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right] = e^{-\alpha|\tau|}$$

$\therefore$ Taking F.T we get

$$R_{xx}(\tau) = \frac{1}{4}\left[U(\tau)\tau e^{\alpha\tau} + U(\tau)\tau e^{-\alpha\tau} + e^{-\alpha|\tau|}\right]$$

$$= \frac{1}{4}\left[U(\tau)\tau\left[e^{\alpha\tau}+e^{-\alpha\tau}\right] + e^{-\alpha|\tau|}\right]$$

Since $X(t)$ is WSS A.P $P_{XX} = E[X^2(t)] = R_{XX}(0)$

$\therefore$ A.P $\boxed{R_{XX}(0) = \frac{1}{4} = 0.25.}$

Prob: A R.P $X(t)$ is given by $X(t) = A\cos at + B\sin at$ where A & B are I.R.V with mean zero & variance σ^2. Find the P.S.D of the process.

Sol: $R_{XX}(\tau) = E[X(t)X(t+\tau)]$

$$= E[(A\cos at + B\sin at)(A\cos a(t+\tau) + B\sin a(t+\tau))]$$

$$\Rightarrow E[A^2\cos at \cos a(t+\tau) + BA\sin at\cos a(t+\tau) + AB\cos at \sin a(t+\tau) + B^2\sin at \sin a(t+\tau)]$$

Given

$E[A] = E[B] = 0$ / $E[A^2] - E[A] = \sigma^2$

$E[A^2] = \sigma^2$

$E[B] = \sigma^2$

$$\Rightarrow \cos at\cos a(t+\tau)E[A^2] + E[BA]\sin at\cos a(t+\tau) + E[AB]\cos at\sin a(t+\tau) + E[B^2]\sin at\sin a(t+\tau)$$

$$\Rightarrow \cos at\cos a(t+\tau)\sigma^2 + \sigma^2\sin at\sin a(t+\tau)$$

Since A & B are IRV

$E[AB] = E[A]E[B] = 0$

$$\sigma^2 \left[\cos at \cos a(t+\tau) + \sin at \sin a(t+\tau) \right]$$

$$\sigma^2 \left[\cos(at - at - a\tau) \right]$$

$$= \sigma^2 \cos a\tau.$$

Prob: Find PSD $R_{xx}(\tau) = e^{-\alpha|\tau|} \cos\beta\tau$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\alpha|\tau|} \cos\beta\tau \, e^{-i\omega\tau} d\tau$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\alpha|\tau|} \cos\beta\tau \left[\cos\omega\tau - i\sin\omega\tau \right] d\tau$$

$$= 2\int_{0}^{\infty} e^{-\alpha|\tau|} \cos\beta\tau \cos\omega\tau \, d\tau$$

$$= \int_{0}^{\infty} e^{-\alpha|\tau|} 2\cos\beta\tau \cos\omega\tau \, d\tau$$

$$\Rightarrow \int_{0}^{\infty} e^{-\alpha|\tau|} \cos(\beta+\omega)\tau \, d\tau + \int_{0}^{\infty} e^{-\alpha|\tau|} \cos(\beta\tau - \omega\tau) \, d\tau$$

$$\Rightarrow \int_{0}^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$$

$$= \int_{0}^{\infty} e^{-\alpha\tau} \cos(\beta\tau+\omega\tau) d\tau + \int_{0}^{\infty} e^{-\alpha\tau} \cos(\beta\tau - \omega\tau) d\tau$$

$$= \frac{\alpha}{\alpha^2 + (\beta\tau+\omega\tau)^2} + \frac{\alpha}{\alpha^2 + (\beta\tau - \omega\tau)^2}$$

Q. Let $x(t)$ & $y(t)$ be both zero-mean & WSS R.P. Consider the R.P $z(t)$ defined by $z(t) = x(t) + y(t)$

Find (1) the A.C.F & the PS of $z(t)$ if $x(t)$ & $y(t)$ are jointly WSS

(2) the power spectrum (PS) of $z(t)$ if $x(t)$ & $y(t)$ are orthogonal.

Sol: Auto correlation function of $z(t)$ is given by

$$R_{zz}(\tau) = E[z(t)\, z(t+\tau)]$$

$$= E[[x(t)+y(t)][x(t+\tau)+y(t+\tau)]]$$

$$= E[x(t)x(t+\tau) + x(t)y(t+\tau) + y(t)x(t+\tau) + y(t)y(t+\tau)]$$

$$= E[x(t)x(t+\tau)] + E[x(t)y(t+\tau)] + E[y(t)x(t+\tau)] + E[y(t)y(t+\tau)]$$

$$= R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

[Since $x(t)$ & $y(t)$ are jointly WSS then $E[x(t)\,y(t+\tau)] = R_{xy}(\tau)$

Taking F.T on both sides, we obtain

Taking F.T on both sides, we obtain

$$S_{zz}(\omega) = S_{xx}(\omega) + S_{xy}(\omega) + S_{yx}(\omega) + S_{yy}(\omega)$$

If $x(t)$ & $y(t)$ are orthogonal then

$$R_{xy}(\tau) = R_{yx}(\tau) = 0$$

Then $R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau)$ & the F.T of this result gives

$$S_{zz}(\omega) = S_{xx}(\omega) + S_{yy}(\omega)$$

Q. If the process $\{x(t)\}$ is defined as $x(t) = y(t)z(t)$ where $\{y(t)\}$ & $\{z(t)\}$ are independent WSS processes P.T

1) $R_{xx}(\tau) = R_{yy}(\tau)\, R_{zz}(\tau)$ and

2) $S_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(\alpha)\, S_{zz}(\omega-\alpha)\, d\alpha$

Sol.

$$R_{xx}(\tau) = E[x(t)\,x(t+\tau)]$$

$$= E[y(t)z(t)\,y(t+\tau)\,z(t+\tau)]$$

$$= E[y(t)\,y(t+\tau)]\,E[z(t)\,z(t+\tau)]$$

$$= R_{yy}(\tau)\cdot R_{zz}(\tau)$$

) We K.T $S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-i\omega\tau}\, d\tau$

$$= R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\omega\tau}\, d\omega$$

Consider

$$F^{-1}\left(\int_{-\infty}^{\infty} S_{YY}(\alpha)\, S_{ZZ}(\omega-\alpha)\, d\alpha\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{YY}(\alpha)\, S_{ZZ}(\omega-\alpha)\, e^{i\omega\tau}\, d\alpha\, d\omega$$

Putting $\alpha = y$ & $\omega - \alpha = z$, we get

$$d\alpha\, d\omega = \begin{vmatrix} \alpha_y & \alpha_z \\ \omega_y & \omega_z \end{vmatrix} dy\,dz$$

$$= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} dy\,dz$$

$$= 2\pi\, R_{YY}(\tau)\, R_{ZZ}(\tau)$$

$$\therefore F\left[R_{YY}(\tau)\, R_{ZZ}(\tau)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\alpha)\, S_{ZZ}(\omega-\alpha)\, d\alpha$$

From the above RIL we get $S_{XX}(\omega)$ is the required form.

If $y(t) = x(t+a) - x(t-a)$, where $x(t)$ is a WSS process, then S.T

1) $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau-2a) - R_{XX}(\tau+2a)$

2) $S_{YY}(\omega) = 4\sin^2 a\omega \, S_{XX}(\omega)$

Sol: Given

$$Y(t) = X(t+a) - X(t-a)$$

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$$

$$= E[(X(t+a) - X(t-a))(X(t+\tau+a) - X(t+\tau-a))]$$

$$= E[X(t+a)X(t+a+\tau)] - E[X(t+a)X(t+\tau-a)]$$
$$- E[X(t-a)X(t+a+\tau)] + E[X(t-a)X(t+\tau-a)]$$

$$\Rightarrow R_{XX}(\tau) - R_{XX}(\tau-2a) - \cancel{E}[R_{XX}(\tau+2a) + R_{XX}(\tau)$$

$$= 2R_{XX}(\tau) - R_{XX}(\tau-2a) - R_{XX}(\tau+2a)$$

We know that $S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) \cdot e^{-i\omega\tau} d\tau$

$$\Rightarrow 2\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau} d\tau - \int_{-\infty}^{\infty} R_{XX}(\tau-2a)e^{-i\omega\tau} d\tau$$
$$- \int_{-\infty}^{\infty} R_{XX}(\tau+2a)e^{-i\omega\tau} d\tau$$

$$S_{YY}(\omega) = 2S_{XX}(\omega) - \int_{-\infty}^{\infty} R_{XX}(z_1)e^{-i\omega(z_1+2a)} dz$$
$$- \int_{-\infty}^{\infty} R_{XX}(z_2)e^{-i\omega(z_2-2a)} dz$$

$$\left[\text{As} \quad \begin{array}{l|l} \tau-2a = z_1 & \tau+2a = z_2 \\ \tau = z_1+2a & \tau = z_2-2a \end{array}\right]$$

$$\Rightarrow 2S_{YY}(\omega) \cancel{=} e^{-i\omega 2a} S_{XX}(\omega) - e^{i\omega 2a} S_{XX}(\omega)$$

$$= 2S_{xx}(\omega) - S_{xx}(\omega)\left[e^{i\omega 2a} + e^{-i\omega 2a}\right]$$

$$2S_{xx}(\omega) - 2S_{xx}(\omega)\cos 2a\omega$$

$$= 2S_{xx}(\omega)\left[1 - \cos 2a\omega\right] = 2S_{xx}(\omega)\, 2\sin^2 a\omega$$

$$= 4\sin^2 a\omega\, S_{xx}(\omega) \quad //$$

Q. The P.S.D function of zero mean WSS process $\{X(t)\}$ is given by $S(\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & \text{otherwise} \end{cases}$

Find $R(\tau)$ & S.T also $X(t)$ & $X\left(t + \frac{\pi}{\omega_0}\right)$ are uncorrelated

Sol:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} 1\, e^{i\omega\tau}\, d\omega \Rightarrow \left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{1}{2\pi}\left[\frac{e^{i\omega_0\tau} - e^{-i\omega_0\tau}}{i\tau}\right] \Rightarrow \frac{1}{\pi\tau}\left[\sin \omega_0\tau\right]$$

$$E\left[X(t)\, X\left(t + \frac{\pi}{\omega_0}\right)\right] = E[X(t)] \cdot E\left[t + \frac{\pi}{\omega_0}\right]$$

$$= 0$$

As $E(X(t)) = 0$

$$R\left(\frac{\pi}{\omega_0}\right) = \frac{1}{\pi\tau}\sin\pi = 0$$

$\therefore$ $X(t)$ & $X\left(t + \frac{\pi}{\omega_0}\right)$ are uncorrelated

Show that the semi-random telegraph signal process is evolutionary. Show that it is not a SSS & it is not a WSS process.

Sol: We know that for Semi-Random telegraph signal process $X(t) = (-1)^{N(t)}$, where

$N(t)$ is a P.P $\quad P[N(t)=r] = \frac{e^{-\lambda t}(\lambda t)^{r}}{r!}$

$r = 0, 1, 2, \ldots$

$P[X(t)=1] = P[N(t) \text{ is even}]$

$$= P[X(t)=0] + P[X(t)=2] + \ldots$$

$$= e^{-\lambda t} + e^{-\lambda t}\frac{(\lambda t)^2}{2!} + e^{-\lambda t}\frac{(\lambda t)^4}{4!} + \ldots$$

$$= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \ldots\right]$$

$$= e^{-\lambda t}\cosh \lambda t$$

$P[X(t)=-1] = P[N(t) \text{ is odd}]$

$$= P[X(t)=1] + P[X(t)=3] + \ldots$$

$$= e^{-\lambda t}\sinh \lambda t$$

$$E[X(t)] = (1)\, e^{-\alpha t} \cos h \alpha t + (-1)\, e^{-\alpha t} \sin h \alpha t$$

$$= e^{-\alpha t} [\cos h \alpha t - \sin h \alpha t]$$

$$= e^{-\alpha t}\, e^{-\alpha t} = e^{-2\alpha t}$$

$$\neq \text{a const.}$$

$\Rightarrow$ $X(t)$ is not a SSS process

$\Rightarrow$ $X(t)$ is not a WSS process

Problems on Cross-power Spectrum

The cross-correlation function of two processes $X(t)$ and $Y(t)$ is given by $R_{XY}(\tau) = \frac{AB}{2}\left[\sin(\omega_0\tau) + \cos(\omega_0(2t+\tau))\right]$ where A, B & ω_0 are constants. Find the power spectrum $S_{XY}(\omega)$

Sol: The time average of cross correlation function is

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(\tau)\, d\tau$$

$$= \frac{AB}{2} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sin\omega_0\tau\, d\tau + \frac{AB}{2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \cos(\omega_0\tau + 2t+\tau)\, d\tau)$$

$$\therefore \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(\tau)\, dt = \frac{AB}{2}\sin\omega_0\tau$$

Consider the F.T of the time averaged cross-correlation function we get

$$S_{XY}(\omega) = F\left[\frac{AB}{2}(\sin\omega_0\tau)\right]$$

$$= \frac{AB}{2}\int_{-\infty}^{\infty} \sin\omega_0\tau\, e^{-i\omega\tau}\, d\tau$$

$$= \frac{AB}{2}\int_{-\infty}^{\infty}\left(\frac{e^{i\omega_0\tau} - e^{-i\omega_0\tau}}{2}\right)e^{-i\omega\tau}\, d\tau$$

$$\frac{AB}{2}\left[\int_{-\alpha}^{\alpha} e^{-i\tau[\omega-\omega_0]}\,d\tau + \int_{-\alpha}^{\alpha} e^{-i\tau[\omega+\omega_0]}\,d\tau\right]$$

$$\frac{AB}{\cancel{2}}\,\cancel{2}\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

∴ Using the Concept of Dirac Delta function i.e $\delta(\omega) = \frac{1}{2\pi}\int_{-\alpha}^{\infty} e^{-i\omega\tau}\,d\tau$

Q. If $x(t)$ is a band limited process $\ni S_{xx}(\omega)=0$ where $|\omega|>\sigma$ P.T $2[R_{xx}(0)-R_{xx}(\tau)] \le \sigma^2\tau^2 R_{xx}(0)$

Sol

$$S_{xx}(\omega)=0 \qquad \omega>\sigma \text{ or } \omega<-\sigma$$

$$R_{xx}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{i\omega\tau}d\omega$$

$$R_{xx}(\tau)=\frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{xx}(\omega)e^{i\omega\tau}d\omega$$

$$R_{xx}(\tau)=\frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{xx}(\omega)\cos\omega\tau\, d\omega$$

$$R_{xx}(0)=\frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{xx}(\omega)\, d\omega$$

$$R_{xx}(0)-R_{xx}(\tau)=\frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{xx}(\omega)d\omega-\frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{xx}(\omega)\cos\omega\tau\, d\omega$$

$$\frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{xx}(\omega)[1-\cos\omega\tau]d\omega$$

$$=\frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{xx}(\omega)\,2\sin^2\left(\frac{\omega\tau}{2}\right)d\omega$$

Now $|\sin\theta|\le\theta$

$\sin^2\theta\le\theta^2$ & hence

$$\le\frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{xx}(\omega)\,2\,\frac{\omega^2\tau^2}{4}\,d\omega \qquad \omega<\sigma$$

$$\frac{1}{2\pi}\,\frac{\sigma^2\tau^2}{2}\int_{-\sigma}^{\sigma} S_{xx}(\omega)\,d\omega$$

$$\frac{\sigma^2\tau^2}{\cancel{2\pi}\;2}\,R_{xx}(0)$$

$$\boxed{2\left[R_{xx}(0)-R_{xx}(\tau)\right] \leq \sigma^2\tau^2\,R_{xx}(0)}$$

Q.1 The cross density spectrum of the processes $\{X(t)\}$ & $\{Y(t)\}$ is given by

$$S_{XY}(\omega) = \begin{cases} a + j\dfrac{b\omega}{W} & -W < \omega < W \\ 0 & \text{otherwise} \end{cases}$$

where $W > 0$, a & b are real constants. Find the cross correlation function $R_{XY}(\tau)$

Sol: $R_{XY}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega\tau}\, d\omega$

$$\Rightarrow \frac{1}{2\pi}\int_{-W}^{W}\left(a + j\frac{b\omega}{W}\right) e^{j\omega\tau}\, d\omega$$

$$\Rightarrow \frac{1}{2\pi}\left[\left(a + j\frac{b\omega}{W}\right)\left(\frac{e^{j\omega\tau}}{j\tau}\right) - \left(\frac{jb}{W}\right)\left(\frac{e^{j\omega\tau}}{(j\tau)^2}\right)\right]_{-W}^{W}$$

$$\Rightarrow \frac{1}{2\pi}\left[\left((a+jb)\left(\frac{e^{jW\tau}}{j\tau}\right) + \left(\frac{jb}{W}\right)\left(\frac{e^{jW\tau}}{\tau^2}\right)\right) - \left((a-jb)\left(\frac{e^{-jW\tau}}{j\tau}\right) + \frac{jb}{W}\left(\frac{e^{-jW\tau}}{\tau^2}\right)\right)\right]$$

$$\frac{1}{2\pi}\left[\frac{a}{j\tau}\left[e^{jW\tau} - e^{-jW\tau}\right] + \frac{\cancel{j}b}{\cancel{j}\tau}\left[e^{jW\tau} + e^{-jW\tau}\right] + \frac{jb}{W\tau^2}\left[e^{jW\tau} - e^{-jW\tau}\right]\right]$$

$$\Rightarrow \frac{1}{2\pi}\left[\frac{2a}{\tau}\sin W\tau + \frac{2b}{\tau}\cos W\tau - \frac{2b}{W\tau^2}\sin W\tau\right]$$

$$= \frac{1}{\pi}\left[\frac{a\sin W\tau}{\tau} + \frac{b\cos W\tau}{\tau} - \frac{b}{W\tau^2}\sin W\tau\right]$$

$$= \frac{1}{\pi W\tau^2}\left[(aW\tau - b)\sin W\tau + bW\tau\cos W\tau\right]$$