UNIT I
PROBABILITY THEORY

1.1 Introduction

The theory of probability deals with averages of mass phenomena occurring sequentially or simultaneously: electron emission, telephone calls, radar detection, quality control, system failure, games of chance, statistical mechanics, turbulence, noise, birth and death rates, and queuing theory, among many others. It has been observed that in these and other fields certain averages approach a constant value as the number of observations increases and this value remains the same if the averages are evaluated over any subsequence specified before the experiment is performed. In the coin experiment, for example, the percentage of heads approaches 0.5 or some other constant, and the same average is obtained if we consider every fourth, say, toss (no betting system can beat the roulette).The purpose of the theory is to describe and predict such averages in terms of probabilities of events. The probability of an event a is a number assigned to this event.

An ***experiment*** is a procedure we perform (quite often hypothetical) that produces some result.

An experiment whose outcome or result can be predicted with certainity is called ***deterministic experiment***. For eg if the potential difference(E) between the two end of a conductor and the resistance(R) is known then the current flowing in the conductor is determined using Ohms law(I=E/R).

An ***outcome*** is a possible result of an experiment.

An ***event*** is a certain set of outcomes of an experiment.

The ***sample space*** is the collection or set of "all possible" distinct (collectively exhaustive and mutually exclusive) outcomes of an experiment. The letter *S* is used to designate the sample space, which is the universal set of outcomes of an experiment. A sample space is called discrete if it is a finite or a countably infinite set otherwise It is called continuous.

Apriori definition of probability

Let S be the sample space and A be an event associated with a random experiment. Let n(S) and n(A) be the number of elements of S and A. Then the probability of event A occurring denoted by P(A) is defined by

$$P(A) = \frac{n(A)}{n(S)} = \frac{number\ of\ cases\ favourable\ to\ A}{Exhaustive\ number\ of\ cases\ in\ S}$$

Consider the example of flipping a fair coin once, where fair means that the coin is not biased in weight to a particular side. There are two possible outcomes, namely, a head or a tail. Thus, the sample space, *S*, consists of two outcomes, *event*1 = *H* to indicate that the outcome of the coin toss was heads and *event*2 = *T* to indicate that the outcome of the coin toss was tails.

1.2 Axioms of Probability

Let S be the sample space and A be an event associated with a random experiment. Then the probability of event A occurring denoted by P(A) is defined as a real number satisfying the following axioms

1.$\mathbf{0 \leq P(A) \leq 1}$

2.$\mathbf{P(S) = 1}$

3.If A and B are mutually excusive events,$P(A \cup B) = P(A) + P(B)$

4.If A_1,A_2,A_3,………A_n are a set of mutually exclusive events, $P(A_1 \cup A_2 \cup \ldots A_n) = P(A_1) + P(A_2) + P(A_n)$.

A set of events is said to be mutually exclusive if occurrence of any one of them excludes the occurrence of the others. i.e., $P(A \cap B) = 0$

1.2.1 Theorem 1

The probability of the impossible event is zero that is if ∅ is the sub set containing no sample point , $P(\emptyset) = 0$

Proof

The certain event S and the impossible event ∅ are mutually exclusive.

Hence $P(S \cup \emptyset) = P(S) + P(\emptyset)$ axiom 3

But $S \cup \emptyset = S$

Therefore $P(S) = P(S) + P(\emptyset), P(\emptyset) = 0.$

1.2.2 **Theorem 2**

If $\bar{A}$ is the complement event of A, $P(\bar{A}) = 1 - P(A) \leq 1$

Proof

$P(A \cup \bar{A}) = P(S) = 1$ axiom 2

That is $P(A) + P(\bar{A}) = 1$ axiom 3

Hence , $P(\bar{A}) = 1 - P(A)$,since , $P(A) \geq 0$ therefore , $P(\bar{A}) \leq 1$

1.2.3 **Theorem 3**(Addition theorem of probability)

If A and B are any two events**,** $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$

Proof

If A is the union of mutually exclusive events $A\bar{B}$ and AB

If A is the union of mutually exclusive events $\bar{A}B$ and AB

Therefore $P(A) = P(A\bar{B}) + P(AB)$ and $P(B) = P(\bar{A}B) + P(AB)$ axiom 3

Hence $P(A) + P(B) = P(A\bar{B}) + P(AB) + P(\bar{A}B) + P(AB) = P(A \cup B) + P(A \cap B)$

The result follows $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$

1.2.4 **Theorem** 4

If $B \in A, P(B) \leq P(A)$

Proof

B and $A\bar{B}$ are mutually exclusive events such that $(B \cup A\bar{B}) = A$

$P(B \cup A\bar{B}) = P(A)$ that is $P(B) + P(A\bar{B}) = P(A)$ axiom 3.

Therefore $P(B) \leq P(A)$

1.3 Conditional probability

The conditional probability of an event B assuming that the event A has happened is denoted by $P(B/A)$ is defined as

$P(B/A) = \frac{P(A \cap B)}{P(A)}$ provided $P(A) \neq 0$.Rearrangiing we get

$P(A \cap B) = P(A) * P(B/A)$ **(Product theorem of probability)**

1.3.1 **Properties based on conditional probability**

1.If $A \epsilon B$, $P(B/A) = 1$ since $A \cap B = A$

2.If $B \epsilon A$, $P(B/A) \geq P(B)$ since $A \cap B = B$ and $\frac{P(B)}{P(A)} \geq P(B)$ as $P(A) \leq P(S) = 1$

3.If A and B are mutually exclusive events$P(B/A) = 0$ since $P(A \cap B) = 0$

4.If P(A)>P(B),P(A/B)>P(B/A)

5.If $A_1 \epsilon A_2$,$P(A_1/B) \leq P(A_2/B)$

1.3.2 **Independent events**

A set of events are said to be independent if the occurrence of any one of them does not depend on the occurrence or non occurrence of the other.

1.3.3 **Theorem1:**

If the events A and B are independent the events $\bar{\mathbf{A}}$ and B are also independent**.**

1.3.4 **Theorem2:**

If the events A and B are independent so also are $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$.

1.3.5 **Theorem of Total Probability**

Let B_1, B_2, . . . , B_n be a set of mutually exclusive and exhaustive events and A is another event associated with Bi then

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i)$$

1.4 Bayes's Theorem

Let B_1, B_2, . . . , B_n be a set of mutually exclusive and exhaustive events and A is another event associated with B_i then

$$P(B_i/A) = \frac{P(B_i)P(A/B_i)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)}$$

1.5 Bernoulli Random Variable

This is the simplest possible random variable and is used to represent experiments that have two possible outcomes. These experiments are called **Bernoulli trials** and the resulting random variable is called a Bernoulli random *variable*. It is most common to associate the values {0,1} with the two outcomes of the experiment. If X is a Bernoulli random variable, its probability mass function is of the form

$P_X(0) = 1 - p$, $P_X(1) = p$

The coin tossing experiment would produce a Bernoulli random variable. In that case, we may map the outcome H to the value $X = 1$ and T to $X = 0$. Also, we would use the value $p = 1/2$ assuming that the coin is fair. Examples of engineering applications might include radar systems where the random variable could indicate the presence ($X = 1$) or absence ($X = 0$) of a target, or a digital communication system where $X = 1$ might indicate a bit was transmitted in error while $X = 0$ would indicate that the bit was received correctly. In these examples, we would probably expect that the value of p would be much smaller than 1/2.

Problems

Example 1

A fair coin is tossed 4 times. Define the sample space corresponding to this random experiment. Also give the subsets corresponding to the following events and find the respective probabilities:

(a) More heads than tails are obtained.
(b) Tails occur on the even numbered tosses.

S = {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}

(a) Let A be the event in which more heads occur than tails.

Then A = {HHHH, HHHT, HHTH, HTHH, THHH}

(b) Let B be the event in which tails occur in the second and fourth tosses.

Then B = {HTHT, HTTT, TTHT, TTTT}

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{16};\ P(B) = \frac{n(B)}{n(S)} = \frac{1}{4}$$

Example 2

There are 4 letters and 4 addressed envelopes. If the letters are placed in the envelopes at random, find the probability that (i) none of the letters is in the correct envelope and (ii) at least 1 letter is in the correct envelope, by explicitly writing the sample space and the event spaces.

Let the envelopes be denoted by A, B, C and D and the corresponding letters by a, b, c and d.

S = {(Aa, Bb, Cc, Dd), (Aa, Bb, Cd, Dc), (Aa, Bc, Cb, Dd), (Aa, Bc, Cd, Db), (Aa, Bd, Cb, Dc), (Aa, Bd, Cc, Db), (Ab, Ba, Cc, Dd), (Ab, Ba, Cd, Dc), (Ab, Bc, Ca, Dd), (Ab, Bc, Cd, Da), (Ab, Bd, Ca, Dc), (Ab, Bd, Cc, Da), (Ac, Ba, Cb Dd), (Ac, Ba, Cd, Db), (Ac, Bb, Ca, Dd), (Ac, Bb, Cd, Da), (Ac, Bd, Ca, Db), (Ac, Bd, Cb, Da), (Ad, Ba, Cb, Dc), (Ad, Ba, Cc, Db), (Ad, Bb, Ca, Dc), (Ad, Bb, Cc, Da), (Ad, Bc, Ca, Db), (Ad, Bc, Cb, Da)}

where 'Aa' means that the letter 'a' is placed in the envelope A.

Let E_1 denote the event in which none of the letters is in the correct envelope.

Then E_1 = {(Ab, Ba, Cd, Dc), (Ab, Bc, Cd, Da), (Ab, Bd, Ca, Dc), (Ac, Ba, Cd, Db), (Ac, Bd, Ca, Db), (Ac, Bd, Cb, Da), (Ad, Ba, Cb, Dc), (Ad, Bc, Ca, Db), (Ad, Bc, Cb, Da)}

Let E_2 denote the event in which at least one of the letters is in the correct envelope.

We note that E_2 is the complement of E_1. Therefore E_2 consists of all the elements of S except those in E_1.

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{9}{24} = \frac{3}{8} \text{ and } P(E_2) = 1 - P(E_1) = \frac{5}{8}$$

Example 3

A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. Two articles are chosen from the lot at random (without replacement). Find the probability that (i) both are good, (ii) both have major defects, (iii) at least 1 is good, (iv) at most 1 is good, (v) exactly 1 is good, (vi) neither has major defects and (vii) neither is good.

Although the articles may be drawn one after the other, we can consider that both articles are drawn simultaneously, as they are drawn without replacement.

(i) $P(\text{both are good}) = \dfrac{\text{No. of ways drawing 2 good articles}}{\text{Total no. of ways of drawing 2 articles}}$

$$= \frac{10C_2}{16C_2} = \frac{3}{8}$$

(ii) $P(\text{both have major defects})$

$$= \frac{\text{No. of ways of drawing 2 articles with major defects}}{\text{Total no. of ways}}$$

$$= \frac{2C_2}{16C_2} = \frac{1}{120}$$

(iii) $P(\text{at least 1 is good}) = P(\text{exactly 1 is good or both are good})$

$$= P(\text{exactly 1 is good and 1 is bad or both are good})$$

$$= \frac{10C_1 \times 6C_1 + 10C_2}{16C_2} = \frac{7}{8}$$

(iv) $P(\text{atmost 1 is good}) = P(\text{none is good or 1 is good and 1 is bad})$

$$= \frac{10C_0 \times 6C_2 + 10C_1 \times 6C_1}{16C_2} = \frac{5}{8}$$

(v) $P(\text{exactly 1 is good}) = P(\text{1 is good and 1 is bad})$

$$= \frac{10C_1 \times 6C_1}{16C_2} = \frac{1}{2}$$

(vi) $P(\text{neither has major defects})$

$$= P(\text{both are non-major defective articles})$$

$$= \frac{14C_2}{16C_2} = \frac{91}{120}$$

(vii) $P(\text{neither is good}) = P(\text{both are defective})$

$$= \frac{6C_2}{16C_2} = \frac{1}{8}$$

Example 4

From 6 positive and 8 negative numbers, 4 numbers are chosen at random (without replacement) and multiplied. What is the probability that the product is positive?

If the product is to be positive, all the 4 numbers must be positive or all the 4 must be negative or 2 of them must be positive and the other 2 must be negative.

No. of ways of choosing 4 positive numbers $= 6C_4 = 15$.

No. of ways of choosing 4 negative numbers $= 8C_4 = 70$.

No. of ways of choosing 2 positive and 2 negative numbers

$$= 6C_2 \times 8C_2 = 420.$$

Total no. of ways of choosing 4 numbers from all the 14 numbers

$$= 14C_4 = 1001.$$

P(the product is positive)

$$= \frac{\text{No. of ways by which the product is positve}}{\text{Total no. of ways}}$$

$$= \frac{15 + 70 + 420}{1001} = \frac{505}{1001}$$

Example 5

A box contains tags marked 1, 2, ..., n. Two tags are chosen at random without replacement. Find the probability that the numbers on the tags will be consecutive integers.

If the numbers on the tags are to be consecutive integers, they must be chosen as a pair from the following pairs.

$$(1, 2); (2, 3); (3, 4); \ldots; (n - 1, n)$$

No. of ways of choosing any one pair from the above $(n - 1)$ pairs $= (n - 1)\, C_1 = n - 1$.

Total No. of ways of choosing 2 tags from the n tags $= nC_2$.

$$\therefore \quad \text{Required probability} = \frac{n-1}{\frac{n\,(n-1)}{2}} = \frac{2}{n}$$

Example 6

If n biscuits are distributed at random among m children, what is the probability that a particular child receives r biscuits, where $r < n$? (MKU — Nov. 96)

The first biscuit can be given to any one of the m children, i.e., in m ways.

Similarly the second biscuit can be given in m ways.

Therefore 2 biscuits can be given in m^2 ways.

Extending, n biscuits can be distributed in m^n ways. The r biscuits received by the particular child can be chosen from the n biscuits in nC_r ways. If this child has got r biscuits, the remaining $(n-r)$ biscuits can be distributed among the remaining $(m-1)$ children in $(m-1)^{n-r}$ ways.

$\therefore$ No. of ways of distributing in the required manner

$$= nC_r(m-1)^{n-r}$$

$$\therefore \quad \text{Required probability} = \frac{nC_r(m-1)^{n-r}}{m^n}$$

Example 7

If $P(A) = P(B) = P(AB)$, show that $P(A\overline{B} + \overline{A}B) = 0$ $[AB \equiv A \cap B]$.

By addition theorem,

$$P(A \cup B) = P(A) + P(B) - P(AB) \tag{1}$$

From the Venn diagram on page (1.3), it is clear that

$$A \cup B = A\overline{B} + \overline{A}B + AB$$

$$\therefore \quad P(A \cup B) = P(A\overline{B}) + P(\overline{A}B) + P(AB) \text{ (by probability axiom)} \tag{2}$$

Using the given condition in (1),

$$P(A \cup B) = P(AB) \tag{3}$$

From (2) and (3), $P(A\overline{B}) + P(\overline{A}B) = 0$

Example 8

If A, B and C are any 3 events such that $P(A) = P(B) = P(C) = 1/4$, $P(A \cap B) = P(B \cap C) = 0$; $P(C \cap A) = 1/8$. Find the probability that at least 1 of the events A, B and C occurs.

P(at least one of A, B and C occurs) $= P(A \cup B \cup C)$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) \tag{1}$$

Since $P(A \cap B) = P(B \cap C) = 0$, $P(A \cap B \cap C) = 0$. Equation (1) becomes

$$P(A \cup B \cup C) = \frac{3}{4} - 0 - 0 - \frac{1}{8} = \frac{5}{8}$$

Example 9

Solve Example 5, if the tags are chosen at random with replacement.

If the tag with '1' is chosen in the first draw, the tag with '2' must be chosen in the second draw. Probability for each $= 1/n$.

$\therefore$ P('1' in the first draw and '2' in the second draw) = $1/n^2$ (Product theorem)
Similarly, P('n' in the first draw and '$n - 1$' in the second draw = $1/n^2$.
If the number drawn first is '2', the number drawn second may be '1' or '3'.

Probability of drawing consecutive numbers in this case

$$= \frac{1}{n} \times \frac{2}{n} = \frac{2}{n^2}$$

Similarly, when the first number drawn is '3', '4', ..., '$(n - 1)$' probability of drawing consecutive numbers will be $2/n^2$.

All the above possibilities are mutually exclusive.

$$\therefore \text{ Required probability} = \frac{1}{n^2} + \frac{1}{n^2} + (n-2) \times \frac{2}{n^2} = \frac{2(n-1)}{n^2}$$

Example 10

A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

Let A = one of the tubes drawn is good and B = the other tube is good.

$$P(A \cap B) = P(\text{both tubes drawn are good})$$

$$= \frac{6C_2}{10C_2} = \frac{1}{3}$$

Knowing that one tube is good, the conditional probability that the other tube is also good is required, i.e., $P(B/A)$ is required.

By definition,

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{6/10} = \frac{5}{9}$$

Example 11

Two defective tubes get mixed up with 2 good ones. *The tubes are tested*, one by one, *until both defectives are found*. What is the probability that the last defective tube is obtained on (i) the second test, (ii) the third test and (iii) the fourth test?

Let D represent defective and N represent non-defective tube.

(i) P(Second D in the II test) = P(D in the I test and D in the II test)

$$= P(D_1 \cap D_2), \text{ say}$$

$$= P(D_1) \times P(D_2) \text{ (by independence)}$$

$$= \frac{2}{4} \times \frac{1}{3} = \frac{1}{6}$$

(ii) $P(\text{second } D \text{ in the III test}) = P(D_1 \cap N_2 \cap D_3 \text{ or } N_1 \cap D_2 \cap D_3)$

$$= P(D_1 \cap N_2 \cap D_3) + P(N_1 \cap D_2 \cap D_3)$$

$$= \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2}$$

$$= \frac{1}{3}$$

(iii) $P(\text{second } D \text{ in the IV test}) = P(D_1 \cap N_2 \cap N_3 \cap D_4) + P(N_1 \cap D_2 \cap N_3 \cap D_4) + P(N_1 \cap N_2 \cap D_3 \cap D_4)$

$$= \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1 + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1 + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1$$

$$= \frac{1}{2}$$

Example 12

In a shooting test, the probability of hitting the target is 1/2 for A, 2/3 for B and 3/4 for C. If all of them fire at the target, find the probability that (i) none of them hits the target and (ii) at least one of them hits the target.

Let $A \equiv$ Event of A hitting the target, and so on.

$$P(\overline{A}) = \frac{1}{2}, P(\overline{B}) = \frac{1}{3}, P(\overline{C}) = \frac{1}{4}$$

$$P(\overline{A} \cap \overline{B} \cap \overline{C}) = P(\overline{A}) \times P(\overline{B}) \times P(\overline{C}) \text{ (by independence)}$$

$$= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} = \frac{1}{24}$$

P(at least one hits the target)

$$= 1 - P(\text{none hits the target})$$

$$= 1 - \frac{1}{24} = \frac{23}{24}$$

Example 13

A and B alternately throw a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, show that his chance of winning is 30/61. (BU — Apr. 96)

Throwing 6 with 2 dice $\equiv$ Getting 6 as the sum of the numbers shown on the upper faces of the 2 dice.

$$P(\text{throwing 6 with 2 dice}) = \frac{5}{36}$$

$$P(\text{throwing 7 with 2 dice}) = \frac{1}{6}$$

Let $A \equiv$ Event of A throwing 6.

Let $B \equiv$ Event of B throwing 7.

A plays in the first, third, fifth, ..., trials.

Therefore A will win, if he throws 6 in the first trial or third trial or in subsequent (odd) trials.

$$\therefore \quad P(A \text{ wins}) = P(A \text{ or } \bar{A}\ \bar{B}\ A \text{ or } \bar{A}\ B\ \bar{A}\ \bar{B}\ A \text{ or } \ldots)$$

$$= P(A) + P(\bar{A}\ \bar{B}\ A) + P\left(\bar{A}\ \bar{B}\ \bar{A}\ \bar{B}\ A\right) + \ldots \text{ (Addition theorem)}$$

$$= \frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)\frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)^2 \times \frac{5}{36} + \ldots \text{ upto } \infty$$

$$= \frac{5/36}{1-(155/216)} \text{ (since the series is an infinite geometric series)}$$

$$= \frac{30}{61}$$

Example 14

Show that $2^n - (n+1)$ equations are needed to establish the mutual independence of n events.

n events are mutually independent, if they are totally independent when considered in sets of 2, 3, ..., n events.

Sets of r events can be chosen from the n events in nC_r ways.

To establish total independence of r events, say, $A_1, A_2, \ldots, A_r$ chosen in any one of the nC_r ways, we need one equation, namely, $P(A_1, A_2, \ldots, A_r) = P(A_1) \times P(A_2) \ldots \times P(A_r)$.

Therefore to establish total independence of all the nC_r sets, each of r events, we need nC_r equations.

Therefore the number of equations required to establish mutual independence

$$= \sum_{r=2}^{n} nC_r$$

$$= (nC_0 + nC_1 + nC_2 + \ldots + nC_n) - (1+n)$$

$$= (1+1)^n - (n+1)$$

$$= 2^n - (n+1)$$

Example 15

Two fair dice are thrown independently. Three events A, B and C are defined as follows.

(a) Odd face with the first die

(b) Odd face with the second die

(c) Sum of the numbers in the 2 dice is odd. Are the events A, B and C mutually independent? (MKU — Apr. 97)

$$P(A) = \frac{3}{6} = \frac{1}{2};\ P(B) = \frac{3}{6} = \frac{1}{2}$$

The outcomes favourable to the event C are (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5) and so on.

$$\therefore \quad P(C) = \frac{1}{2}$$

$$P(A \cap B) = P(B \cap C) = P(A \cap C) = \frac{1}{4}$$

$$\therefore \quad P(A \cap B) = P(A)\,P(B), \text{ and so on}$$

But $P(A \cap B \cap C) = 0$, since C cannot happen when A and B occur. Therefore $P(A \cap B \cap C) \neq P(A) \times P(B) \times P(C)$.

Therefore the events are pairwise independent, but not mutually independent.

Example 16

If A, B and C are random subsets (events) in a sample space and if they are pairwise independent and A is independent of $(B \cup C)$, prove that A, B and C are mutually independent. (MU — Nov. 96)

Given: $P(AB) = P(A) \times P(B)$ (1)

$P(BC) = P(B) \times P(C)$ (2)

$P(CA) = P(C) \times P(A)$ (3)

$P[A\,(B \cup C)] = P(A) \times P(B \cup C)$ (4)

Consider $P[A(B \cup C)] = P(AB \cup AC)$

$= P(AB) + P(AC) - P(AB \cap AC)$ (by addition theorem)

$= P(A) \times P(B) + P(A) \times P(C) - P(ABC)$ [by (1) and (3)] (5)

Therefore from (4) and (5), we get

$$P(ABC) = P(A) \times P(B) + P(A) \times P(C) - P(A) \times P(B \cup C)$$

$$= P(A) \times [P(B) + P(C) - P(B \cup C)]$$

$$= P(A) \times P(B \cap C) \text{ (by addition theorem)}$$

$$= P(A) \times P(B) \times P(C) \text{ [by (2)]} \qquad (6)$$

From (1), (2), (3) and (6), the required result follows.

Baye's theorem

Example 1

A coin is tossed an infinite number of times. If the probability of a head in a single toss is p, show that the probability that kth head is obtained at the nth tossing, but not earlier is $(n-1)C_{k-1}p^k q^{n-k}$, where $q = 1 - p$.

k heads should be obtained at the nth tossing, but not earlier.

Therefore $(k-1)$ heads must be obtained in the first $(n-1)$ tosses and 1 head must be obtained at the nth toss.

$$\therefore \quad \text{Required probability} = P[k-1 \text{ heads in } (n-1) \text{ tosses}] \times P(1 \text{ head in 1 toss})$$

$$= (n-1)C_{k-1}p^{k-1}q^{n-k} \times p$$

$$= (n-1)C_{k-1}p^k q^{n-k}$$

Example 2

Each of two persons A and B tosses 3 fair coins. What is the probability that they obtain the same number of heads?

P(A and B get the same no. of heads)

$= P$(they get no head each or 1 head each or 2 heads each or 3 heads each)

$= P$(each gets 0 head) + P(each gets 1 head) + P(each gets 2 heads) + P(each gets 3 heads) (since the events are mutually exclusive)

$= P$(A gets 0 head) $\times$ P(B gets 0 head) + ... (since A and B toss independently)

$$= \left[3C_0\left(\frac{1}{2}\right)^3\right]^2 + \left[3C_1\left(\frac{1}{2}\right)^3\right]^2 + \left[3C_2\left(\frac{1}{2}\right)^3\right]^2 + \left[3C_3\left(\frac{1}{2}\right)^3\right]^2$$

$$= \frac{1}{64}(1 + 9 + 9 + 1) = \frac{5}{16}$$

Example 3

A coin with $P(\text{head}) = p = 1 - q$ is tossed n times. Show that the probability that the number of heads obtained is even is $0.5[1 + (q - p)^n]$.

$$P(\text{even no. of heads are obtained})$$
$$= P(0 \text{ head or } 2 \text{ heads or } 4 \text{ heads or } \ldots)$$
$$= nC_0 q^n p^0 + nC_2 q^{n-2} p^2 + nC_4 q^{n-4} p^4 + \ldots \quad (1)$$

Consider $1 = (q + p)^n = nC_0 q^n p^0 + nC_1 q^{n-1} p^1 + nC_2 q^{n-2} p^2 + \ldots \quad (2)$

$$(q - p)^n = nC_0 q^n p^0 - nC_1 q^{n-1} p^1 + nC_2 q^{n-2} p^2 - \ldots \quad (3)$$

Adding (2) and (3), we get

$$1 + (q - p)^n = 2[nC_0 q^n p^0 + nC_2 q^{n-2} p^2 + nC_4 q^{n-4} p^4 + \ldots] \quad (4)$$

Using (4) in (1), the required probability $= 0.5[1 + (q - p)^n]$.

Example 4

If at least 1 child in a family with 2 children is a boy, what is the probability that both children are boys? (MKU - Apr. 96)

$$p = \text{Probability that a child is a boy} = \frac{1}{2}$$

$$\therefore \quad q = \frac{1}{2} \text{ and } n = 2$$

$$P(\text{at least one boy}) = p(\text{exactly 1 boy}) + p(\text{exactly 2 boys})$$

$$= 2C_1 \left(\frac{1}{2}\right)^2 + 2C_2 \left(\frac{1}{2}\right)^2 \quad \text{(by Bernoulli's theorem)}$$

$$= \frac{3}{4}$$

$$P(\text{both are boys/at least one is a boy})$$

$$= \frac{P(\text{both are boys} \cap \text{at least one is a boy})}{P(\text{at least one is a boy})}$$

$$= \frac{P(\text{both are boys})}{P(\text{at least one is a boy})}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Example 5

Find the probability of getting at least 60 heads, when 100 fair coins are tossed.

Bernoulli's trials

Example 1

A bolt is manufactured by 3 machines A, B and C. A turns out twice as many items as B, and machines B and C produce equal number of items. 2% of bolts produced by A and B are defective and 4% of bolts produced by C are defective. All bolts are put into 1 stock pile and 1 is chosen from this pile. What is the probability that it is defective?

Let A = the event in which the item has been produced by machine A, and so on.

Let D = the event of the item being defective.

$$P(A) = \frac{1}{2},\ P(B) = P(C) = \frac{1}{4}$$

$$P(D/A) = P(\text{an item is defective, given that } A \text{ has produced it})$$

$$= \frac{2}{100} = P(D/B)$$

$$P(D/C) = \frac{4}{100}$$

By theorem of total probability,

$$P(D) = P(A) \times P(D/A) + P(B) \times P(D/B) + P(C) \times P(D/C)$$

$$= \frac{1}{2} \times \frac{2}{100} + \frac{1}{4} \times \frac{2}{100} + \frac{1}{4} \times \frac{4}{100}$$

$$= \frac{1}{40}$$

Example 2

An urn contains 10 white and 3 black balls. Another urn contains 3 white and 5 black balls. Two balls are drawn at random from the first urn and placed in the second urn and then 1 ball is taken at random from the latter. What is the probability that it is a white ball?

The two balls transferred may be both white or both black or 1 white and 1 black.

Let B_1 = event of drawing 2 white balls from the first urn, B_2 = event of drawing 2 black balls from it and B_3 = event of drawing 1 white and 1 black ball from it.

Clearly B_1, B_2 and B_3 are exhaustive and mutually exclusive events.

Let A = event of drawing a white ball from the second urn after transfer.

$$P(B_1) = \frac{10C_2}{13C_2} = \frac{15}{26};\ P(B_2) = \frac{3C_2}{13C_2} = \frac{1}{26};\ P(B_3) = \frac{10 \times 3}{13C_2} = \frac{10}{26},$$

$$\begin{aligned}P(A/B_1) &= P(\text{drawing a white ball/2 white balls have been transferred})\\ &= P(\text{drawing a white ball/urn II contains 5 white and 5 black balls})\\ &= \frac{5}{10}\end{aligned}$$

Similarly, $P(A/B_2) = \frac{3}{10}$ and $P(A/B_3) = \frac{4}{10}$.

By theorem of total probability,

$$\begin{aligned}P(A) &= P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2) + P(B_3) \times P(A/B_3)\\ &= \frac{15}{26} \times \frac{5}{10} + \frac{1}{26} \times \frac{3}{10} + \frac{10}{26} \times \frac{4}{10}\\ &= \frac{59}{130}\end{aligned}$$

Example 3

In a coin tossing experiment, if the coin shows head, 1 die is thrown and the result is recorded. But if the coin shows tail, 2 dice are thrown and their sum is recorded. What is the probability that the recorded number will be 2?

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When a single die is thrown, $P(2) = \frac{1}{6}$.

When 2 dice are thrown, the sum will be 2, only if each die shows 1.

$\therefore$ P(getting 2 as sum with 2 dice) $= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ (since independence)

By theorem of total probability,

$$\begin{aligned}P(2) &= P(H) \times P(2/H) + P(T) \times P(2/T)\\ &= \frac{1}{2} \times \frac{1}{6} + \frac{1}{2} \times \frac{1}{36}\\ &= \frac{7}{72}\end{aligned}$$

Example 4

A bag contains 5 balls and it is not known how many of them are white. Two balls are drawn at random from the bag and they are noted to be white. What is the chance that all the balls in the bag are white?

Since 2 white balls have been drawn out, the bag must have contained 2, 3, 4 or 5 white balls.

Let B_1 = Event of the bag containing 2 white balls, B_2 = Events of the bag containing 3 white balls, B_3 = Event of the bag containing 4 white balls and B_4 = Event of the bag containing 5 white balls.

Let A = Event of drawing 2 white balls.

$$P(A/B_1) = \frac{2C_2}{5C_2} = \frac{1}{10},\ P(A/B_2) = \frac{3C_2}{5C_2} = \frac{3}{10}$$

$$P(A/B_3) = \frac{4C_2}{5C_2} = \frac{3}{5},\ P(A/B_4) = \frac{5C_2}{5C_2} = 1$$

Since the number of white balls in the bag is not known, B_i's are equally likely.

$$\therefore \quad P(B_1) = P(B_2) = P(B_3) = P(B_4) = \frac{1}{4}$$

By Baye's theorem,

$$P(B_4/A) = \frac{P(B_4) \times P(A/B_4)}{\sum_{i=1}^{4} P(B_i) \times P(A/B_i)} = \frac{\frac{1}{4} \times 1}{\frac{1}{4} \times \left(\frac{1}{10} + \frac{3}{10} + \frac{3}{5} + 1\right)} = \frac{1}{2}$$

Example 5

There are 3 true coins and 1 false coin with 'head' on both sides. A coin is chosen at random and tossed 4 times. If 'head' occurs all the 4 times, what is the probability that the false coin has been chosen and used?

$$P(T) = P(\text{the coin is a true coin}) = \frac{3}{4}$$

$$P(F) = P(\text{the coin is a false coin}) = \frac{1}{4}$$

Let A = Event of getting all heads in 4 tosses

Then $\quad P(A/T) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$ and $P(A/F) = 1$.

By Baye's theorem,

$$P(F/A) = \frac{P(F) \times P(A/F)}{P(F) \times P(A/F) + P(T) \times P(A/T)}$$

$$= \frac{\frac{1}{4} \times 1}{\frac{1}{4} \times 1 + \frac{3}{4} \times \frac{1}{16}} = \frac{16}{19}$$