UNIT - I.

RANDOM VARIABLES

Defn. of a Random Variable

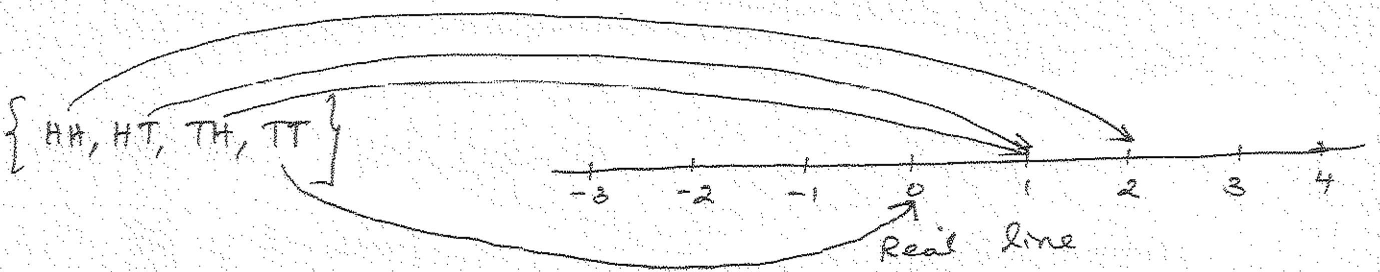
A random variable X can be considered as a function that maps all elements in the sample space S into points on the real line.

Illustration

Experiment - Throw a coin twice.

Possible outcomes - {HH, HT, TH, TT}

Define a rule - No. of Heads.



Here the random variable X takes the values 0, 1, 2.

Overall, random variable X is just a rule of association from sample space S to a set of real numbers.

Types of Random Variable

- Discrete random variable (X)
- Continuous random variable (X).

Discrete Random Variable (X)

A discrete random variable 'X' is a random variable which takes finite (or) countable set of values.

(for eg): Mark scored by a student out of 100 ($X = 0, 1, 2, \dots 100$)

$X = 1, 2, 3, 4.$ (finite)

$X = -5, -4, -3, -1, 0, 1, 2$ (finite)

$X = 0, 1, 2, 3, \dots\dots$ (countable).

Probability Mass function (or) Probability function $[P(X = x_i)]$

Let X be a discrete random variable which takes the values $x_1, x_2, x_3, \dots$

For each values (outcomes) 'x_i', we define a number $P(X = x_i)$ called the probability of x_i, which should satisfy the following conditions.

(i) $P(X = x_i) \geq 0, \quad \forall i$ (ii) $\sum_{i=1}^{\infty} P(X = x_i) = 1.$

(i.e) (i) $P(X = x_1) \geq 0,\ P(X = x_2) \geq 0,\ P(X = x_3) \geq 0,$ & so on

(i.e All Probabilities should be non-negative)

(ii) $P(X = x_1) + P(X = x_2) + P(X = x_3) + \dots = 1$

(ie sum of all probabilities $= 1$)

Illustration:

1) Suppose a random variable X takes the values 1, 2, 3, 4.

X : 1 2 3 4.

$P(X=x_i)$: $P(X=1)$ $P(X=2)$ $P(X=3)$ $P(X=4)$

Now i) $P(X=1) \geq 0$, $P(X=2) \geq 0$, $P(X=3) \geq 0$, $P(X=4) \geq 0$

(ii) $P(X=1) + P(X=2) + P(X=3) + P(X=4) = 1$

2) Suppose a random variable X takes the values $-1, 0, 1$

(i) X : -1 0 1

$P(X=x_i)$: $P(X=-1)$ $P(X=0)$ $P(X=1)$

Now (i) $P(X=-1) \geq 0$, $P(X=0) \geq 0$, $P(X=1) \geq 0$

(ii) $P(X=-1) + P(X=0) + P(X=1) = 1$

3) Suppose the random variable X takes the values 0, 1, 2, 3, ... (infinite but countable values).

Now (i) $P(X=0) \geq 0$, $P(X=1) \geq 0$, $P(X=2) \geq 0$, & so on

(ii) $P(X=0) + P(X=1) + P(X=2) + \ldots\ldots = 1$.

Continuous Random Variable (X)

A continuous random variable X is a random variable which can take all possible values in an interval.

eg: Speed of a motor bike when operated in first gear (0 – 30 kmph)

Temperature of a new born baby (36°c – 37°c).

(e.g) $1 \le x < 3$, $2 < x \le 5$, $-\infty < x < \infty$, $0 < x < \infty$.

Probability Density function (p.d.f) ($f(x)$)

Let X be a continuous random variable taking the values between 'a' and 'b'.

The probability function $f(x)$ which has the following properties (i) $f(x) \geq 0$, $a < x < b$

(ii) $\int_a^b f(x)\,dx = 1$.

is called a probability density function.

Illustration

(i) If a continuous R.V takes values between 1 to 3, ($1 < x < 3$)

then (i) $f(x) \geq 0$

(ii) $\int_1^3 f(x)\,dx = 1$.

Summary

Random Variable (X)

Discrete R.V (X)
($X = 1, 2, 3, \ldots$)

↓ Prob. fn

Prob. Mass fn $\{P(X = x_i)\}$

Properties

(i) $P(x = x_i) \geq 0$

(ii) $\Sigma P(x = x_i) = 1$

Continuous R.v (X)
($x \geq 2$, $1 < x \le 2$)

↓ Prob. fn.

Prob. Density fn. $\{f(x)\}$

Properties:

(i) $f(x) \geq 0$

(ii) $\int_a^b f(x)\,dx = 1$

Distribution function: (or) Cumulative Distribution function

The distribution function of a random variable X is given by $F(x) = P(X \le x)$.

Illustration (for a Discrete Random variable X).

Let the Prob. Mass function be defined by

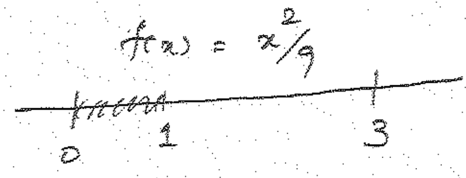
X:	-1	0	1	2	3
$P(X=x_i)$:	$P(X=-1)$	$P(X=0)$	$P(X=1)$	$P(X=2)$	$P(X=3)$

Now
$$F(-1) = P(X \le -1) = P(X=-1)$$
$$F(0) = P(X \le 0) = P(X=-1) + P(X=0)$$
$$F(1) = P(X \le 1) = P(X=-1) + P(X=0) + P(X=1)$$
$$F(2) = P(X \le 2) = P(X=-1) + P(X=0) + P(X=1) + P(X=2)$$
$$F(3) = P(X \le 3) = P(X=-1) + P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

Illustration (for a Continuous random variable X)

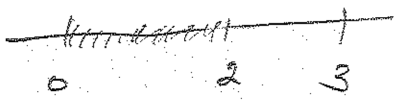
Let the Prob. density function be defined by

$$f(x) = \frac{x^2}{9}, \quad 0 \le x \le 3.$$

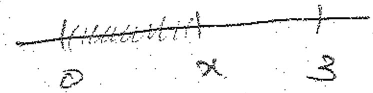


Now
$$F(1) = P(X \le 1) = \int_0^1 f(x)\,dx.$$

$$F(2) = P(X \le 2) = \int_0^2 f(x)\,dx$$



In general
$$F(x) = P(X \le x) = \int_0^x f(x)\,dx.$$



Possible Questions:

Data: Probability Mass function $[P(X = x_i)]$

Probability Density function $[f(x)]$

01) Verify the given is a P.M.F (or) P.D.F

02) Find the unknown constants.

[Hint for Qn 1, Qn 2 : Use the properties of P.M.F & P.D.F]

03) Find the Distribution Function.

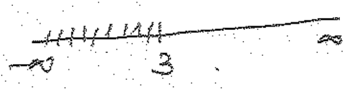
04) Find the Probabilities.

Note: Hint for finding Probabilities

① $P(X < 3)$

D.R.V → $\dots + P(X=0) + P(X=1) + P(X=2)$

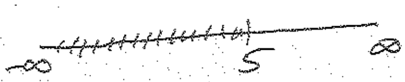
C.R.V → $\int_{-\infty}^{3} f(x)\,dx$



② $P(X \leq 5)$

D.R.V → $\dots + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

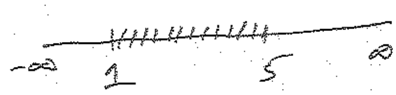
C.R.V → $\int_{-\infty}^{5} f(x)\,dx$



③ $P(1 < X \leq 5)$

D.R.V → $P(X=2) + P(X=3) + P(X=4) + P(X=5)$

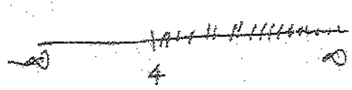
C.R.V → $\int_{1}^{5} f(x)\,dx$



④ $P(X > 4)$

D.R.V → $P(X=5) + P(X=6) + \dots$

C.R.V → $\int_{4}^{\infty} f(x)\,dx$



$\boxed{\text{Note: } |x| \le a \Rightarrow -a \le x \le a}$

5 $P(|x| < 2)$

$\longrightarrow P(-2 < x < 2) = P(X = -1) + P(X = 0) + P(X = 1)$

$\longrightarrow P(-2 < x < 2) = \int_{-2}^{2} f(x)\,dx.$

6 $P(|x| \ge 3) = 1 - P(|x| < 3)$

$= 1 - P(-3 < x < 3)$

7 $P(2x + 5 > 4) = P(2x > -1) = P(x > -\tfrac{1}{2})$

8 $P(|x + 5| > 6) = 1 - P(|x + 5| \le 6)$

$= 1 - P(-6 \le x + 5 \le 6)$

$= 1 - P(-11 \le x \le 1)$

9 $P(|x - 2| \le 2) = P(-2 \le x - 2 \le 2)$

$= P(0 \le x \le 4).$

Conditional Probabilities

Note: $P(A/B) = \dfrac{P(A \cap B)}{P(B)}.$

Note 2: Read $P(A/B)$ as: Probability of A given B.

Conditional Probability

Discrete R.V X | Continuous R.V X.

$P(X>1 / X \le 4)$

$= \dfrac{P(X>1 \cap X \le 4)}{P(X \le 4)}$

$= \dfrac{P(X=2,3,4,\ldots \cap X = \ldots,0,1,2,3,4)}{P(X \le 4)}$

$= \dfrac{P(X=2,3,4)}{P(X \le 4)} = \dfrac{P(X=2)+P(X=3)+P(X=4)}{P(X \le 4)}$

$P(X>1 / X \le 4)$

$-\infty \quad 1 \quad 4 \quad \infty$

$= \dfrac{P(X>1 \cap X \le 4)}{P(X \le 4)}$

$= \dfrac{P(1<X<4)}{P(X \le 4)}$

$= \dfrac{\int_1^4 f(x)\,dx}{\int_{-\infty}^4 f(x)\,dx}$

Note:

<u>Mean (or) Expectation of a random Variable X.</u>

$E(X) = \sum x \cdot P(X=x)$ [for Discrete r.v X]

$= \int_a^b x f(x)\,dx$ [for cont. r.v X]

<u>Variance of a random Variable X</u>

$Var(X) = E(X^2) - [E(X)]^2$

where $E(X^2) = \sum x^2 P(X=x)$ [for Discrete r.v X]

$= \int_a^b x^2 f(x)\,dx$ [for cont. r.v X]

Problems based on above results [It's a Foundation]
(for Discrete Random Variable - finite values).

01. A random variable X has the following probability distribution:

X:	0	1	2	3	4	5	6	7
$P(X=x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$
	$[P(X=0)]$	$P(X=1)$	$P(X=2)$	$P(X=3)$	$P(X=4)$	$P(X=5)$	$P(X=6)$	$P(X=7)$

Find (i) the value of k

(ii) $P(1.5 < X < 4.5 / X > 2)$

(iii) $P(|X-1| > 3)$

(iv) Mean and variance

(v) Distribution function

(vi) the smallest value of λ such that $P(X \le \lambda) > \frac{1}{2}$.

Soln:

(i) We know that $\sum P(X=x) = 1$

$\Rightarrow P(X=0) + P(X=1) + P(X=2) + \cdots + P(X=7) = 1$

$\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow 10k^2 + 9k - 1 = 0$

$\Rightarrow k = -1, \ 1/10$

$a = 10, \ b = 9, \ c = -1$

$k = \dfrac{-9 \pm \sqrt{81+40}}{20}$

$= \dfrac{-9 \pm 11}{20} = -1, \ 1/10$

$k = -1$ is not acceptable.

$[\because P(X=1) = -1, \ P(X=2) = -2, \ldots]$ is not acceptable.

$\therefore \boxed{k = 1/10}$

(ii) $P(1.5 < X < 4.5 / X > 2) = \dfrac{P(1.5 < X < 4.5 \cap X > 2)}{P(X > 2)}$

$= \dfrac{P(X = 2, 3, 4 \cap X = 3, 4, 5, 6, 7, ..)}{P(X = 3, 4, 5, 6, 7)}$

$= \dfrac{P(X = 3, 4)}{P(X = 3, 4, 5, 6, 7)} = \dfrac{2K + 3K}{2K + 3K + K^2 + 2K^2 + 7K^2 + K}$

$= \dfrac{5K}{10K^2 + 6K} = \dfrac{5/10}{\frac{10}{100} + \frac{6}{10}} = \dfrac{5/10}{7/10} = 5/7$

$\therefore \boxed{P(1.5 < X < 4.5 / X > 2) = \frac{5}{7}}$

(iii) $P(|X - 1| > 3) = 1 - P(|X - 1| \leq 3)$

$= 1 - P(-3 \leq X - 1 \leq 3) = 1 - P(-2 \leq X \leq 4)$

$= 1 - \{P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)\}$

$= 1 - \{0 + K + 2K + 2K + 3K\} = 1 - 8K$

$= 1 - 8/10 = 2/10 = 1/5$

$\therefore \boxed{P(|X - 1| > 3) = 1/5}$

(iv) Mean and variance.

$E(X) = \sum x P(X = x)$

$= (0) P(X = 0) + 1 P(X = 1) + 2 P(X = 2) + \cdots + 7 P(X = 7)$

$= 0 + 1(K) + 2(2K) + 3(2K) + 4(3K) + 5(K^2) + 6(2K^2)$
$\quad + 7(7K^2 + K)$

$= K + 4K + 6K + 12K + 5K^2 + 12K^2 + 49K^2 + 7K$

$= 66K^2 + 30K = \frac{66}{100} + \frac{30}{10} = \frac{366}{100} = 3.66.$

$\therefore \boxed{E(X) = 3.66}$

$$E(x^2) = \sum x^2 P(X=x) = (0)^2 \cancel{P(X=0)} + (1)^2 P(X=1) + \cdots + (7)^2 P(X=7)$$

$$= (1)(K) + (4)(2K) + 9(2K) + 16(3K) + 25(K^2) + 36(2K^2) + 49(7K^2+K)$$

$$= K + 8K + 18K + 48K + 25K^2 + 72K^2 + 343K^2 + 49K$$

$$= 124K + 440K^2 = \frac{124}{10} + \frac{44\cancel{0}}{10\cancel{0}} = \frac{168}{10} = 16.8.$$

$$\boxed{E(x^2) = 16.8}$$

$$Var(x) = E(x^2) - [E(x)]^2 = 16.8 - (3.66)^2$$

$$\simeq 3.41$$

$$\therefore \boxed{Var(x) = 3.41}$$

(V) <u>Distribution Function</u> (Conceptual Approach)

$F(0) = P(X \le 0) = P(X=0) = 0$

$F(1) = P(X \le 1) = P(X=0) + P(X=1) = 0 + K = 1/10$

$F(2) = P(X \le 2) = P(X=0) + P(X=1) + P(X=2) = 0 + K + 2K = \frac{3}{10}$

$F(3) = P(X \le 3) = P(X=0) + \cdots + P(X=3)$

$= 0 + K + 2K + 2K = 5K = 5/10$

$F(4) = P(X \le 4) = P(X=0) + P(X=1) + \cdots + P(X=4)$

$= 0 + K + 2K + 2K + 3K = 8K = 8/10$

$F(5) = P(X \le 5) = P(X=0) + P(X=1) + \cdots + P(X=5)$

$= 0 + K + 2K + 2K + 3K + K^2 = 8K + K^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$

$F(6) = P(X \le 6) = P(X=0) + P(X=1) + \cdots + P(X=6)$

$= 0 + K + 2K + 2K + 3K + K^2 + 2K^2 = 8K + 3K^2 = 83/100$

$F(7) = P(X \le 7) = P(X=0) + P(X=1) + \cdots + P(X=7)$

$= 1.$ (sum of all probabilities)

Distribution function [Exam point of view]

x	$P(X=x)$	$F(x)$
0	0	0
1	k	$k = 1/10$
2	$2k$	$3k = 3/10$
3	$2k$	$5k = 5/10$
4	$3k$	$8k = 8/10$
5	k^2	$8k + k^2 = 81/100$
6	$2k^2$	$8k + 3k^2 = 83/100$
7	$7k^2 + k$	$10k^2 + 9k = 1$

(vi) Smallest value of 'λ' such that $P(X \le \lambda) > \frac{1}{2}$.

In General, What is λ?

λ may 0, 1, 2, 3, 4, 5, 6, 7 (for this problem).

Now $P(X \le 0) = 0 < 1/2$ $(\therefore \lambda \ne 0)$

$P(X \le 1) = 1/10 < 1/2$ $(\therefore \lambda \ne 1)$

$P(X \le 2) = 3/10 < 1/2$ $(\therefore \lambda \ne 1)$

$P(X \le 3) = 5/10 \not> 1/2$ $(\therefore \lambda \ne 3)$

$P(X \le 4) = 8/10 > 1/2$ $(\therefore \lambda = 4)$

$P(X \le 5) = 81/100 > 1/2$ $(\therefore \lambda = 5)$

$P(X \le 6) = 83/100 > 1/2$ $(\therefore \lambda = 6)$

$P(X \le 7) = 1 > 1/2$ $(\therefore \lambda = 7)$

$\therefore$ $\boxed{\text{Smallest value of } \lambda = 4}$

02). Let X be a random variable such that $P(X=-2) = P(X=-1) = P(X=1) = P(X=2)$ and $P(X<0) = P(X=0) = P(X>0)$. Determine the probability mass function of X and the distribution function of X.

Soln: Given: Values of $X = -2, -1, 0, 1, 2$

WKT $P(X=-2) + P(X=-1) + P(X=0) + P(X=1) + P(X=2) = 1$. $\longrightarrow$ ①

Given: $P(X=-2) = P(X=-1) = P(X=1) = P(X=2) = K$ (say).

Also, $P(X<0) = P(X=0)$

$\Rightarrow P(X=-2) + P(X=-1) = P(X=0)$

$\therefore P(X=0) = 2K$.

$\therefore$ Probability Distribution (P.M.F)

X:	-2	-1	0	1	2
$P(X=x)$:	K	K	$2K$	K	K

From ① $K + K + 2K + K + K = 1 \Rightarrow 6K = 1 \Rightarrow K = 1/6$

$\therefore$ Probability Distribution is given by

X:	-2	-1	0	1	2
$P(X=x)$:	$1/6$	$1/6$	$2/6$	$1/6$	$1/6$.

Distribution function of X

x	$P(X=x)$	$F(x)$
-2	$1/6$	$1/6$
-1	$1/6$	$2/6$
0	$2/6$	$4/6$
1	$1/6$	$5/6$
2	$1/6$	1

——— x ——— x ———

Sums for Practice.

01) A random variable X has the following probability distribution

x:	-2	-1	0	1	2	3
$p(x)$:	0.1	K	0.2	$2K$	0.3	$3K$

Find (i) the value of 'K' (ii) $P(X<2)$ & $P(|X|<2)$ (iii) cumulative distribution of X (iv) Mean & variance of X.

02) The Probability mass function of a random variable is defined as $P(X=0)=3c^2$, $P(X=1)=4c-10c^2$ & $P(X=2)=5c-1$ where $c>0$ & $P(X=r)=0$ if $r\neq 0,1,2$.

Find (i) the value of c (ii) $p(0<X<2/X>0)$

(iii) Distribution function of X (iv) largest Value of 'x' for which $F(x)<1/2$.

03) If the random Variable X takes the values 1, 2, 3, 4 such that $2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)$. Find the probability distribution & cumulative distribution of X.

04) If $P(X = x) = \frac{x}{15}$, $x = 1, 2, 3, 4, 5$.

Find (i) $P(X = 1 \text{ or } X = 2)$ (ii) $P\left(\frac{1}{2} < X < \frac{5}{2} \middle/ X > 1\right)$

05) If $P(X = x) = -K$, $x = 1, 2, 3$ find the value of K.

Problems based on Discrete R.V X (infinite values)

Note:

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{\lfloor 2}x^2 + \frac{n(n+1)(n+2)}{\lfloor 3}x^3 + \frac{n(n+1)(n+2)(n+3)}{\lfloor 4}x^4 + \cdots$$

In particular,

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \cdots$$

$$(1-x)^{-4} = 1 + 4x + 10x^2 + 20x^3 + \cdots$$

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01) The probability function of an infinite discrete distribution is given by $P(X = j) = \frac{1}{2^j}$, $j = 1, 2, 3, \ldots$ Verify it is a probability mass function (p.m.f) and find its mean and variance. Also find $P(X$ is even$)$, $P(X \geq 5)$ and $P(X$ is divisible by $3)$.

Soln: Given $P(X = j) = \frac{1}{2^j}$, $j = 1, 2, 3, \dots$

(i.e) $P(X = x) = \frac{1}{2^x}$, $x = 1, 2, 3, \dots$

(i) Verification

To prove $\sum_{x=1}^{\infty} P(X = x) = 1$

Now $\sum_{x=1}^{\infty} P(X = x) = \sum_{x=1}^{\infty} \frac{1}{2^x}$

$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$

$= \frac{1}{2}\left\{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\right\}$

Assign, $x = 1/2$.

$= \frac{1}{2}\left\{1 + x + x^2 + \dots\right\}$

$= \frac{1}{2}(1 - x)^{-1}$

$= \frac{1}{2}(1 - 1/2)^{-1} = \frac{1}{2}(1/2)^{-1}$

$= 1.$

$\therefore$ the Given $P(X = x) = \frac{1}{2^x}$ is a p.m.f.

(ii) Mean and Variance

$E(X) = \sum_{x=1}^{\infty} x \cdot \frac{1}{2^x} = (1)\left(\frac{1}{2}\right) + 2\left(\frac{1}{2^2}\right) + 3\left(\frac{1}{2^3}\right) + 4\left(\frac{1}{2^4}\right) + \dots$

$= \frac{1}{2}\left\{1 + 2(1/2) + 3(1/2^2) + 4(1/2^3) + \dots\right\}$

Assign, $x = 1/2$.

$= \frac{1}{2}\left\{1 + 2x + 3x^2 + \dots\right\}$

$= \frac{1}{2}(1 - x)^{-2}$

$= \frac{1}{2}(1 - 1/2)^{-2} = (1/2)(1/2)^{-2}$

$= \frac{1}{2}(4) = 2$

$\therefore \boxed{E(X) = 2}$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{x=1}^{\infty} x^2 \cdot P(X=x) = \sum_{x=1}^{\infty} [x(x+1) - x] P(X=x)$$

$$= \sum_{x=1}^{\infty} x(x+1) \cdot \frac{1}{2^x} - \sum_{x=1}^{\infty} x \cdot \frac{1}{2^x} \rightarrow (=2.)$$

$$= \left[(1)(2) \cdot \frac{1}{2} + (2)(3)\left(\frac{1}{2^2}\right) + (3)(4)\frac{1}{2^3} + (4)(5)\frac{1}{2^4} + \cdots\right] - 2$$

$$= (2)\left(\frac{1}{2}\right)\left\{1 + 3(1/2) + 6(1/2^2) + (10)(1/2^3) + \cdots\right\} - 2.$$

$$X = 1/2$$

$$= \{1 + 3X + 6X^2 + 10X^3 + \cdots\} - 2$$

$$= (1-X)^{-3} - 2. = (1 - 1/2)^{-3} - 2 = (1/2)^{-3} - 2$$

$$= 8 - 2 = 6$$

$$\therefore \boxed{E(X^2) = 6}$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2$$

$$\therefore \boxed{Var(X) = 2}$$

(iii) P(X is even)

$$P(X \text{ is even}) = P(X = 2, 4, 6, 8, \ldots)$$

$$= \sum_{x=2,4,6}^{\infty} P(X=x) = \sum_{x=2,4,6}^{\infty} \frac{1}{2^x}$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots$$

$$= \frac{1}{2^2}\left\{1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots\right\}$$

$x = 1/2^2 = 1/4$

$\frac{1}{2^4} = \left(\frac{1}{2^2}\right)^2 = x^2$

$\frac{1}{2^6} = \left(\frac{1}{2^2}\right)^3 = x^3$

$$= \frac{1}{4}\left\{1 + x + x^2 + x^3 + \cdots\right\}$$

$$= \frac{1}{4}(1-x)^{-1} = \frac{1}{4}\left(1 - \frac{1}{4}\right)^{-1}$$

$$= \frac{1}{4}\left(\frac{3}{4}\right)^{-1} = \frac{1}{4}\left(\frac{4}{3}\right) = \frac{1}{3}.$$

$$\therefore \boxed{P(x \text{ is even}) = \frac{1}{3}}$$

Note: $P(x \text{ is odd}) = 1 - P(x \text{ is even}) = 1 - 1/3 = 2/3.$

(iv) $P(x \geq 5)$

$$P(x \geq 5) = \sum_{x=5}^{\infty} \frac{1}{2^x}$$

$$= \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \cdots$$

$$= \frac{1}{2^5}\left\{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right\} \quad (x = 1/2)$$

$$= \frac{1}{2^5}(1 + x + x^2 + x^3 + \cdots) = \frac{1}{32}(1-x)^{-1}$$

$$= \frac{1}{32}\left(1 - \frac{1}{2}\right)^{-1} = \frac{1}{32}\left(\frac{1}{2}\right)^{-1} = \frac{1}{32}(2) = \frac{1}{16}.$$

$$\therefore \boxed{P(x \geq 5) = \frac{1}{16}}$$

(v) P(X is divisible by 3)

$P(X$ is divisible by $3) = P(X = 3, 6, 9, 12, \dots)$

$$= \sum_{X=3,6,9}^{\infty} P(X = x) = \sum_{x=3,6,9}^{\infty} \frac{1}{2^x}$$

$$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \frac{1}{2^{12}} + \dots$$

$$= \frac{1}{2^3}\left[1 + \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \dots\right]$$

$$= \frac{1}{8}\left[1 + X + X^2 + X^3 + \dots\right]$$

$$= \frac{1}{8}(1-X)^{-1} = \frac{1}{8}\left(1 - \frac{1}{8}\right)^{-1} = \frac{1}{8}\left(\frac{7}{8}\right)^{-1} = \frac{1}{8}\cdot\frac{8}{7} = \frac{1}{7}$$

$X = 1/2^3 = \frac{1}{8}$

$\frac{1}{2^6} = \left(\frac{1}{2^3}\right)^2 = X^2$

$\frac{1}{2^9} = \left(\frac{1}{2^3}\right)^3 = X^3$

$\therefore$ $\boxed{P(X \text{ is divisible by } 3) = \frac{1}{7}}$

Q2) Let the random variable X assume the value 'r' with the probability law $P(X = r) = q^{r-1}p$, $r = 1, 2, 3, \dots$ $q + p = 1$. Verify it is a p.m.f and also find its mean and variance.

Soln: Given $P(X = r) = q^{r-1}p$, $r = 1, 2, 3, \dots$

$\Rightarrow P(X = x) = q^{x-1}\cdot p$, $x = 1, 2, 3, \dots$ $(q + p = 1)$

(i) Verification of P.M.F

To prove $\sum_{x=1}^{\infty} P(X = x) = 1$.

Now $\sum_{x=1}^{\infty} P(X=x) = \sum_{x=1}^{\infty} q^{x-1}\cdot p = p\sum_{x=1}^{\infty} \frac{q^x}{q}$

$$= \frac{p}{q}\left\{q + q^2 + q^3 + \cdots\right\}$$

$$= \frac{p}{\cancel{q}}\cdot\cancel{q}\left\{1 + q + q^2 + \cdots\right\}$$

$$= p(1-q)^{-1} = p(p)^{-1} = 1. \qquad (\because p+q=1)$$

$\therefore$ the given $P(X=x) = q^{x-1}p,\ x = 1, 2, 3, \ldots$ is a p.m.f.

(ii) <u>Mean and Variance.</u>

$$\text{Mean} = E(X) = \sum_{x=1}^{\infty} x\,P(X=x)$$

$$= \sum_{x=1}^{\infty} x\,q^{x-1}\cdot p = p\sum_{x=1}^{\infty} \frac{x\,q^x}{q}$$

$$= \frac{p}{q}\left\{q + 2q^2 + 3q^3 + 4q^4 + \cdots\right\}$$

$$= \frac{p}{\cancel{q}}\cdot\cancel{q}\left\{1 + 2q + 3q^2 + 4q^3 + \cdots\right\}$$

$$= p(1-q)^{-2} = p(p^{-2}) = 1/p.$$

$$\therefore \boxed{E(X) = 1/p}$$

$$E(X^2) = \sum_{x=1}^{\infty} x^2 P(X=x) = \sum_{x=1}^{\infty}\left[x(x+1) - x\right]P(X=x)$$

$$= \sum_{x=1}^{\infty} x(x+1)\,q^{x-1}p - \sum_{x=1}^{\infty} x\,q^{x-1}\cdot p \rightarrow = 1/p.$$

$$= p\sum_{x=1}^{\infty} x(x+1)\frac{q^x}{q} - \frac{1}{p}$$

$$= \frac{p}{q}\left\{(1)(2)(q)+(2)(3)(q^2)+(3)(4)q^3+\cdots\right\} - \frac{1}{p}$$

$$= \frac{p}{q}(2q)\left\{1+3q+6q^2+\cdots\right\} - \frac{1}{p}$$

$$= 2p(1-q)^{-3} - \frac{1}{p} = 2p(p)^{-3} - \frac{1}{p}$$

$$\boxed{E(X^2) = \frac{2}{p^2} - \frac{1}{p}}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2} = q/p^2$$

$$\therefore \boxed{var(X) = q/p^2}$$

---

03) A random variable X takes values 0, 1, 2, ... with probability proportional to $\frac{x+1}{5^x}$. Find $P(X \le 5)$. Also find its Mean and variance.

Soln: Given P.M.F $\propto \frac{x+1}{5^x}$

$$\therefore P(X=x) = K\left(\frac{x+1}{5^x}\right), \quad x = 0,1,2,3,\ldots$$

To find 'k'

$\because \quad P(X = x) = 1$

$\Rightarrow \sum_{x=0}^{\infty} k\left(\frac{x+1}{5^x}\right) = 1$

$\Rightarrow k\left\{1 + \frac{2}{5} + \frac{3}{5^2} + \frac{4}{5^3} + \cdots\right\} = 1$

$X = 1/5$

$\Rightarrow k\{1 + 2X + 3X^2 + 4X^3 + \cdots\} = 1$

$\Rightarrow k(1-X)^{-2} = 1$

$\Rightarrow k\left(1 - \frac{1}{5}\right)^{-2} = 1 \quad \Rightarrow$

$\Rightarrow k(4/5)^{-2} = 1$

$(4/5)^{-2} = (5/4)^2 = 25/16$

$\Rightarrow k(25/16) = 1$

$\Rightarrow \boxed{k = 16/25}$

$\therefore \boxed{P(X = x) = \frac{16}{25}\left(\frac{x+1}{5^x}\right), \quad x = 0, 1, 2, \ldots}$

$P(X \le 5) = P(X=0) + P(X=1) + P(X=2) + \cdots + P(X=5)$

$= \frac{16}{25}(1) + \frac{16}{25}\left(\frac{2}{5}\right) + \frac{16}{25}\left(\frac{3}{5^2}\right) + \frac{16}{25}\left(\frac{4}{5^3}\right) + \frac{16}{25}\left(\frac{5}{5^4}\right) + \frac{16}{25}\left(\frac{6}{5^5}\right)$

$= \frac{16}{25}\left\{1 + \frac{2}{5} + \frac{3}{25} + \frac{4}{125} + \frac{5}{625} + \frac{6}{3125}\right\}$

$=$

(ii) <u>Mean and Variance</u>

$$E(X) = \sum_{x=0}^{\infty} x \cdot P(X=x) = \frac{16}{25}\sum_{x=0}^{\infty} x\left(\frac{x+1}{5^x}\right)$$

$$= \frac{16}{25}\left\{0 + (1)(2)\left(\frac{1}{5}\right) + (2)(3)\left(\frac{1}{5^2}\right) + (3)(4)\cdot\frac{1}{5^3} + \cdots\right]$$

$$= \frac{16}{25}\left(\frac{2}{5}\right)\left\{1 + 3\left(\tfrac{1}{5}\right) + 6\left(\tfrac{1}{5^2}\right) + \cdots\right] \quad (X = 1/5)$$

$$= \frac{32}{125}\left\{1 + 3X + 6X^2 + \cdots\right]$$

$$= \frac{32}{125}(1-X)^{-3} = \frac{32}{125}\left(1 - 1/5\right)^{-3} = \frac{32}{125}\left(4/5\right)^{-3}$$

$$= \frac{32}{125}\left(\frac{125}{64}\right) = 1/2.$$

$$\therefore \boxed{E(X) = 1/2}$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 P(X=x) = \frac{16}{25}\sum_{x=0}^{\infty} x^2\left(\frac{x+1}{5^x}\right)$$

$$= \frac{16}{25}\sum_{x=0}^{\infty}\left[x(x+2) - 2x\right]\left[\frac{x+1}{5^x}\right]$$

$$= \frac{16}{25}\sum_{x=0}^{\infty} x(x+1)(x+2)\left(\frac{1}{5^x}\right) - (2)\underbrace{\boxed{\frac{16}{25}\sum_{x=0}^{\infty}\frac{x(x+1)}{5^x}}}_{=1/2}$$

$$= \frac{16}{25}\left\{0 + (1)(2)(3)\left(\frac{1}{5}\right) + (2)(3)(4)\frac{1}{5^2} + (3)(4)(5)\left(\frac{1}{5^3}\right) + \cdots\right]$$

$$- 2\left(1/2\right)$$

$$= \frac{16}{25}\left(\frac{6}{5}\right)\left[1 + 4\left(\tfrac{1}{5}\right) + 10\left(\tfrac{1}{5^2}\right) + \cdots\right]$$

$$= \frac{96}{125}\left\{1 + 4x + 10x^2 + \cdots\right\}$$

$$= \frac{96}{125}(1-x)^{-4} = \frac{96}{125}\left(1-\frac{1}{5}\right)^{-4} = \frac{96}{125}\left(\frac{4}{5}\right)^{-4}$$

$$= \frac{96}{125}\cdot\frac{5^4}{4^4} = \frac{\overset{3}{\cancel{96}}}{\cancel{125}}\cdot\frac{\overset{5}{\cancel{625}}}{\underset{8}{\cancel{256}}} = \frac{15}{8}$$

$$\therefore \boxed{E(x^2) = 15/8}$$

$$\therefore \operatorname{var}(x) = E(x^2) - [E(x)]^2$$

$$= 15/8 - \frac{1}{4} = 13/8$$

$$\boxed{\operatorname{var}(x) = 13/8}$$

Problems based on Probability Density function

01) In a Continuous distribution, density function is given by $f(x) = kx(2-x)$, $0 < x < 2$. Find

(i) Value of k, mean and variance

(ii) Mean Deviation

(iii) Distribution function.

(iv) $P\left(X \le \frac{1}{2} \,/\, \frac{1}{3} \le X \le \frac{2}{3}\right)$

(V) the value of 'a' such that $P(X < a) = P(X > a)$

Soln:

$f(x) = kx(2-x)$

0   2

(i) Value of 'k'

we know that $\int_0^2 f(x)\,dx = 1$

$$\Rightarrow k\int_0^2 x(2-x)\,dx = 1$$

$$\Rightarrow k\int_0^2 (2x - x^2)\,dx = 1$$

$$\Rightarrow k\left(x^2 - \frac{x^3}{3}\right)_0^2 = 1$$

$$\Rightarrow k\left\{\left(4 - \frac{8}{3}\right) - (0)\right\} = 1 \quad \Rightarrow k\left(\frac{4}{3}\right) = 1$$

$$\Rightarrow \boxed{k = \frac{3}{4}}$$

$$\therefore \boxed{f(x) = \frac{3}{4}x(2-x), \quad 0 < x < 2}$$

(ii) Mean and variance

$$E(X) = \int_0^2 x f(x)\,dx = \frac{3}{4}\int_0^2 x \cdot x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx = \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{3}{4}\left[\left(\frac{16}{3} - \frac{16}{4}\right) - (0)\right] = \frac{3}{4}\left[\frac{16}{3} - 4\right]$$

$$= \frac{3}{4}\left\{\frac{4}{3}\right\} = 1$$

$$\therefore \boxed{E(X) = 1}$$

$$E(X^2) = \int_0^2 x^2 f(x)\,dx = \frac{3}{4}\int_0^2 x^2 \cdot x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx = \frac{3}{4}\left(\frac{\cancel{2}x^4}{\cancel{4}_2} - \frac{x^5}{5}\right)_0^2$$

$$= \frac{3}{4}\left(\frac{16}{2} - \frac{32}{5}\right) = \frac{3}{4}\left(\frac{8}{5}\right) = \frac{6}{5}$$

$$\therefore \boxed{E(X^2) = 6/5}$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2 = 6/5 - 1 = 1/5$$

$$\therefore \boxed{Var(X) = 1/5}$$

(iii) Mean Deviation

$$E\{|X - E(X)|\} = E\{|X - 1|\} = \int_0^2 |x-1| f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 |x-1| \cdot x(2-x)\,dx$$

$|x-1| = -(x-1) = 1-x$ $\quad$ $|x-1| = x-1.$

$0 \quad x-1<0 \quad 1 \quad x-1>0 \quad 2$

$$= \frac{3}{4}\left\{ \int_0^1 x(1-x)(2-x)\,dx + \int_1^2 x(x-1)(2-x)\,dx \right\}$$

$$= \frac{3}{4}\left\{ \int_0^1 (x^3 - 3x^2 + 2x)\,dx + \int_1^2 (-x^3 + 3x^2 - 2x)\,dx \right\}.$$

$$= \frac{3}{4}\left[\frac{x^4}{4} - \frac{\cancel{3}x^3}{\cancel{3}} + \frac{\cancel{2}x^2}{\cancel{2}} \right]_0^1 + \frac{3}{4}\left[\frac{-x^4}{4} + \frac{\cancel{3}x^3}{\cancel{3}} - \frac{\cancel{2}x^2}{\cancel{2}} \right]_1^2$$

$$= \frac{3}{4}\left(\frac{1}{4} - 1 + 1\right) + \frac{3}{4}\left\{\left(\cancel{\frac{-16}{4}} + \cancel{8} - \cancel{4}\right) - \left(-\frac{1}{4} + \cancel{1} - \cancel{1}\right)\right\}.$$

$$= \frac{3}{4}\left(\frac{1}{4}\right) + \frac{3}{4}\left(\frac{1}{4}\right) = \frac{3}{16} + \frac{3}{16} = \frac{3}{8}$$

$\therefore$ $\boxed{\text{Mean Deviation} = 3/8}$

(iv) <u>Distribution function</u>

$0 \quad x \quad 2.$

$$F(x) = P(X \leq x)$$

$$= \int_0^x f(x)\,dx = \frac{3}{4}\int_0^x x(2-x)\,dx$$

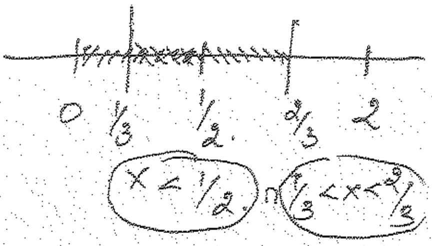
$$= \frac{3}{4}\int_0^x (2x - x^2)\,dx = \frac{3}{4}\left\{ x^2 - x^3/3 \right\}_0^x$$

$$= 3/4\,(x^2 - x^3/3) = \frac{\cancel{3}}{4}\left(\frac{3x^2 - x^3}{\cancel{3}}\right) = \frac{1}{4}(3x^2 - x^3)$$

$$\therefore \boxed{F(x) = \frac{1}{4}(3x^2 - x^3)}$$

(v) $P\left(X \le \frac{1}{2} \,\middle|\, \frac{1}{3} < x < \frac{2}{3}\right)$

$$= \frac{P\left(x \le \frac{1}{2} \cap \frac{1}{3} < x < \frac{2}{3}\right)}{P\left(\frac{1}{3} < x < \frac{2}{3}\right)}$$



$$= \frac{P\left(\frac{1}{3} < x < \frac{1}{2}\right)}{P\left(\frac{1}{3} < x < \frac{2}{3}\right)} \quad \text{———} \quad (1)$$

Now $P\left(\frac{1}{3} < x < \frac{1}{2}\right) = \int_{1/3}^{1/2} f(x)\,dx = \frac{3}{4}\int_{1/3}^{1/2} x(2-x)\,dx$

$$= \frac{3}{4}\int_{1/3}^{1/2} (2x - x^2)\,dx = \frac{3}{4}\left[\left(x^2 - \frac{x^3}{3}\right)\right]_{1/3}^{1/2}$$

$$= \frac{3}{4}\left\{\left(\frac{1}{4} - \frac{1}{24}\right) - \left(\frac{1}{9} - \frac{1}{81}\right)\right] = \frac{3}{4}\left(\frac{5}{24} - \frac{8}{81}\right).$$

$$= \frac{3}{4}\left(\frac{71}{648}\right) = \frac{71}{864}$$

$$\therefore \boxed{P\left(\frac{1}{3} < x < \frac{1}{2}\right) = \frac{71}{864}}$$

$P\left(\frac{1}{3} < x < \frac{2}{3}\right) = \int_{1/3}^{2/3} f(x)\,dx = \frac{3}{4}\int_{1/3}^{2/3} x(2-x)\,dx$

$$= \frac{3}{4}\int_{1/3}^{2/3} (2x - x^2)\,dx = \frac{3}{4}\left(x^2 - \frac{x^3}{3}\right)_{1/3}^{2/3}$$

$$= \frac{3}{4}\left\{\left(\frac{4}{9} - \frac{8}{81}\right) - \left(\frac{1}{9} - \frac{1}{81}\right)\right] = \frac{3}{4}\left\{\frac{28}{81} - \frac{8}{81}\right\} = \frac{3}{4}\left(\frac{2}{81}\right.$$

$$= \frac{5}{27} \quad \therefore \boxed{P\left(\frac{1}{3} < x < \frac{2}{3}\right) = \frac{5}{27}}$$

① $\Rightarrow P(x \le \frac{1}{2} \mid \frac{1}{3} < x < \frac{2}{3}) = \dfrac{\frac{71}{864}}{\frac{5}{27}} = \dfrac{71}{160}$

$\therefore \boxed{P(x \le \frac{1}{2} \mid \frac{1}{3} < x < \frac{2}{3}) = \frac{71}{160}}$

(vi) Value of 'a'

$P(x < a) = P(x > a)$

$\Rightarrow P(x<a) = 1 - P(x \le a)$

$\Rightarrow 2P(x<a) = 1$

$\Rightarrow P(x<a) = \frac{1}{2}$

$\Rightarrow \int_0^a f(x)\,dx = \frac{1}{2}$

$\Rightarrow \frac{3}{4}\int_0^a x(2-x)\,dx = \frac{1}{2}.$

$\Rightarrow \frac{3}{4}\int_0^a (2x - x^2)\,dx = \frac{1}{2}$

$\Rightarrow \left(x^2 - \frac{x^3}{3}\right)_0^a = \frac{2}{3}$

$\Rightarrow a^2 - \frac{a^3}{3} = \frac{2}{3}.$

x^{ly} by ③ $3a^2 - a^3 = 2.$

$\Rightarrow a^3 - 3a^2 + 2 = 0.$

$0 \quad a \quad 2$

1	1	−3	0	2
	1	1	−2	−2
	1	−2	−2	0

$\therefore \boxed{a=1}$

$a^2 - 2a - 2 = 0$

$a = \frac{2 \pm \sqrt{4+8}}{2}$

$= \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2}.$

$= 1 \pm \sqrt{3}.$

$= 1+1.7,\ 1-1.7$

$\boxed{a = 2.7,\ 0.3}$

$\therefore a = 1,\ 2.7,\ 0.3$

Here $\boxed{a = 1,\ 0.3}$ is possible

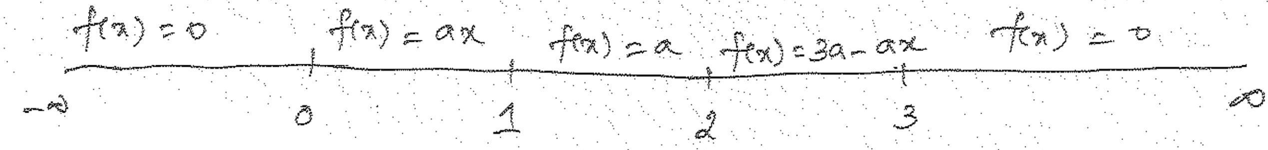
—— × —— ✳ —— × ——

(*) (02) If the density function of a continuous random variable X is given by $f(x) = \begin{cases} ax, & 0 \le x \le 1/2 \\ a, & 1 \le x \le 2. \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$

Find the value of 'a' and c.d.f of X.

Also find $P(X \le 1.5)$.

Soln: Given p.d.f



To find 'a'

we know that $\int_{-\infty}^{\infty} f(x)\, dx$

$$\Rightarrow \int_0^1 ax\, dx + \int_1^2 a\, dx + \int_2^3 (3a - ax)\, dx = 1$$

$$\Rightarrow \left(\frac{ax^2}{2}\right)_0^1 + (ax)_1^2 + \left(3ax - \frac{ax^2}{2}\right)_2^3 = 1$$

$$\Rightarrow \frac{a}{2} + 2a - a + \left(9a - \frac{9a}{2}\right) - (6a - 2a) = 1$$

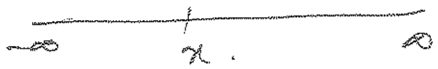
$$\Rightarrow a/2 + a + 9a - \frac{9a}{2} - 4a = 1$$

$$\Rightarrow 2a = 1$$

$$\Rightarrow \boxed{a = 1/2}$$

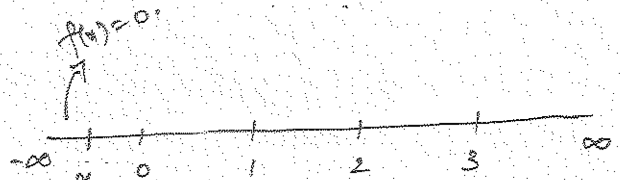
Distribution function of X

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$



What is 'x'? 'x' may lie anywhere in between '$-\infty$' & '∞'.

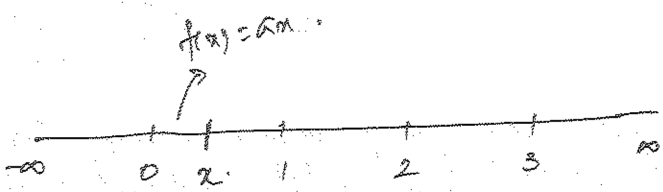
Case 1: (If $-\infty < x < 0$)



$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{x} 0\,dx = 0 \qquad \therefore \boxed{F(x) = 0}$$

Case 2: (If $0 < x < 1$)



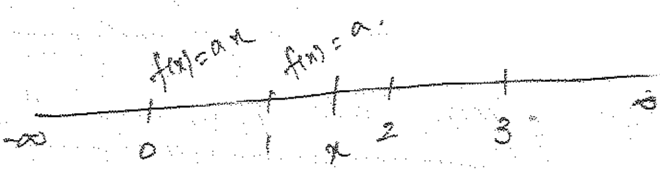
$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} ax\,dx = \left(\frac{ax^2}{2}\right)_0^x = \frac{ax^2}{2} = \frac{x^2}{4}$$

$$(\because a = 1/2)$$

$$\therefore \boxed{F(x) = \frac{x^2}{4}}$$

Case 3: (If $1 < x < 2$)



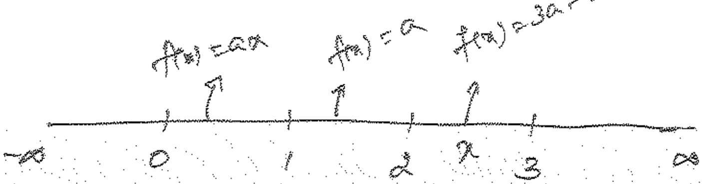
$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} ax\,dx + \int_{1}^{x} a\,dx$$

$$= \left(\frac{ax^2}{2}\right)_0^1 + (ax)_1^x = \frac{a}{2} + ax - a = ax - \frac{a}{2}.$$

$$\therefore \boxed{F(x) = \frac{x}{2} - \frac{1}{4}}$$

Case 4: $(2 < x < 3)$



$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \cancel{\int_{-\infty}^{0} 0\,dx} + \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{x} (3a - ax)\,dx$$

$$= \left(\frac{ax^2}{2}\right)_0^1 + (ax)_1^2 + \left(3ax - \frac{ax^2}{2}\right)_2^x$$

$$= \frac{a}{2} + 2a - a + \left(3ax - \frac{ax^2}{2}\right) - (6a - 2a)$$

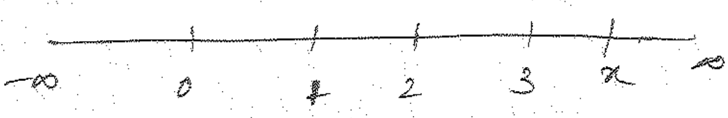
$$= \frac{a}{2} + a + 3ax - \frac{ax^2}{2} - 4a.$$

$$= 3ax - \frac{ax^2}{2} - \frac{5a}{2}$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}.$$

$$\therefore \boxed{F(x) = \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}}$$

Case 5: $(3 < x < \infty)$



$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{3} (3a - ax)\,dx + \cancel{\int_{3}^{x} 0\,dx}$$

$$= 1 \qquad \therefore \boxed{F(x) = 1}$$

$$\therefore F(x) = \begin{cases} 0, & -\infty < x < 0 \\ x^2/4, & 0 < x < 1 \\ x/2 - 1/4, & 1 < x < 2 \\ 3x/2 - x^2/4 - 5/4, & 2 < x < 3 \\ 1, & 3 < x < \infty. \end{cases}$$

(iii) $\underline{P(X \le 1.5)}$

$$P(X \le 1.5) = \int_{-\infty}^{1.5} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^1 ax\,dx + \int_1^{1.5} a\,dx$$

$$= \left(\frac{ax^2}{2}\right)_0^1 + (ax)_1^{1.5}$$

$$= \left(\frac{a}{2} - 0\right) + (1.5a - a)$$

$$= a/2 + 0.5a = a = 1/2.$$

$$\therefore \boxed{P(X \le 1.5) = 1/2}$$

———— × ———— × ————

Q3) Experience has shown that while walking in a certain park, the time X (in mins), between seeing two people smoking has a density function of the form $f(x) = \lambda x e^{-x}$, $x > 0$. Find the value of λ, the distribution function of X, Mean and Variance. Also find the probability that Jeff, who has just seen a person smoking, will see another person smoking in 2 to 5 minutes?

Soln:

(i) To find λ

WKT, $\int_0^\infty f(x)\,dx = 1$

$$\Rightarrow \lambda \int_0^\infty x e^{-x}\,dx = 1$$

$$\Rightarrow \lambda \left\{ x(-e^{-x}) - (1)(e^{-x}) \right\}_0^\infty = 1$$

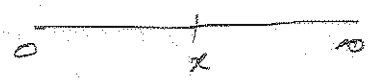
$$\Rightarrow \lambda \left\{ -xe^{-x} - e^{-x} \right\}_0^\infty = 1$$

$$\Rightarrow \lambda \left\{ (0) - (0 - 1) \right\} = 1$$

$$\Rightarrow \boxed{\lambda = 1}$$

$$\therefore f(x) = xe^{-x}, \quad x > 0$$

(ii) Distribution function of X



$$F(x) = P(X \le x) = \int_0^x f(x)\,dx$$

$$= \int_0^x x e^{-x}\,dx = \left[x(-e^{-x}) - (1)(e^{-x}) \right]_0^x$$

$$= \left[-xe^{-x} - e^{-x} \right]_0^x$$

$$= \left(-xe^{-x} - e^{-x} \right) - (-1)$$

$$= 1 - xe^{-x} - e^{-x}$$

$$\therefore \boxed{F(x) = 1 - xe^{-x} - e^{-x}}$$

(iii) Mean & Variance of X

$$E(X) = \int_0^\infty x\, f(x)\, dx = \int_0^\infty x \cdot x e^{-x}\, dx$$

$$= \int_0^\infty x^2 e^{-x}\, dx$$

$$= \left\{ x^2(-e^{-x}) - (2x)(e^{-x}) + (2)(-e^{-x}) \right\}_0^\infty$$

$$= \left(-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right)_0^\infty$$

$$= (0) - (-2) = 2$$

$$\therefore \boxed{E(X) = 2}$$

$$E(X^2) = \int_0^\infty x^2 f(x)\, dx = \int_0^\infty x^2 \cdot x e^{-x}\, dx$$

$$= \int_0^\infty x^3 e^{-x}\, dx$$

$$= \left\{ x^3(-e^{-x}) - (3x^2)(e^{-x}) + (6x)(-e^{-x}) - 6(e^{-x}) \right\}_0^\infty$$

$$= \left\{ -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6e^{-x} \right\}_0^\infty$$

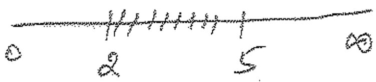
$$= (0) - (-6) = 6 \qquad \therefore \boxed{E(X^2) = 6}$$

$$\therefore \mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 6 - 4 = 2$$

$$\therefore \boxed{\mathrm{Var}(X) = 2}$$

(iv) $P(2 < x < 5)$



$$P(2 < x < 5) = \int_2^5 x e^{-x}\, dx$$

$$= \left\{ x(-e^{-x}) - (1)(e^{-x}) \right\}_2^5$$

$$= \left\{ -xe^{-x} - e^{-x} \right\}_2^5$$

$$= (-5e^{-5} - e^{-5}) - (-2e^{-2} - e^{-2})$$

$$= 2e^{-2} + e^{-2} - 5e^{-5} - e^{-5}.$$

$$= 3e^{-2} - 6e^{-5}.$$

$$\therefore \boxed{P(2 < x < 5) = 3e^{-2} - 6e^{-5}}$$

——— x ——— x ——— x ———

Q4) The length of time (in minutes) that a certain lady speaks on the telephone is found to be random phenomenon, with probability function specified by the p.d.f

$$f(x) = \begin{cases} A e^{-x/5}, & x \geq 0 \\ 0, & \text{otherwsise} \end{cases}$$

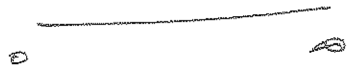
(a) Find the value of A that makes $f(x)$ a p.d.f.

(b) What is the probability that the number of minutes she will talk over the phone is (i) more than 10 minutes (ii) less than 5 minutes and (iii) between 5 and 10 minutes ?

Solution:

ⓐ Value of A

$f(x) = Ae^{-x/5}$



WKT $\int_0^\infty f(x)\,dx = 1$

$\Rightarrow A\int_0^\infty e^{-x/5}\,dx = 1$

$\Rightarrow A\left\{\frac{e^{-x/5}}{-1/5}\right\}_0^\infty = 1$

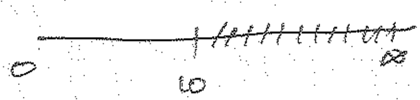
$\Rightarrow A\left\{-5e^{-x/5}\right\}_0^\infty = 1$

$\Rightarrow A\{0 + 5\} = 1$

$\Rightarrow \boxed{A = 1/5}$

$\therefore \boxed{f(x) = \frac{1}{5}e^{-\frac{1}{5}x}, \quad x \geq 0}$

ⓑ (i) $P(X > 10)$



$P(X > 10) = \int_{10}^\infty f(x)\,dx$

$= \frac{1}{5}\int_{10}^\infty e^{-x/5}\,dx = \frac{1}{5}\left\{\frac{e^{-x/5}}{-1/5}\right\}_{10}^\infty$

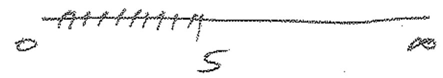
$= \frac{1}{5}\left\{-5e^{-\frac{x}{5}}\right\}_{10}^\infty = \frac{1}{5}\left\{(0) - (-5e^{-2})\right\}$

$= \frac{1}{5}\cdot 5e^{-2} = e^{-2}.$

$\therefore \boxed{P(X > 10) = e^{-2} =}$

(ii) $P(X<5)$

$$P(x<5) = \int_0^5 \frac{1}{5} e^{-x/5}\,dx$$



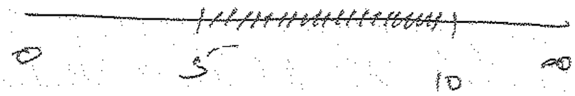
$$= \frac{1}{5}\left\{\frac{e^{-x/5}}{-1/5}\right\}_0^5 = \frac{1}{5}\left\{-5e^{-x/5}\right\}_0^5$$

$$= \frac{1}{5}\left\{(-5e^{-1}) - (-5)\right\}$$

$$= \frac{1}{5}\left\{5 - 5e^{-1}\right\} = 1 - e^{-1}$$

$$\therefore \boxed{P(x<5) = 1 - e^{-1} = }$$

(iii) $P(5<X<10)$



$$P(5<x<10)$$

$$= \frac{1}{5}\int_5^{10} e^{-x/5}\,dx$$

$$= \frac{1}{5}\left\{\frac{e^{-x/5}}{-1/5}\right\}_5^{10} = \frac{1}{5}\left\{-5e^{-x/5}\right\}_5^{10}$$

$$= \frac{1}{5}\left\{(-5e^{-2}) - (-5e^{-1})\right\}$$

$$= \frac{1}{5}\left\{5e^{-1} - 5e^{-2}\right\} = e^{-1} - e^{-2}$$

$$\therefore \boxed{P(5<x<10) = e^{-1} - e^{-2} = }$$

——— × ——— × ——— × ———

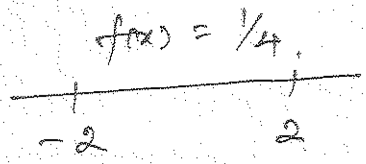
(05) If a Random variable X has the p.d.f

$$f(x) = \begin{cases} \frac{1}{4}, & |x| < 2 \\ 0, & \text{otherwise} \end{cases}$$, obtain (i) $P(X < 1)$

(ii) $P(|X| > 1)$ and (iii) $P(2X + 3 > 5)$

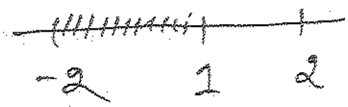
Soln:
Given $f(x) = \begin{cases} 1/4, & |x| < 2 \\ 0, & \text{otherwise.} \end{cases}$

(ie) $f(x) = \begin{cases} \frac{1}{4}, & -2 < x < 2 \\ 0, & \text{otherwise} \end{cases}$



(i) $P(X < 1)$

$$P(X < 1) = \int_{-2}^{1} f(x)\, dx = \int_{-2}^{1} \frac{1}{4}\, dx$$



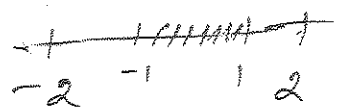
$$= \frac{1}{4}(x)_{-2}^{1} = \frac{1}{4}\left\{(1) - (-2)\right\}$$

$$= 3/4 \qquad \therefore \boxed{P(X < 1) = 3/4}$$

(ii) $P(|X| > 1)$

$$P(|X| > 1) = 1 - P(|X| \leq 1) = 1 - P(-1 \leq X \leq 1)$$

$$= 1 - \int_{-1}^{1} f(x)\, dx$$



$$= 1 - \int_{-1}^{1} 1/4\, dx = 1 - \frac{1}{4}(x)_{-1}^{1}$$

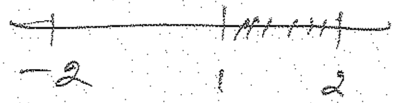
$$= 1 - \frac{1}{4}(1 + 1) = 1 - 1/2 = 1/2.$$

$$\therefore \boxed{P(|X| > 1) = 1/2}$$

(iii) $P(2X + 3 > 5)$

$$P(2X+3>5) = P(2X>2) = P(X>1)$$

$$= \int_1^2 f(x)\,dx = \int_1^2 \frac{1}{4}\,dx$$



$$= \frac{1}{4}(x)\Big|_1^2 = \frac{1}{4}(2-1) = 1/4$$

$$\therefore \boxed{P(2X+3>5) = \frac{1}{4}}$$

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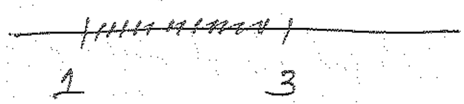
(06) Given $f(x)$ is the probability density function of a random variable X: $f(x) = \frac{1}{x^2}$, $1 < x < \infty$.

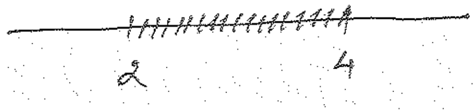
Find $P(A \cup B)$ and $P(A \cup B \cup C)$ where $A = \{x : 1 < x < 3\}$, $B = \{x : 2 < x < 4\}$ and $C = \{x : 4 < x < 5\}$.

Soln:

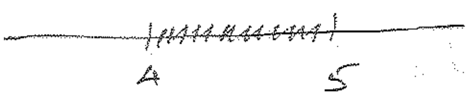
A :



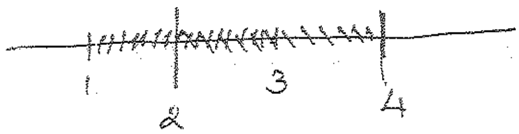
B :



C :



$A \cup B$:



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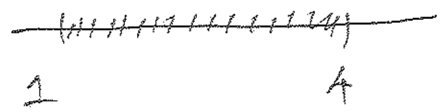
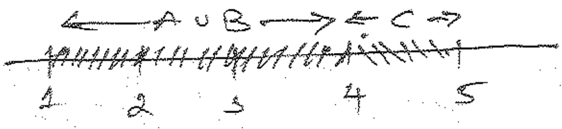


$A \cup B \cup C$:



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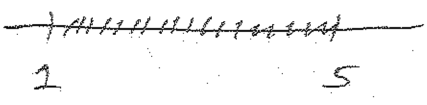


(i) $P(A \cup B)$

$$P(A \cup B) = P(1 < X < 4) = \int_1^4 f(x)\,dx$$

$$= \int_1^4 \frac{1}{x^2}\,dx$$

$$\boxed{\int \frac{1}{x^n}\,dx = \frac{1}{(1-n)\,x^{n-1}}}$$

$$= \left(\frac{1}{(-1)x}\right)_1^4 = \left(-\frac{1}{x}\right)_1^4$$

$$= \left(-\frac{1}{4}\right) - (-1) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\therefore \boxed{P(A \cup B) = \frac{3}{4}}$$

(ii) $P(A \cup B \cup C)$

$$P(A \cup B \cup C) = P(1 < X < 5) = \int_1^5 f(x)\,dx.$$

$$= \int_1^5 \frac{1}{x^2}\,dx = \left(-\frac{1}{x}\right)_1^5$$

$$= \left(-\frac{1}{5}\right) - (-1) = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\therefore \boxed{P(A \cup B \cup C) = \frac{4}{5}}$$

—— x —— x —— x ——

(07) A continuous R.V has the p.d.f $f(x) = \frac{k}{1+x^2}$, $-\infty < x < \infty$.

Find the value of K and the distribution function of X. Also find $P(X \geq 0)$.

Soln:

(i) To find 'k'

WKT $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow k \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = 1$$

$$\Rightarrow k \left(\tan^{-1} x \right)_{-\infty}^{\infty} = 1$$

$$\Rightarrow k \left(\tan^{-1} \infty - \tan^{-1}(-\infty) \right) = 1$$

$$\Rightarrow k \left(\pi/2 - \left(-\pi/2\right) \right) = 1$$

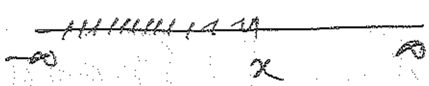
$$\Rightarrow k(\pi) = 1$$

$$\Rightarrow \boxed{k = 1/\pi}$$

$$\therefore \boxed{f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad -\infty < x < \infty}$$

(ii). Distribution function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\, dx$$



$$= \frac{1}{\pi} \int_{-\infty}^{x} \frac{1}{1+x^2}\, dx = \frac{1}{\pi} \left(\tan^{-1} x \right)_{-\infty}^{x}$$

$$= \frac{1}{\pi} \left(\tan^{-1} x - \tan^{-1}(-\infty) \right) = \frac{1}{\pi} \left(\tan^{-1} x + \pi/2 \right)$$

$$\therefore \boxed{F(x) = \frac{1}{\pi} \left(\tan^{-1} x + \pi/2 \right)}$$

(08) If $p(x) = \begin{cases} x e^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(a) Show that $p(x)$ is a p.d.f

(b) Find its distribution function.

Soln:

(a) To show $p(x)$ is a p.d.f (To prove $\int_0^\infty f(x)dx = 1$).

$f(x) = x e^{-x^2/2}$.

Now $\int_0^\infty x e^{-x^2/2} dx$

$= \int_0^\infty e^{-t} dt$

put $x^2/2 = t$

$\Rightarrow \frac{\cancel{2} x dx}{\cancel{2}} = dt$

$\therefore x dx = dt$

$= (-e^{-t})_0^\infty$

$= (0) - (-1) = 1$

Limits for 't'

Limits for x	0	∞
Limits for t	0	∞

$\therefore$ the given $p(x)$ is a p.d.f.

(b) Distribution function

$F(x) = P(X \leq x) = \int_0^x p(x) dx$

$= \int_0^x x e^{-x^2/2} dx = \int_0^{x^2/2} e^{-t} dt$

x	0	x
t	0	$x^2/2$

$= (-e^{-t})_0^{x^2/2} = -e^{-x^2/2} + 1$

$\therefore \boxed{F(x) = 1 - e^{-x^2/2}}$

Sums for Practice

01) A continuous R.V X has a p.d.f $f(x) = Kx^2 e^{-x}$, $x \geq 0$. Find K, mean and variance.

02) A continuous R.V X has a p.d.f $f(x) = 3x^2$, $0 \leq x \leq 1$. Find 'a' & 'b' such that (a) $P(X \leq a) = P(X > a)$ and (b) $P(X > b) = 0.05$.

03) The diameter of an electric cable X is a continuous R.V with p.d.f $f(x) = Kx(1-x)$, $0 \leq x \leq 1$. Find (i) the value of K (ii) the cumulative distribution function of X (iii) $P\left(X \leq \frac{1}{2} \middle| \frac{1}{3} \leq X \leq \frac{2}{3}\right)$ (iv) a number 'b' such that $P(X < b) = P(X > b)$.

04) If X is a continuous R.V with density function

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ \frac{3}{2}(x-1)^2, & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

, find the cumulative distribution function of X. Also find $P\left(\frac{3}{2} < X < \frac{5}{2}\right)$

05) A continuous R.V X can assume any value between $x = 2$ and $x = 5$ has the density function given by $f(x) = K(1+x)$. Find $P(X < 4)$ and $P(3 < X < 4)$.

———— x ———— x ———— x ————

01) For the following density function $f(x) = ae^{-|x|}, -\infty < x < \infty$. find the value of 'a', mean and variance.

Soln:

Note 1:

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx, \text{ if } f(x) \text{ is even}$$

$$\int_{-a}^{a} f(x)\,dx = 0, \text{ if } f(x) \text{ is odd.}$$

Note 2:

* $f(x)$ is even if $f(-x) = +f(x)$
* $f(x)$ is odd if $f(-x) = -f(x)$

To find 'a'

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow a\int_{-\infty}^{\infty} e^{-|x|}\,dx = 1$$

$$\Rightarrow 2a\int_{0}^{\infty} e^{-|x|}\,dx = 1$$

$$\Rightarrow 2a\int_{0}^{\infty} e^{-x}\,dx = 1$$

$$\Rightarrow 2a\left(-e^{-x}\right)_{0}^{\infty} = 1$$

$$\Rightarrow 2a\left[(0) - (-1)\right] = 1$$

$$\Rightarrow 2a = 1 \Rightarrow \boxed{a = 1/2}$$

$f(x) = e^{-|x|}$

$f(-x) = e^{-|-x|}$

$= e^{-|x|}$

$= f(x)$

$\therefore f(-x) = f(x)$

$\Rightarrow f(x)$ is even

$|x| = x$.

$0 \longrightarrow \infty$

Mean and Variance

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \frac{1}{2}\int_{-\infty}^{\infty} x e^{-|x|}\, dx$$

$f(x) = x e^{-|x|}$

$f(-x) = -x e^{-|-x|}$

$= -x e^{-|x|}$

$= -f(x)$

$f(x)$ is odd

$$= 0$$

$$\therefore \boxed{E(X) = 0}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}\, dx$$

$f(x) = x^2 e^{-|x|}$

$f(-x) = (-x)^2 e^{-|-x|}$

$= x^2 e^{-|x|}$

$= f(x)$

$\therefore f(x)$ is even

$$= \frac{1}{\cancel{2}} \cdot \cancel{2} \int_0^{\infty} x^2 e^{-|x|}\, dx$$

$$= \int_0^{\infty} x^2 e^{-x}\, dx$$

$$= \left[x^2(-e^{-x}) - (2x)(e^{-x}) + 2(-e^{-x}) \right]_0^{\infty}$$

$$= \left(-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right)_0^{\infty}$$

$$= (0) - (-2) = 2. \qquad \therefore \boxed{E(X^2) = 2}$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2$$

$$= 2 - 0$$

$$= 2$$

$$\therefore \boxed{Var(X) = 2}$$

——— × ——— × ——— × ———

Problems based on Distribution function

Possible Questions

Q1) Find probability density function: $f(x)$

[Data - Distribution function $F(x)$]

Hint:- $f(x) = \frac{d}{dx} F(x)$

Q2) Verify the given is a Distribution function.

Hint: Verify these points

(i) $F(-\infty) = 0$ (ii) $F(\infty) = 1$

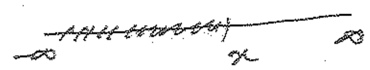
(iii) $\frac{d}{dx} F(x) = f(x)$, where $f(x)$ is a density function

[Note: For proving (iii), * we must differentiate $F(x)$

* Prove $\frac{d}{dx} F(x) = f(x)$ is a density function]

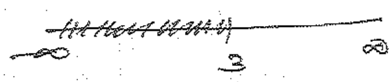
Q3) Finding Probabilities.

Hint: Always Make use of this Definition: $F(x) = P(X \leq x)$



Examples:

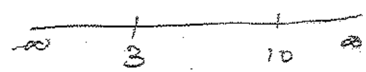
(i) $P(X \leq 3) = F(3)$



(ii) $P(X > 2) = 1 - P(X \leq 2) = 1 - F(2)$.

(iii) $P(3 < X < 10) = P(X < 10) - P(X < 3)$

$= F(10) - F(3)$.



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01) The cumulative distribution function of a random variable X is given by
$$F(x) = \begin{cases} 0 & , x < 0 \\ x^2 & , 0 \le x < \frac{1}{2} \\ 1 - \frac{3}{25}(3-x)^2 & , \frac{1}{2} \le x < 3 \\ 1 & , x \ge 3 \end{cases}$$

Verify it is a cdf and find the pdf of X.

Evaluate $P(|X| \le 1)$ using both the pdf and cdf.

Soln:

(i) Verification of a c.d.f

$F(-\infty) = 0$, $F(\infty) = 1$

$$f(x) = \frac{d}{dx} F(x) = \begin{cases} 0 & , x < 0 \\ 2x & , 0 \le x < 1/2, \\ 0 - \frac{3}{25} \cdot 2(3-x)(-1) & , 1/2 \le x < 3 \\ 0 & , x \ge 3 \end{cases}$$

$$\therefore f(x) = \begin{cases} 2x & , 0 \le x < 1/2 \\ 6/25\,(3-x) & , \frac{1}{2} \le x < 3 \\ 0 & , \text{otherwise} \end{cases}$$

To prove f(x) is a p.d.f

Now $\int_0^{1/2} 2x\, dx + \frac{6}{25} \int_{1/2}^{3} (3-x)\, dx$

$= (x^2)_0^{1/2} + \frac{6}{25}\left(3x - x^2/2\right)_{1/2}^{3}$

$= \frac{1}{4} + \frac{6}{25}\left\{\left(9 - 9/2\right) - \left(3/2 - \frac{1}{8}\right)\right\}$

$= \frac{1}{4} + \frac{6}{25}\left(\frac{25}{8}\right) = 1$ $\therefore \frac{d}{dx}F(x) = f(x)$ is a p.d.f.

(ii) pdf of X

$$f(x) = \frac{d}{dx} F(x) = \begin{cases} 2x, & 0 \le x < 1/2 \\ 6/25\,(3-x), & 1/2 \le x < 3 \\ 0, & \text{otherwise} \end{cases}$$

(iii) To find $P(|X| \le 1)$

Using c.d.f

$$P(|X| \le 1) = P(-1 \le X \le 1) = F(1) - F(-1)$$

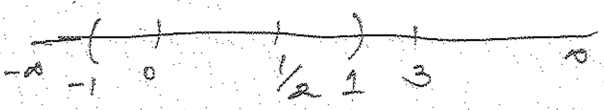
$$= \left[1 - \frac{3}{25}(3-1)^2\right] - 0.$$

$$= 1 - \frac{12}{25} = \frac{13}{25}.$$

$$\therefore \boxed{P(|X| \le 1) = \frac{13}{25}}$$

Using pdf

$$P(X \le 1) = P(-1 \le X \le 1)$$



$$= \cancel{\int_{-1}^{0} 0\,dx} + \int_0^{1/2} 2x\,dx + \int_{1/2}^{1} \frac{6}{25}(3-x)\,dx$$

$$= \left(x^2\right)_0^{1/2} + \frac{6}{25}\left(3x - \frac{x^2}{2}\right)_{1/2}^{1}$$

$$= \frac{1}{4} + \frac{6}{25}\left\{\left(3 - \frac{1}{2}\right) - \left(\frac{3}{2} - \frac{1}{8}\right)\right]$$

$$= \frac{1}{4} + \frac{6}{25}\left(\frac{9}{8}\right) = \frac{1}{4} + \frac{27}{100} = \frac{52}{100} = \frac{13}{25}$$

$$\therefore P(|X| \le 1) = 13/25$$

—— x ——

Q2) Suppose that the amount of money that a person has saved is found to be a random variable

with $F(x) = \begin{cases} \frac{1}{2} e^{-(x/50)^2}, & x < 0 \\ 1 - \frac{1}{2} e^{-(x/50)^2}, & x \geq 0 \end{cases}$

(a) What is the p.d.f

(b) What is the probability that the amount of saving possessed by him will be (i) more than 50 (ii) equal to 50 rupees.

(c) What is the conditional probability that the amount of savings will be less than Rs. 100 given that it is more than Rs 50 ? .

Soln:

(a) Finding p.d.f

$$f(x) = \frac{d}{dx} F(x) = \begin{cases} \frac{1}{\cancel{2}} e^{-(x/50)^2} \left(\frac{-\cancel{2}x}{50}\right)\left(\frac{1}{50}\right), & x < 0 \\ -\frac{1}{\cancel{2}} e^{-(x/50)^2} \left(-\frac{\cancel{2}x}{50}\right)\left(\frac{1}{50}\right), & x \geq 0 \end{cases}$$

$$= \begin{cases} \frac{-x}{2500} e^{-(x/50)^2}, & x < 0 \\ \frac{x}{2500} e^{-(x/50)^2}, & x \geq 0. \end{cases}$$

(b) $P(X > 50)$ & $P(X = 50)$

$$P(X > 50) = 1 - F(50) = 1 - \left(1 - \frac{1}{2} e^{-(50/50)^2}\right)$$

$$= 1 - 1 + \frac{1}{2} e^{-1} = \frac{1}{2e}$$

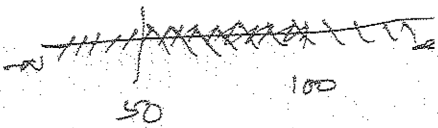
$\therefore \boxed{P(X > 50) = 1/2e}$

$P(X = 50) = 0$ ($\because$ 'X' is continuous)

(iii) $\underline{P(X < 100 \,/\, X > 50)}$

$$P(X < 100 / X > 50) = \frac{P(X < 100 \cap X > 50)}{P(X > 50)}$$

$$= \frac{P(50 < x < 100)}{P(X > 50)} \quad \text{--- (1)}$$



Now $P(50 < X < 100) = F(100) - F(50)$

$$= \left(1 - \frac{1}{2} e^{-(100/50)^2}\right) - \left(1 - \frac{1}{2} e^{-(50/50)^2}\right)$$

$$= 1 - \frac{1}{2} e^{-4} - 1 + \frac{1}{2} e^{-1}$$

$$= \frac{1}{2}\left(e^{-1} - e^{-4}\right).$$

$$P(X > 50) = \frac{1}{2} e^{-1}$$

$$\therefore (1) \Rightarrow P(X < 100 / X > 50) = \frac{e^{-1} - e^{-4}}{e^{-1}}$$

$$= 1 - e^{-3}.$$

—— x —— x —— x ——

(03) The sales of a convenience store on a randomly selected day are X thousand dollars, where X is a random variable with a distribution function of the following form :
$$F(x) = \begin{cases} 0 & , x < 0 \\ x^2/2 & , 0 \le x < 1 \\ K(4x - x^2) & , 1 \le x < 2 \\ 1 & , x \ge 2. \end{cases}$$

Suppose this convenience store's total sales on any day are less than \$2000.

(1) Find the value of K.

(2) Let A and B be the events that tomorrow the store's total sales are between 500 and 1500 dollars, and over 1000 dollars respectively. Find P(A) & P(B).

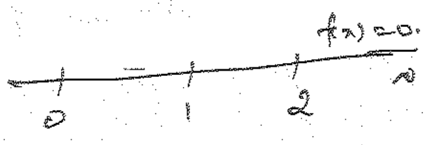
(3) Are A and B independent events?

Soln:

(i) Value of 'K'

WKT $f(x) = \frac{d}{dx} F(x)$ is a p.d.f.

Now $f(x) = \begin{cases} x & , 0 \le x < 1 \\ K(4-2x) & , 1 \le x < 2 \\ 0 & , \text{otherwise}. \end{cases}$



$\therefore \int_0^2 f(x)\,dx = 1$

$\Rightarrow \int_0^1 x\,dx + K\int_1^2 (4-2x)\,dx = 1.$

$\Rightarrow \left(\frac{x^2}{2}\right)_0^1 + K\left(4x - x^2\right)_1^2 = 1$

$\Rightarrow \frac{1}{2} + K\{(8-4) - (4-1)\} = 1$

$$\Rightarrow \frac{1}{2} + K = 1$$

$$\Rightarrow \boxed{K = 1/2}$$

$$\therefore f(x) = \begin{cases} x, & 0 \le x < 1 \\ \frac{1}{2}(4-2x), & 1 \le x < 2. \end{cases}$$

$$\text{Also } F(x) = \begin{cases} 0, & x < 0 \\ x^2/2, & 0 \le x < 1 \\ \frac{1}{2}(4x - x^2), & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}.$$

(ii) $\underline{P(A) \;\&\; P(B)}$

$A \longrightarrow$ between 500 & 1500 dollars

(ie) $0.5 < X < 1.5$ ($\because X$ is in terms of thousands)

(ie) $1/2 < x < 3/2$.

$B \longrightarrow$ over 1000 dollars

(ie) $X > 1$.

Now $P(A) = P(1/2 < x < 3/2) = F(3/2) - F(1/2)$

$$= \frac{1}{2}\left(4\left(3/2\right) - 9/4\right) - \frac{1}{2}\left(1/4\right)$$

$$= \frac{1}{2}\left(\frac{15}{4}\right) - \frac{1}{8} = 14/8 > 1.$$

Given Data is Wrong

Sums for practice

01). If the cumulative distribution function of a R.V X is given by $F(x) = \begin{cases} 1 - 4/x^2, & x \geq 2 \\ 0, & x < 2. \end{cases}$

Find $P(X < 3)$, $P(4 < X < 5)$ and $P(X \geq 3)$.

02) If X is a continuous R.V with p.d.f

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ \frac{3}{2}(x-1)^2, & 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

. Find the cumulative distribution function of X and use its to find $P\left(\frac{3}{2} < X < \frac{5}{2}\right)$.

03) The distribution function of a R.V X is given by $F(x) = 1-(1+x)e^{-x}$, $x \geq 0$. Find the density function, mean and Variance of X.

04) If $dF = Ke^{-|x|}\,dx$, $-\infty < x < \infty$, Show that $K = 1/2$. Also find mean and variance

[Hint: Given $\frac{dF}{dx} = Ke^{-|x|}$ (ie) $f(x) = Ke^{-|x|}$]

———— x ———— * ————

Problems based on properties of Expectation and Variance

Note:

1) Properties of Expectation

* $E(c) = c$, where 'c' is a constant.

* $E(aX) = aE(X)$

(eg) $E(5) = 5$; $E(2X+3) = 2E(X)+3$

$E(5-9X) = 5-9E(X)$.

2) Properties of Variance

* $Var(c) = 0$, where 'c' is a constant

* $Var(aX) = a^2 \, Var X$.

(eg) $Var(5) = 0$

$Var(5-2X) = 0 + 4 \, Var X = 4 \, Var X$

Note:

1) r^{th} moment about the origin $= E(X^r)$

2) Mean $= E(X)$

3) Variance $= E(X^2) - [E(X)]^2$

01) Let X be a R.V with $E(X) = 2$ & $E\{X(X-1)\} = 9$. Find $Var(X/2)$ and $var(2-3X)$.

Soln: Given $\boxed{E(X) = 2}$

$$E\{X(X-1)\} = 9$$

$$\Rightarrow E(X^2 - X) = 9$$

$$\Rightarrow E(X^2) - E(X) = 9$$

$$\Rightarrow E(X^2) - 2 = 9 \Rightarrow \boxed{E(X^2) = 11}$$

$$Var\left(\frac{X}{2}\right) = \frac{1}{4} Var\, X = \frac{1}{4}\left\{E(X^2) - (E(X))^2\right]$$

$$= \frac{1}{4}(11 - 4) = 7/4.$$

$$\therefore \boxed{Var(X/2) = 7/4}$$

$$Var(2-3X) = 0 + 9\, var\, X$$

$$= 9\left\{E(X^2) - (E(X)]^2\right]$$

$$= 9(11-4) = 63.$$

$$\therefore \boxed{Var(2-3X) = 63}$$

02) Let X be a R.V with $E(X) = 10$ and $V(X) = 25$. Find the positive values of 'a' and 'b' such that $Y = aX - b$ has expectation '0' and variance 1.

Soln: Given $E(X) = 10$, $V(X) = 25$.

$\therefore \quad E(y) = 0$

$\Rightarrow \quad E(ax - b) = 0$

$\Rightarrow \quad aE(x) - b = 0$

$\Rightarrow \quad \boxed{10a - b = 0} \quad \text{——} \; (1)$

Also $\quad V(y) = 1$

$\Rightarrow \quad V(ax - b) = 1$

$\Rightarrow \quad a^2 \operatorname{var} x = 1$

$\Rightarrow \quad 25a^2 = 1$

$\therefore \quad a^2 = 1/25 \quad \Rightarrow \quad \boxed{a = 1/5}$

$(1) \Rightarrow \quad 10\left(\frac{1}{5}\right) - b = 0$

$\Rightarrow \quad 2 - b = 0$

$\therefore \quad \boxed{b = 2}$

Sums for Practice

01) A continuous random variable 'x' is distributed over the interval $(0, 1)$ with p.d.f $ax^2 + bx$ where 'a' & 'b' are constants. If the arithmetic mean of x is 0.5, find the values of 'a' & 'b'.

[Hint: * use p.d.f property
* use $E(x)$ property]

02) If x & y are random variables with means $2, 3$ and variance $1, 2$ respectively. Find the mean and variance of $Z = 2x - 3y$.

01) If the probability density function of X is given by $f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$. Show that

$E(X^r) = \dfrac{2}{(r+1)(r+2)}$. Use this result to find

$E\{(2X+1)^2\}$.

<u>Soln</u>: $E(X^r) = \int_0^1 x^r f(x)\,dx = 2\int_0^1 x^r(1-x)\,dx$

$= 2\int_0^1 (x^r - x^{r+1})\,dx$

$= 2\left(\dfrac{x^{r+1}}{r+1} - \dfrac{x^{r+2}}{r+2}\right)_0^1$

$= 2\left(\dfrac{1}{r+1} - \dfrac{1}{r+2}\right) = 2\left(\dfrac{\cancel{r}+2-\cancel{r}-1}{(r+1)(r+2)}\right)$

$= \dfrac{2}{(r+1)(r+2)}$

$\therefore \boxed{E(X^r) = \dfrac{2}{(r+1)(r+2)}}$

$E(2X+1)^2 = E(4X^2+4X+1)$

$= 4E(X^2) + 4E(X) + 1$

$= 4\left(\dfrac{2}{(3)(4)}\right) + 4\left(\dfrac{2}{(2)(3)}\right) + 1$

$= 2/3 + 4/3 + 1 = 3$

$\therefore \boxed{E(2X+1)^2 = 3}$

— x — x —

Q2) A random variable X has the density function given by $f(x) = \frac{1}{K}$, $0 < x < K$. Find the r^{th} moment. Also find its mean and variance.

Soln:

$$E(X^r) = \int_0^K x^r f(x)\,dx = \int_0^K x^r \cdot \frac{1}{K}\,dx$$

$$= \frac{1}{K}\left(\frac{x^{r+1}}{r+1}\right)_0^K = \frac{1}{K} \cdot \frac{K^{r+1}}{r+1}$$

$$\boxed{E(X^r) = \frac{K^r}{r+1}}$$

$$\boxed{\text{Mean} = E(X) = K/2}$$

$$E(X^2) = K^2/3.$$

$$\text{var}(X) = E(X^2) - \{E(X)\}^2 = K^2/3 - K^2/4 = K^2/12$$

$$\therefore \boxed{\text{var}(X) = K^2/12}$$

Moment Generating Function.

The Moment Generating function of a random variable 'x' is given by

$$M_X(t) = E(e^{tX}) = \sum e^{tx} p(X=x)$$

$$= \int_a^b e^{tx} f(x)\, dx.$$

—— x —— x —— x ——

Note: $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\cdots$

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01)* Prove that the r^{th} moment of the R.V 'x' about origin is the co-efficient of $\frac{t^r}{r!}$ in Moment Generating function.

(or)

Prove that $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$.

Soln: $M_X(t) = E(e^{tX}) = E\left\{1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \cdots + \frac{(tX)^r}{r!} + \cdots\right\}$

$$= 1 + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots$$

$$= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

$\therefore$ μ_1' = Co-efft of $\frac{t}{1!}$ in $M_X(t)$

μ_2' = Co-efft of $\frac{t^2}{2!}$ in $M_X(t)$

μ_r' = Co-efft of $\frac{t^r}{r!}$ in $M_X(t)$

Q2) Find μ_1' $(E(x))$ & μ_2' $(E(x^2))$ from $M_x(t)$.

Soln: WKT $M_x(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots$

Diff. w.r.to 't'.

$$M_x'(t) = \mu_1' + \frac{2t}{2!}\mu_2' + \frac{3t^2}{3!}\mu_3' + \cdots \quad \text{(A)}$$

Put $t=0$

$$\boxed{\{M_x'(t)\}_{t=0} = \mu_1'}$$

Diff. (A) w.r.to 't'

$$M_x''(t) = \mu_2' + \frac{6t}{6!}\mu_3' + \cdots$$

Put $t=0$

$$\boxed{\{M_x''(t)\}_{t=0} = \mu_2'}$$

$$\therefore \quad \mu_1' = E(x) = \{M_x'(t)\}_{t=0}$$

$$\mu_2' = E(x^2) = \{M_x''(t)\}_{t=0}$$

Q3) Find the m.g.f of the random variable whose moments are $\mu_r' = (r+1)!\, 2^r$.

Soln: WKT $M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$.

$$M_x(t) = \sum_{r=0}^{\infty} (r+1)!\, 2^r \frac{t^r}{r!} = \sum_{r=0}^{\infty} \frac{(r+1)!}{r!} (2t)^r$$

$$= \sum_{r=0}^{\infty} \frac{(r+1)\cancel{r!}}{\cancel{r!}} (2t)^r = \sum_{r=0}^{\infty} (r+1)(2t)^r$$

$$= (1)(1) + (2)(2t) + 3(2t)^2 + (4)(2t)^3 + \cdots$$

$$= 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \cdots$$

$$(2t = x)$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= (1-x)^{-2} = (1-2t)^{-2}$$

$$\boxed{M_x(t) = \frac{1}{(1-2t)^2}}$$

x ——— x ——— x

Q4)* If the moments of a random variable 'X' are defined by $E(X^r) = 0.6$, $r = 1, 2, 3, \ldots$

Show that $P(X=0) = 0.4$, $P(X=1) = 0.6$, $P(X \geq 2) = 0$

Soln: WKT $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$

Also $M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} P(X=x)$

$$\therefore \sum_{x=0}^{\infty} e^{tx} P(X=x) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

$$P(X=0) + e^t P(X=1) + e^{2t} P(X=2) + \cdots = 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!} (0.6)$$

$$P(X=0) + e^{t}P(X=1) + e^{2t}P(X=2) + \cdots$$

$$= 1 + (0.6)\left\{\frac{t}{\lfloor 1} + \frac{t^2}{\lfloor 2} + \frac{t^3}{\lfloor 3} + \cdots\right\}$$

$$= 1 + (0.6)\left\{1 + \frac{t}{\lfloor 1} + \frac{t^2}{\lfloor 2} + \frac{t^3}{\lfloor 3} + \cdots - 1\right\}$$

$$= 1 + 0.6\left(e^{t} - 1\right)$$

$$= 1 + 0.6e^{t} - 0.6$$

$$= 0.4 + 0.6e^{t}.$$

Comparing the co-effts, (of Constant, e^{t}, e^{2t}, etc..)

$P(X=0) = 0.4$, $P(X=1) = 0.6$

$P(X=2) = 0$, $P(X=3) = 0$ & so on.

---

05) Find the m.g.f of the random variable with the Probability law $P(X=x) = q^{x-1}p$, $x = 1, 2, 3, \ldots$ Also find the mean and variance. $(p+q=1)$

Soln: $M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} P(X=x)$

$$= \sum_{x=1}^{\infty} (e^{t})^{x} \cdot q^{x-1} p = \sum_{x=1}^{\infty} (e^{t})^{x} q^{x} \cdot \frac{p}{q}$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^{t})^{x}$$

$$= \frac{p}{q}\left\{ qe^t + (qe^t)^2 + (qe^t)^3 + \cdots \right\}$$

$$= \frac{p}{\cancel{q}} \cdot \cancel{q}e^t \left\{ 1 + qe^t + (qe^t)^2 + \cdots \right\}$$

$(x = qe^t)$

$$= \frac{pe^t}{1}\left\{ 1 + x + x^2 + \cdots \right\}$$

$$= pe^t(1-x)^{-1} = pe^t(1-qe^t)^{-1}$$

$$\therefore \boxed{M_x(t) = \frac{pe^t}{1-qe^t}}$$

<u>Mean & Variance</u>

$$E(x) = \left\{M_x'(t)\right\}_{t=0} \quad \& \quad E(x^2) = \left\{M_x''(t)\right\}_{t=0}.$$

$$M_x'(t) = \frac{(1-qe^t)(pe^t) - (pe^t)(-qe^t)}{(1-qe^t)^2}$$

$$= \frac{pe^t - \cancel{pqe^{2t}} + \cancel{pqe^{2t}}}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2}$$

$$\therefore \quad M_x'(t) = \frac{pe^t}{(1-qe^t)^2}$$

Now $$E(x) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = 1/p$$

$$\therefore \boxed{E(x) = \frac{1}{p}}$$

$$M_x''(t) = \frac{(1-qe^t)^2(pe^t) - pe^t \cdot 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$E(x^2) = \frac{(1-q)^2(p) - 2p(1-q)(-q)}{(1-q)^4}$$

$$= \frac{p^2 \cdot p - 2p(p)(-q)}{p^4} = \frac{p^3 + 2p^2q}{p^4}$$

$$\therefore \ Var(x) = E(x^2) - \{E(x)\}^2$$

$$= \frac{p^3 + 2p^2q}{p^4} - \frac{1}{p^2} = \frac{p^3 + 2p^2q - p^2}{p^4}$$

$$= \frac{\cancel{p^2}(p + 2q - 1)}{\cancel{p^4}\ p^2} = \frac{p + q + q - 1}{p^2}$$

$$= \frac{\cancel{1} + q - \cancel{1}}{p^2} = q/p^2$$

$$\therefore \ \boxed{var(x) = q/p^2}$$

Q6) The probability function of an infinite discrete distribution is given by $P(X = x) = \frac{1}{2^x}$, $x = 1, 2, \ldots \infty$. Also find the mean and variance of the distribution.

R

Soln:

we know that $M_x(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} \cdot \frac{1}{2^x}$.

$$= \sum_{x=1}^{\infty} (e^t)^x \cdot \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(e^t/2\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots$$

$$= \frac{e^t}{2}\left\{1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right\} \qquad (X = e^t/2)$$

$$= \frac{e^t}{2}\left\{1 + X + X^2 + \cdots\right\}$$

$$= \frac{e^t}{2}(1-X)^{-1} = \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1} = \frac{e^t}{2}\left(\frac{2-e^t}{2}\right)^{-1}$$

$$= \frac{e^t}{2} \cdot \frac{2}{2-e^t} = \frac{e^t}{2-e^t}.$$

$$\therefore \boxed{M_x(t) = \frac{e^t}{2-e^t}}$$

Mean & Variance

$E(X) = \left\{M_x'(t)\right\}_{t=0}$ & $E(X^2) = \left\{M_x''(t)\right\}_{t=0}$.

$$M_x'(t) = \frac{(2-e^t)(e^t) - (e^t)(-e^t)}{(2-e^t)^2} = \frac{2e^t - \cancel{e^{2t}} + \cancel{e^{2t}}}{(2-e^t)^2}$$

$$= \frac{2e^t}{(2-e^t)^2}$$

$\therefore E(X) = \frac{2}{(2-1)^2} = 2 \quad \therefore \boxed{E(X) = 2}$.

$$M_X''(t) = \frac{(2-e^t)^2(2e^t) - (2e^t)2(2-e^t)(-e^t)}{(2-e^t)^4}$$

$$\therefore E(X^2) = \frac{(2-1)^2(2) - (2)(2)(2-1)(-1)}{(2-1)^4}$$

$$= \frac{2+4}{1} = 6.$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2.$$

$$\therefore \boxed{Var(X) = 2}$$

---

07) If 'x' represents the outcome, when a fair die is tossed, find the m.g.f of 'x' & hence find $E(X)$ and $Var(X)$.

Soln: The Prob. Mass fn. is

$X = x$:	1	2	3	4	5	6
$P(X=x)$:	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$M_X(t) = E(e^{t}) = \sum_{x=1} e^{tx} P(X=x)$$

$$= \sum_{x=1}^{6} e^{tx}\left(\frac{1}{6}\right) = \frac{1}{6}\left(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\right)$$

$$M_X'(t) = \frac{1}{6}\left(e^{t} + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}\right)$$

$$M_X''(t) = \frac{1}{6}\left(e^{t} + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}\right)$$

Now $E(X) = \left\{M_X'(t)\right\}_{t=0} = \frac{1}{6}(1+2+3+4+5+6)$

$$= \frac{21}{6}.$$

$E(X^2) = \left\{M_X''(t)\right\}_{t=0} = \frac{1}{6}(1+4+9+16+25+36)$

$$= \frac{91}{6}.$$

$\therefore Var(X) = E(X^2) - [E(X)]^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{35}{12}$

$$\therefore \boxed{Var(X) = \frac{35}{12}}$$

Sums for Practice

01) If a random variable 'X' has the m.g.f $M_X(t) = \frac{3}{3-t}$, find the S.D of X.

[S.D — Standard Deviation $= \sqrt{\text{variance}}$]

——— x ———

Moment Generating function for Continuous Random Variables

01) Let X be a R.V with p.d.f. $f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

Find the moment generating function, mean & variance.

Soln: $M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)\, dx$

$$= \frac{1}{3}\int_0^\infty e^{tx} \cdot e^{-x/3}\, dx = \frac{1}{3}\int_0^\infty e^{-(1/3 - t)x}\, dx.$$

$$= \frac{1}{3}\left[\frac{e^{-(1/3-t)x}}{-(1/3-t)}\right]_0^\infty = \frac{1}{3}\left[0 - \left(\frac{-1}{1/3 - t}\right)\right]$$

$$= \frac{1}{3}\left(\frac{1}{\frac{1-3t}{3}}\right) = \frac{1}{3}\cdot\frac{3}{1-3t} = \frac{1}{1-3t}.$$

$$\therefore \boxed{M_X(t) = \frac{1}{1-3t}}$$

$$M_X'(t) = \frac{\cancel{(1-3t)}(0) - (1)(0-3)}{(1-3t)^2} = \frac{3}{(1-3t)^2}$$

$$M_X''(t) = \frac{(1-3t)^2(0) - 3\cdot(2)(1-3t)(-3)}{(1-3t)^4} = \frac{18(1-3t)}{(1-3t)^4},$$

$$E(X) = \left\{M_X'(t)\right\}_{t=0} = \frac{3}{(1-0)^2} = 3.$$

$$E(X^2) = \left\{M_X''(t)\right\}_{t=0} = \frac{18\,(1)}{(1)^4} = 18.$$

$$Var(X) = E(X^2) - [E(X)]^2 = 18 - 9 = 9.$$

$$\therefore \boxed{Var(X) = 9}$$

Sums for Practice

01) A random variable X has density function given by $f(x) = \frac{1}{K}$, $0 < x < K$. Find the m.g.f, mean & variance.

02) Find the m.g.f of $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$

Standard Distributions

Important Discrete Distributions.

* Binomial Distribution.
* Poisson Distribution.
* Geometric Distribution.
* Negative Binomial Distribution.

Distribution	Binomial	Poisson	Geometric	Negative Binomial
Probability Mass function	$P(X=x) = nC_x p^x q^{n-x}$ $x = 0, 1, 2, \ldots, n$	$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0, 1, 2, \ldots$	$P(X=x) = q^{x-1}p$ $x = 1, 2, 3, \ldots$	$P(X=x) = (x+r-1)C_{r-1} p^r q^x$, $x = 0, 1, 2, \ldots$
M.G.F	$(q + pe^t)^n$	$e^{\lambda(e^t-1)}$	$\frac{pe^t}{1-qe^t}$	$p^r(1-qe^t)^{-r}$
Mean	np	λ.	$1/p$	$\frac{rq}{p}$.
Variance	npq	λ	$\frac{q}{p^2}$	$\frac{rq}{p^2}$
'x' represents. (X → random variable)	no. of successes	no. of successes	no. of trials	x → no. of failures r → no. of successes x+r → no. of trials.

Binomial Distribution

Defn: A random variable 'x' is said to follow Binomial Distribution if its probability mass function is given by

$$P(X=x) = nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2, \dots, n$$

Note:

$$nC_0 a^n + nC_1 a^{n-1} b + nC_2 a^{n-2} b^2 + \dots + nC_n b^n = (a+b)^n.$$

* Find the Moment Generating function, Mean and Variance of a Binomial Distribution.

Soln: $P(X=x) = nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2, \dots, n$

$$M_X(t) = E\{e^{tX}\} = \sum_{x=0}^{n} e^{tx} P(X=x)$$

$$= \sum_{x=0}^{n} (e^t)^x\, nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$$

$$= nC_0 q^n + nC_1 (pe^t) q^{n-1} + nC_2 (pe^t)^2 q^{n-2} + \dots + nC_n (pe^t)^n$$

$$= (q + pe^t)^n$$

$$\therefore \boxed{M_X(t) = (q + pe^t)^n}$$

$$M_x'(t) = n(q+pe^t)^{n-1}(pe^t)$$

$$= npe^t(q+pe^t)^{n-1}.$$

$$E(x) = \{M_x'(t)\}_{t=0} = np(1)(q+p)^{n-1} = np(1)^{n-1}$$

$$= np$$

$$\therefore \boxed{E(x) = np}$$

$$M_x''(t) = np\left\{e^t(n-1)(q+pe^t)^{n-2}(pe^t) + (q+pe^t)^{n-1}e^t\right\}$$

$$E(x^2) = \{M_x''(t)\}_{t=0} = np\left\{(1)(n-1)(\underbrace{q+p}_{=1})^{n-2}(p) + (q+p)^{n-1}(1)\right\}$$

$$= np\left\{(n-1)p + 1\right\} = n^2p^2 - np^2 + np.$$

$$Var(x) = E(x^2) - \{E(x)\}^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p) = npq$$

$$\therefore \boxed{var(x) = npq} \qquad \therefore \boxed{S.D = \sqrt{npq}}$$

——— × ——— × ———

Poisson Distribution

Defn: A random variable 'X' is said to follow Poisson Distribution if its probability mass function is given by

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, 2, \dots$$

* Find the m.g.f, mean and variance of Poisson Distribution.

Soln: $M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \cdot P(X=x)$

$$= \sum_{x=0}^{\infty} (e^t)^x \cdot \frac{e^{-\lambda}\lambda^x}{\lfloor x} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{\lfloor x}$$

$$= e^{-\lambda}\left\{1 + \frac{\lambda e^t}{\lfloor 1} + \frac{(\lambda e^t)^2}{\lfloor 2} + \frac{(\lambda e^t)^3}{\lfloor 3} + \dots\right\}$$

$$= e^{-\lambda}\left\{1 + \frac{X}{\lfloor 1} + \frac{X^2}{\lfloor 2} + \frac{X^3}{\lfloor 3} + \dots\right\} \qquad X = \lambda e^t$$

$$= e^{-\lambda} \cdot e^{X} = e^{-\lambda} \cdot e^{\lambda e^t}$$

$$\therefore \boxed{M_X(t) = e^{-\lambda} \cdot e^{\lambda e^t}}$$

$$M_X'(t) = e^{-\lambda} \cdot e^{\lambda e^t}(\lambda e^t) = \lambda e^{-\lambda} e^t \cdot e^{\lambda e^t}$$

$$\therefore E(X) = \left\{M_X'(t)\right\}_{t=0} = \lambda e^{-\lambda}(1)e^{\lambda} = \lambda$$

$$\therefore \boxed{E(X) = \lambda}$$

$$M_X''(t) = \lambda e^{-\lambda}\left\{ e^t \cdot e^{\lambda e^t} \cdot \lambda e^t + e^{\lambda e^t} \cdot e^t \right\}$$

$$E(X^2) = \left\{ M_X''(t) \right\}_{t=0} = \lambda e^{-\lambda}\left\{ \lambda e^{\lambda} + e^{\lambda} \right\}$$

$$= \lambda^2 + \lambda .$$

$$\therefore Var(X) = E(X^2) - \{E(X)\}^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\therefore \boxed{Var(X) = \lambda} \qquad \boxed{S.D = \sqrt{\lambda}}$$

Geometric Distribution

Defn: A random variable 'X' is said to follow Geometric Distribution if its probability mass function is given by

$$P(X = x) = q^{x-1} p , \quad x = 1, 2, 3, \ldots$$

* Find the m.g.f, mean and variance of Geometric Distribution.

Soln: $M_X(t) = E\left[e^{tX}\right] = \sum_{x=1}^{\infty} e^{tx} P(X = x)$

$$= \sum_{x=1}^{\infty} \left(e^t\right)^x q^{x-1} \cdot p = \sum_{x=1}^{\infty} \left(e^t\right)^x \frac{q^x}{q} \cdot p$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} \left(qe^t\right)^x = \frac{p}{q}\left\{ qe^t + \left(qe^t\right)^2 + \left(qe^t\right)^3 + \cdots \right\}$$

$$= \frac{p}{\cancel{q}} \cdot \cancel{q} e^t \{1 + qe^t + (qe^t)^2 + (qe^t)^3 + \cdots\}$$

$$= pe^t \{1 + x + x^2 + x^3 + \cdots\} \qquad x = qe^t$$

$$= pe^t (1-x)^{-1} = pe^t (1-qe^t)^{-1} = \frac{pe^t}{1-qe^t}$$

$$\therefore \boxed{M_x(t) = \frac{pe^t}{1-qe^t}}$$

$$M_x'(t) = \frac{(1-qe^t)(pe^t) - (pe^t)(-qe^t)}{(1-qe^t)^2}$$

$$= \frac{pe^t - \cancel{pqe^{2t}} + \cancel{pqe^{2t}}}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2}$$

$$E(x) = \{M_x'(t)\}_{t=0} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\therefore \boxed{E(x) = \frac{1}{p}}$$

$$M_x''(t) = \frac{(1-qe^t)^2 pe^t - pe^t\, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$E(x^2) = \{M_x''(t)\}_{t=0} = \frac{(1-q)^2 p - 2p(1-q)(-q)}{(1-q)^4}$$

$$= \frac{p^2 \cdot p - 2p(p)(-q)}{p^4} = \frac{p^3 + 2p^2 q}{p^4} = \frac{p+2q}{p^2}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{p+2q}{p^2} - \frac{1}{p^2}$$

$$= \frac{p+2q-1}{p^2} = \frac{p+q+q-1}{p^2} = \frac{1+q-1}{p^2} = \frac{q}{p^2}$$

$$\therefore \boxed{Var(X) = q/p^2}$$

Negative Binomial Distribution

Defn: A random variable X is said to follow a negative binomial distribution if its probability mass function is given by

$$P(X=x) = (x+r-1)C_{r-1}\, p^r q^x, \quad x = 0, 1, 2, \ldots$$

* Find the m.g.f, mean and variance of a Negative Binomial Distribution.

Soln: $M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \cdot P(X=x)$

$$= \sum_{x=0}^{\infty} (e^t)^x \cdot (x+r-1)C_{r-1} \cdot p^r q^x$$

$$= p^r \sum_{x=0}^{\infty} (x+r-1)C_{r-1} (qe^t)^x$$

$$= p^r \left\{ (r-1)C_{r-1} + rC_{r-1}\, qe^t + (r+1)C_{r-1} (qe^t)^2 + \cdots \right]$$

$$= p^r \left\{ 1 + rC_1\, qe^t + (r+1)C_2 (qe^t)^2 + \cdots \right\}$$

$x = qe^t$

$$= p^r \left\{ 1 + rx + \frac{(r+1)(r)}{2!} x^2 + \cdots \right\}$$

$$= p^r (1-x)^{-r} = p^r (1-qe^t)^{-r}$$

$$\therefore \boxed{M_X(t) = p^r (1-qe^t)^{-r}}$$

$$M_X'(t) = p^r (-r)(1-qe^t)^{-r-1}(-qe^t)$$

$$= rqe^t \cdot p^r (1-qe^t)^{-r-1}$$

$$E(X) = \left\{ M_X'(t) \right\}_{t=0} = rq \cdot p^r (1-q)^{-r-1}$$

$$= rq \cdot p^r \cdot p^{-r-1} = rq \cdot p^{-1} = \frac{rq}{p}.$$

$$\therefore \boxed{E(X) = \frac{rq}{p}}$$

$$M_X''(t) = rqp^r \left\{ e^t (-r-1)(1-qe^t)^{-r-2}(-qe^t) + (1-qe^t)^{-r-1}(e^t) \right\}$$

$$E(X^2) = \left\{ M_X''(t) \right\}_{t=0} = rqp^r \left\{ (-r-1)(1-q)^{-r-2}(-q) + (1-q)^{-r-1} \right\}$$

$$= rqp^r \left\{ q(r+1)p^{-r-2} + p^{-r-1} \right\}$$

$$= r(r+1)q^2 p^{-2} + rqp^{-1} = \frac{r(r+1)q^2}{p^2} + \frac{rq}{p}.$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{r(r+1)q^2}{p^2} + \frac{rq}{p} - \frac{r^2q^2}{p^2}$$

$$= \frac{\cancel{r^2q^2} + rq^2 + rpq - \cancel{r^2q^2}}{p^2} = \frac{rq(q+p)}{p^2}$$

$$= \frac{rq}{p^2} \quad \therefore \boxed{Var(X) = \frac{rq}{p^2}}$$

Points to remember:

I. Binomial Distribution:

This distribution is used when we need 'x' successes out of 'n' trials.

Eg Getting 5 heads on tossing a coin 12 times.

Notations: $n \rightarrow$ no. of trials

$p \rightarrow$ probability of success

$q \rightarrow$ probability of failure

$x \rightarrow$ no. of successes (out of 'n' trials)

$\therefore$ Possible values of $x = 0, 1, 2, \ldots n$.

II. Poisson Distribution:

This distribution is as similar as Binomial distribution where 'n' is very large & 'p' is small.

Geometric Distribution:

This distribution is used when an experiment is repeated till we get the first success.

Notations: $x \rightarrow$ no. of attempts to make first success.

$p \rightarrow$ probability of success

$q \rightarrow$ probability of failure.

Negative Binomial Distribution:

This distribution is used when an experiment is repeated till we get 'r' successes out of $x+r$ trials.

(ie) till we get '4' (r) successes out of $10 \, (\overset{x+r}{6+4})$ trials.

Notation:

$x \rightarrow$ no. of failures

$r \rightarrow$ no. of successes

$x+r \rightarrow$ no. of trials

$p \rightarrow$ probability of success

$q \rightarrow$ probability of failure

Problems based on Binomial Distribution

Q1) The mean and variance of a binomial variate are 8 and 6. Find $P(X \geq 2)$

Soln: Given: X follows binomial distribution

Mean $= 8$, Variance $= 6$.

To find: $P(X \geq 2)$

Now $np = 8$, $npq = 6$ $\Rightarrow \frac{\cancel{n}\cancel{p}q}{\cancel{n}\cancel{p}} = \frac{6}{8}$

$\Rightarrow \boxed{q = 3/4}$

$p = 1 - q = 1 - 3/4 = 1/4 \Rightarrow \boxed{p = 1/4}$

$\because np = 8 \Rightarrow n\left(\frac{1}{4}\right) = 8 \Rightarrow \boxed{n = 32}$

$\therefore$ Binomial Distribution is given by

$$P(X = x) = 32C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{32-x}, \quad x = 0, 1, 2, \ldots 32$$

$$\left(\because P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \ldots n\right)$$

Now $P(X \geq 2) = 1 - P(X < 2) = 1 - (P(X=0) + P(X=1))$

$$= 1 - \left[32C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{32} + 32C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{31}\right]$$

$$= 1 - \left(\frac{3}{4}\right)^{32} - 32\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{31}$$

$= 0.9988$

02) With the usual notation find 'p' for a binomial random variate 'x' if $n = 6$ and if $9P(X=4) = P(X=2)$

Soln: WKT for binomial random variable 'x'

$$P(X=x) = nC_x \, p^x q^{n-x}$$

Given, $9P(X=4) = P(X=2)$

$$\Rightarrow 9\{6C_4 \, p^4 q^{6-4}\} = 6C_2 \, p^2 q^{6-2}$$

$$\Rightarrow 9\{\cancel{6C_2} \, p^4 q^2\} = \cancel{6C_2} \, p^2 q^4$$

$$\boxed{nC_r = nC_{n-r}}$$

$$\Rightarrow 9p^2 = q^2$$

$$\Rightarrow 9p^2 = (1-p)^2 = 1 + p^2 - 2p$$

$$\Rightarrow 8p^2 + 2p - 1 = 0$$

$$\Rightarrow p = \frac{1}{4}, \ -\frac{1}{2}$$

$a = 8$, $b = 2$, $c = -1$

$$p = \frac{-2 \pm \sqrt{4 + 32}}{16} = \frac{-2 \pm 6}{16} = \frac{1}{4}, \ -\frac{1}{2}$$

$\therefore p \neq -\frac{1}{2}$, $\boxed{p = \frac{1}{4}}$

---

03) A pair of dice is thrown 4 times. If getting a doublet is considered as a success, find the probability of 2 successes.

Soln: No. of trials $= 4$. $\therefore \boxed{n = 4.}$

Success = getting a doublet $\{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

Failure = not getting a doublet

$\therefore p = \frac{6}{36} = \frac{1}{6} \Rightarrow q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

$\therefore \boxed{p = \frac{1}{6}, \; q = \frac{5}{6}}$

In Binomial Distribution: $x \rightarrow$ no. of successes.

We need P(2 successes) i.e $P(X = 2)$

WKT $P(X = x) = 4C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{4-x}, \; x = 0, 1, 2, 3, 4.$

$\therefore P(X = 2) = 4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 = 6\left(\frac{1}{36}\right)\left(\frac{25}{36}\right) = 0.1157$

$\therefore \boxed{P(X = 2) = 0.1157}$

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04) Find the probability that in tossing a fair coin 5 times, there will appear (i) 3 heads (ii) 3 tails & 2 heads (iii) atleast 1 head (iv) not more than 1 tail.

Soln: No. of trials = 5 ie $\boxed{n = 5}$

Let the success be head. $\therefore p = \frac{1}{2} \Rightarrow q = 1/2$

$\therefore P(X = x) = 5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}, \; x = 0, 1, 2, 3, 4, 5$

(i) P(3 heads)

$\because$ Head is assumed as a success, we need $P(X = 3)$.

Now $P(X = 3) = 5C_3 (1/2)^3 (1/2)^2 = 10\left(\frac{1}{8}\right)\left(\frac{1}{4}\right) = \frac{5}{16}$

$\therefore \boxed{P(\text{getting 3 heads}) = \frac{5}{16}}$

(ii) P(3 tails and 2 heads)

$\because$ we need the Probability for success (head) in Binomial Distribution, getting 3 tails and 2 heads is similar to getting 2 heads. (ie) $P(X=2)$

$\therefore P(X=2) = 5C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = 10\left(\frac{1}{4}\right)\left(\frac{1}{8}\right) = \frac{5}{16}$.

$\therefore \boxed{P(X=2) = \frac{5}{16}}$ ie

(iii) P(atleast 1 head)

$P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$

$= 1 - 5C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = 1 - \frac{1}{32} = \frac{31}{32}$

$\therefore \boxed{P(X \geq 1) = \frac{31}{32}}$

(iv) P(not more than 1 tail)

Note: We need the problem in terms of success (ie) heads.

Not more than 1 tail $\Rightarrow$ 0 tail (or) 1 tail.

(ie) 5 heads (or) 4 heads.

$\therefore$ P(not more than 1 tail) $= P(X=5) + P(X=4)$

$= 5C_5\left(\frac{1}{2}\right)^5(1/2)^0 + 5C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1$

$= \left(1/32\right) + 5\left(1/32\right) = 6/32 = 3/16$

$\therefore \boxed{P(\text{not more than 1 tail}) = 3/16}$

——— x ——— x ———

05) If 10% of the screws produced by an automatic machine are defective, find the probability that out of 20 screws selected at random, there are (i) exactly 2 defective (ii) atmost 3 defective (iii) atleast 2 defective (iv) between 1 and 3 defective (inclusive) (v) exactly 5 non-defective.

Soln: Here $n = 20$.

Success $\longrightarrow$ defective screws.

$$p = 10\% = \frac{10}{100} = \frac{1}{10} \Rightarrow q = 1-p = 1-\frac{1}{10} = \frac{9}{10}$$

$$\therefore p = \frac{1}{10},\ q = \frac{9}{10}.$$

$$\therefore P(X = x) = 20C_x \left(\frac{1}{10}\right)^x \left(\frac{9}{10}\right)^{20-x},\quad x = 0, 1, 2, \ldots 20.$$

(i) $P(\text{exactly 2 defective}) = P(X = 2)$

$$= 20C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^{18} = 190 \times \frac{9^{18}}{10^{20}}.$$

(ii) $P(\text{atmost 3 defective}) = P(X \le 3)$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 20C_0\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{20} + 20C_1\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^{19} + 20C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{18}$$

$$+ 20C_3\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^{17}$$

$$= \left(\frac{9}{10}\right)^{20} + 20\left(\frac{9^{19}}{10^{20}}\right) + \frac{20\times 19}{1\times 2}\left(\frac{9^{18}}{10^{20}}\right) + \frac{20\times 19\times 18}{1\times 2\times 3}\left(\frac{9^{17}}{10^{20}}\right)$$

$$= \frac{9^{17}}{10^{20}} \times 5199.$$

(iii) P(atleast 2 defectives) $= P(X \geq 2)$

$$= 1 - P(X < 2) = 1 - \{P(X=0) + P(X=1)\}$$

$$= 1 - 20C_0\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{20} - 20C_1\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^{19}$$

$$= 1 - \left(\frac{9}{10}\right)^{20} - 20\left(\frac{9^{19}}{10^{20}}\right)$$

$$= 1 - \frac{9^{19}}{10^{20}}(29)$$

(iv) P(between 1 & 3 defective) $= P(1 \leq X \leq 3)$

$$= P(X=1) + P(X=2) + P(X=3)$$

$$= 20C_1\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^{19} + 20C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{18} + 20C_3\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^{17}$$

$$= 20 \times \left(\frac{9^{19}}{10^{20}}\right) + \frac{20 \times 19}{1 \times 2}\cdot\left(\frac{9^{18}}{10^{20}}\right) + \frac{20 \times 19 \times 18}{1 \times 2 \times 3}\left(\frac{9^{17}}{10^{20}}\right)$$

$$= \frac{9^{17}}{10^{20}}(4470)$$

(v) P(exactly 5 non-defective)

$$= P(\text{exactly } 15 \text{ defective})$$

$$= 20C_{15}\left(\frac{1}{10}\right)^{15}\left(\frac{9}{10}\right)^{5}$$

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06) Assume that half of the population is vegetarian so that the chance of an individual being a vegetarian is $\frac{1}{2}$. Assuming that 100 investigators take samples of 10 invidual each to see whether they are vegetaria how many investigators would you expect to report that three people or less were vegetarians?

Soln:

Success — being a vegetarian.

$p = \frac{1}{2}, \quad q = 1 - \frac{1}{2} = \frac{1}{2} \qquad \therefore \boxed{p = \frac{1}{2}, \; q = \frac{1}{2}}$

n = no. of individuals = 10

$\therefore P(X = x) = 10C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x}, \quad x = 0, 1, 2, \ldots, 10.$

P(3 people or less were vegetarians) $= P(X \leq 3)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 10C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10} + 10C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^9 + 10C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^8$

$+ 10C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^7$

$= \left(\frac{1}{2}\right)^{10} \left\{ 10C_0 + 10C_1 + 10C_2 + 10C_3 \right\} = \frac{176}{1024}.$

$\therefore$ No. of investigators reported 3 or less vegeterians

$= 100 \times \frac{176}{1024}$

$\simeq 17.$

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07) An irregular six faced die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many times in 10,000 sets of 10 throws each, would you expect it to give no even number.

Soln:

$n = 10$

Success $\rightarrow$ getting even number.

Given: P(getting 5 even numbers) = 2P(getting 4 even nos)

$\Rightarrow P(X=5) = 2P(X=4)$

$\Rightarrow {}^{10}C_5\, p^5 q^5 = 2\left({}^{10}C_4\, p^4 q^6\right)$

$\Rightarrow \dfrac{\cancel{10\times 9\times 8\times 7}\times \cancel{6}^{3}}{\cancel{1\times 2\times 3}\times 4\times 5}\cdot p^5 q^5 = \cancel{2}\times \dfrac{\cancel{10\times 9\times 8\times 7}}{\cancel{1\times 2\times 3\times 4}}\, p^4 q^6$

$\Rightarrow \dfrac{3p}{5} = q \quad \Rightarrow 3p = 5(1-p) \quad \Rightarrow 3p = 5 - 5p$

$\Rightarrow 8p = 5 \quad \Rightarrow \boxed{p = 5/8}$

$\therefore q = 1 - p = 1 - \dfrac{5}{8} = \dfrac{3}{8} \qquad \therefore \boxed{q = \dfrac{3}{8}}$

$\therefore P(X=x) = {}^{10}C_x \left(\dfrac{5}{8}\right)^x \left(\dfrac{3}{8}\right)^{10-x}, \quad x = 0, 1, 2, \ldots 10$

Now P(getting no even no.) $= P(X=0) = {}^{10}C_0 \left(\dfrac{5}{8}\right)^0 \left(\dfrac{3}{8}\right)^{10}$

$= \left(\dfrac{3}{8}\right)^{10}.$

$\therefore$ In 10,000 sets, no. of times getting no even number $= 10000 \times \dfrac{3^{10}}{8^{10}}$

$\simeq 1.$

Sums for Practice

01) Check whether the following data follow a Binomial distribution or not. Mean = 3, variance = 4.

02) Let 'x' follows a binomial distribution. Suppose $P(X=0) = 1 - P(X=1)$. If $E(X) = 3 \, \text{Var} \, X$, find $P(X=0)$.

03) The probability of a bomb hitting a target is $\frac{1}{5}$. Two bombs are enough to destroy a bridge. If six bombs are aimed at the bridge, find the probability that the bridge is destroyed?

04) Five fair coins are flipped. If the outcomes are assumed independent find the probability mass function of the number of heads obtained.

05) In a binomial distribution consisting of 5 independent trials, probability of 1 & 2 successes are 0.4096 and 0.2048 respectively. Find the parameter 'p' of the distribution.

06) 6 dice are thrown 729 times. How many times do you expect atleast three dice to show a five or a six?

07) The moment generating fn. of a binomial distribution is given by $\left(\frac{1}{4} + \frac{3}{4}e^t\right)^7$. Find $P(X>7)$.

Problems on Poisson Distribution

01) If X and Y are independent Poisson variate such that $P(X=1) = P(X=2)$ and $P(Y=2) = P(Y=3)$. find the variance of $X - 2Y$.

Soln: WKT, $P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$, $x = 0, 1, 2, \ldots$

$\because P(X=1) = P(X=2) \Rightarrow \frac{\cancel{e^{-\lambda}}\lambda^{1}}{1!} = \frac{\cancel{e^{-\lambda}}\lambda^{2}}{2!}$

$\Rightarrow \lambda = \frac{\lambda^2}{2} \Rightarrow \boxed{\lambda = 2}$

Also, $P(Y = y) = \frac{e^{-\mu}\mu^{y}}{y!}$, $y = 0, 1, 2, \ldots$

$\because P(Y=2) = P(Y=3) \Rightarrow \frac{\cancel{e^{-\mu}}\mu^{2}}{2!} = \frac{\cancel{e^{-\mu}}\mu^{3}}{3!}$

$\Rightarrow \frac{\cancel{\mu^2}}{2} = \frac{\mu^{\cancel{3}}}{6} \Rightarrow \frac{1}{2} = \frac{\mu}{6} \Rightarrow \boxed{\mu = 3}$

$Var(X - 2Y) = Var X + 4 Var Y$

$= 2 + 4(3) = 14$

$\therefore \boxed{Var(X - 2Y) = 14}$

02) One percent of jobs arriving at a computer system need to wait until weekends for scheduling, owing to core-size limitations. Find the probability that among a sample of 200 jobs there are no jobs that have to wait until weekends.

Soln: $x \rightarrow$ jobs need to wait.
(success)

$p = 1\% = \frac{1}{100}$, $n = 200$

$\therefore \lambda = np = 200\left(\frac{1}{100}\right) = 2 \qquad \therefore \boxed{\lambda = 2}$

$$\therefore P(X = x) = \frac{e^{-2}(2)^x}{x!}$$

$$P(\text{no jobs have to wait}) = P(X=0) = \frac{e^{-2}2^0}{0!}$$

$$= e^{-2} = 0.1353.$$

$$\therefore \boxed{P(X=0) = 0.1353}$$

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03) A manufacture of pins knows that 2% of his products are defective. If he sells pins in boxes of 100 and guarantees that not more than 4 pins will be defective what is the probability that a box will fail to meet the guaranteed quality?

Soln: $x \rightarrow$ defective products.
(success)

$p = 2\% = \frac{2}{100}$, $n = 100$ $\qquad \therefore \lambda = np = \frac{2}{100} \times 100 = 2$

$\therefore \boxed{\lambda = 2}$

$$\therefore P(X = x) = \frac{e^{-2} 2^{x}}{x!}.$$

P(a box will fail to meet the guarantee)

= P(more than 4 pins will be defective)

$= P(X > 4) = 1 - P(X \leq 4)$

$= 1 - \{P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)\}$

$$= 1 - \left\{ \frac{e^{-2} 2^{0}}{0!} + \frac{e^{-2} 2^{1}}{1!} + \frac{e^{-2} 2^{2}}{2!} + \frac{e^{-2} 2^{3}}{3!} + \frac{e^{-2} 2^{4}}{4!} \right\}$$

$= 1 - 0.9473$

$\cong 0.0527$

Sams for Practice

01) An insurance company found that only 0.01% of the population is involved in a certain type of accident each year. If its 1000 policy holders were randomly selected from the population, what is the probability that not more than two of its clients are involved in such an accident next year?

02) A manufacturer knows that the condensors he makes contain on the average 1% of defectives. He packs them in boxes of 100. What is the probability that a box picked at random will contain 3 or more faulty condensers.

03) The m.g.f of a random variable X is $e^{3(e^{t}-1)}$. Find $P(X=1)$.

Problems on Geometric Distribution

01) If the probability that a target is destroyed on any one shot is 0.5, what is the probability that it would be destroyed on 6^{th} attempt.

Soln: Success $\rightarrow$ target is destroyed

$x \rightarrow$ no. of attempts to make a first success.

Given $p = 0.5$, $q = 0.5$

Now $P(X = x) = q^{x-1} \cdot p$.

$\therefore$ P(target destroyed on 6^{th} attempt)

$= P(X = 6) = q^5 \cdot p = \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right) = \frac{1}{32}$.

$\therefore \boxed{P(X = 6) = \frac{1}{32}}$

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02) A die is cast until 6 appears. What is the probability that the die must be cast more than 5 times.

Soln: Success $\longrightarrow$ getting '6'

$\therefore p = \frac{1}{6} \Rightarrow q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$.

$x \rightarrow$ no. of attempts to get '6' (first success)

$$P(X > 5) = \sum_{5}^{\infty} q^{x-1} \cdot p = \sum_{x=5}^{\infty} q^{x} \cdot \frac{p}{q}$$

$$= \frac{p}{q} \left\{ q^5 + q^6 + q^7 + q^8 + \cdots \right\}$$

$$P(X > 5) = \frac{p}{q} \cdot q^5 (1 + q + q^2 + \cdots)$$

$$= pq^4 (1-q)^{-1} = \frac{pq^4}{1-q} = \frac{pq^4}{p} = q^4$$

$$= \left(\frac{5}{6}\right)^4$$

$$\therefore \boxed{P(X > 5) = \left(\frac{5}{6}\right)^4}$$

Sums for practice:

Q1) If X is a Geometric variate taking values $1, 2, \ldots \infty$, find $P(X$ is odd$)$.

Q2). Let one copy of a magazine out of 10 copies bears a special prize following geometric distribution. Determine its mean and variance.

Q3) If the probability is 0.05 that a certain kind of measuring device will show excessive drift, what is the probability that the sixth of these measuring devices tested will be the first to show excessive drift?

Problems on Negative - Binomial Distribution.

01) An item is produced in large numbers. The machine is known to produce 5% defective. A quality control inspector is examining the items by taking them at random. What is the probability that atleast 4 items are to be examined in order to get 2 defectives.

Soln: Success $\rightarrow$ producing defective items.

Here $p = 5\% = \frac{5}{100} = \frac{1}{20}$ $\quad \boxed{p = \frac{1}{20}}$

$q = 1 - p = 1 - \frac{1}{20} = \frac{19}{20}$ $\quad \boxed{q = \frac{19}{20}}$

Here we need 2 defective (ie) 2 successes.

$\therefore \boxed{r = 2.}$

Also '4' items are to be examined

(ie) no. of trials $= 4$ $\Rightarrow x + r = 4$

$\Rightarrow \boxed{x = 2}$

Here x — no. of failures $= 2$.

$\therefore P(X = x) = (x+r-1)C_{r-1} \cdot p^r q^x, \quad x = 0, 1, 2, \ldots$

P(atleast 4 items are to be examined)

$= P(x + r \geq 4) = P(x \geq 2)$

$= 1 - P(X < 2) = 1 - (P(X = 0) + P(X = 1))$

$= 1 - \left\{ 1C_1 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^0 + 2C_1 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^1 \right\}$.

$$= 1 - \left\{ (0.05)^2 + 2(0.05)^2(0.95) \right\}$$

$$= 0.995.$$

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② Sharon and Ann play a series of backgammon games until one of them wins five games. Suppose that the games are independent and the probability that Sharon win a game is 0.58.

(a) Find the probability that the series ends in 7 games.

(b) If the series ends in 7 games, what is the probability that Sharon wins?.

Soln: Here $r = 5$.

$X \rightarrow$ Sharon probability of success ($p = 0.58$, $q = 0.42$)

$Y \rightarrow$ Ann probability of success ($p = 0.42$, $q = 0.58$)

ⓐ P(Series ends in 7 games)

= P(Sharon wins in 7 games or Ann wins in 7 games)

$$= P(X + r = 7) + P(Y + r = 7)$$

$$= P(X = 2) + P(Y = 2)$$

$$= (2+2-1)C_{2-1}(0.58)^5(0.42)^2 + (2+2-1)C_{2-1}(0.42)^5(0.58)^2$$

$$= 3\cdot(0.58)^5(0.42)^2 + 3(0.42)^5(0.58)^2$$

$$=$$

$P(X = x)$
$= (x+r-1)C_{r-1}\, p^r q^x$
Here $p = 0.58$, $q = 0.42$

$P(Y = y)$
$= (y+r-1)C_{r-1}\, p^r q^y$
Here $p = 0.42$, $q = 0.58$