SUBJECT:**ENGINEERING MATHEMATICS-I**

SUBJECT CODE :**SMT1101**

UNIT –V THREE DIMENSIONAL ANALYTICAL GEOMETRY

Definition

1. Direction Cosines :

If a line makes an angle α,β and ɣ with the positive direction of the axes respectively then cosα,cosβ and cosɣ are called the Direction Cosines(D.C's).

Hence $\cos^2\alpha+\cos^2\beta + \cos^2\gamma = 1$.

2.Direction Ratios :

Any three numbers which are proportional to the D.C's are called Direction Ratios(D.R's).

Problems :

1. Find the angle made by a line with any one of the co-ordinate axes if it makes equal angles with the axes?

   Let the line make an angle α with the co-ordinate axis.Given that α=β=γ.

   Hence $\cos^2\alpha+\cos^2\alpha + \cos^2\alpha = 1$

   $$3\cos^2\alpha = 1$$

   $$\cos^2\alpha = \frac{1}{3}$$

   $$\cos\alpha = \frac{1}{\sqrt{3}}$$

   $$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

2. Using D.C's, prove that the points A(3,1,3), B(1,-2,-1) and C(-1,-5,-5) are collinear

   The D.R's of AB are (-2,-3,-4)

   D.C's of AB are $\left(\frac{-2}{\sqrt{29}},\frac{-3}{\sqrt{29}},\frac{-4}{\sqrt{29}}\right)$

   Similarly D.R's of BC are (-2,-3,-4)

   D.C's of BC are $\left(\frac{-2}{\sqrt{29}},\frac{-3}{\sqrt{29}},\frac{-4}{\sqrt{29}}\right)$

The D.C's of AB are equal to the D.C's of BC. The two lines are equal and B is the midpoint of AC. Hence the three points A,B and C are Collinear.

3. Find the D.C's of the line perpendicular to the two lines whose D.R's are (1,2,3) and (-2,1,4)

Let the D.R's of the required line be (a,b,c)

Since the required line is perpendicular to a line whose D.R's are (1,2,3) by applying the condition for the perpendicularity of two lines.

$a+2b+3c=0$

since the required line is perpendicular to another line with D.R's (-2,1,4)

$-2a+b+4c = 0$

Solving by the method of cross multiplication

$$\frac{a}{8-3} = \frac{-b}{4+6} = \frac{c}{1+4}$$

$$\frac{a}{5} = \frac{-b}{10} = \frac{c}{5}$$

The D.R's are (5,-10,5)

D.R's of the required line are (1,-2,1)

The D.C's of the required line are $\left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

4. A line makes angles 30˚ and 60˚ with the X and Y axes. Find the angle made by the line with the Z axis.

Let α be the angle made by the line with the z axis.

We know that if a line makes an angle α,β and ɣ with the co-ordinate axes then

$\cos^2\alpha+\cos^2\beta + \cos^2\gamma= 1.$

Here $\alpha= 30^\circ$, $\beta= 60^\circ$, $\gamma=\alpha$

$\cos^2 30^\circ + \cos^2 60^\circ + \cos^2\alpha = 1$

$$\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\alpha = 1$$

$\frac{3}{4}+\frac{1}{4}$ +cos^{2}α = 1

cos^{2}α = 1-1=0

cos α = 0

α = cos^{-1}(0) $=\frac{\pi}{2}$

Therefore, the line makes an angle 90˚ with the z-axis

5. Find the angles of a triangle whose vertices are A(1,0,-1),B(2,1,3) and C(3,2,1)

D.R's of AB are (1,1,4)

D.R's of BC are (1,1,-2)

D.R's of AC are (2,2,2)

Let α be the angle between AB and BC, then

$$\cos\alpha = \frac{1+1-8}{\sqrt{1+1+16}\ \sqrt{1+1+4}} = \frac{-6}{\sqrt{18}\ \sqrt{6}} = \frac{-\sqrt{6}}{\sqrt{18}} = \frac{-1}{\sqrt{3}}$$

$$\alpha = \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

Let β be the angle between BC and AC, then

$$\cos\beta = \frac{2+2-4}{\sqrt{1+1+4}\ \sqrt{4+4+4}} = 0 \Rightarrow \beta = \cos^{-1}(0) = \frac{\pi}{2}$$

Let ɣ be the angle between AB and AC, then

$$\cos\gamma = \frac{2+2+8}{\sqrt{18}\ \sqrt{12}} = \frac{12}{\sqrt{18}\ \sqrt{12}} = \frac{\sqrt{12}}{\sqrt{18}} = \sqrt{\frac{2}{3}}$$

$$\alpha = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

PLANES

Problems:

1. Find the angle between the planes 2x-y+z+7=0 and x+y+2z-11=0

The D.R's of the normal to the plane 2x-y+z+7=0 are (2,-1,1)

D.R's of the normal to the plane x+y+2z-11=0 are (1,1,2)

Let θ be the angle between the 2 planes then

$$\cos\theta = \frac{2-1+2}{\sqrt{4+1+1}\ \sqrt{1+1+4}} = \frac{3}{\sqrt{6}\ \sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^{\circ}$$

2. Find the equation of the plane passing through the origin and parallel to the plane x+2y-3z-4=0

   Let the required plane be ax+by+cz+d=0 ---------------- (1)

   This plane passes through the origin (0,0,0)

   Substituting in (1) we have d = 0

   Hence the required plane is ax+by+cz=0-----------------(2)

   (2) is parallel to the plane x+2y-3z-4=0

   Therefore the required plane will be x+2y-3z=0

   (As the planes are parallel the normal to the planes and also parallel. Hence the DR's of the normal to the planes are equal)

3. Find the equation of the plane through the points (0,0,2), (0,-1,0) and (-3,0,0)

   Let the required plane be ax+by+cz+d=0 -----------(1)

   (1) Passes through the point (0,0,2)

   Substituting the point in (1) we have

   2c+d = 0 ---------------(2)

   (1)Passes through the point (0,-1,0)

   -b+d = 0 ---------------(3)

   (1)Passes through the point (-3,0,0)

   -3a+d =0 ---------------(4)

   (2) $\Rightarrow$ d = -2c

   (3) $\Rightarrow$ d = b

   (4) $\Rightarrow$ d = 3a

   3a = b = -2c

$$\frac{a}{1/3}=\frac{b}{1}=\frac{c}{-1/2}$$

$$\Rightarrow a=\frac{1}{3}, \quad b=1, \quad c=-\frac{1}{2}$$

Sub c value in (2) we get d = 1

The required plane be $\frac{1}{3}x+y-\frac{1}{2}z+1=0$

2x + 6y – 3z +6 = 0

4. Find the equation of the plane through the point (-1,2,-3) and perpendicular to the line joining (-3,2,4) and (5,4,1)

Let (a,b,c) are the D.R's of the normal to the plane. Hence the required plane be

ax+by+cz+d=0 ------------(1)

(1)is perpendicular to the line joining the points (-3,2,4) and (5,4,1)

D.R's of the line is (8,2,-3)

This line is parallel to the normal to the plane

Hence a = 8, b = 2, c = -3

Sub these points in (1) we have 8x +2y -3z +d =0 --------(2)

Substitute the point (-1,2,-3) in (2)

8(-1) + 2(2) – 3(-3) + d = 0

d = - 5

The required plane is 8x +2y -3z – 5 =0

5. Find the distance between the parallel planes 3x+6y+2z = 22 and 3x + 6y + 2z = 27

The distance between the parallel plane is the length of the perpendicular from a point on one plane to the another plane.

Let (x_1,y_1,z_1) be a point on 3x+6y+2z = 22, then $3x_1+6y_1+2z_1 = 22$ ------(1)

The perpendicular distance from (1) to the plane 3x+6y+2z – 27 =0 is

$$d = \frac{|3x_1+6y_1+2z_1-27|}{\sqrt{3^2+6^2+2^2}}$$

$$= \frac{|(3x_1+6y_1+2z_1)-27|}{\sqrt{49}} = \frac{|22-27|}{7} = \frac{5}{7}\,units.$$

6. Find the equation of the plane through the point (1,0,-2) and perpendicular to the plane 2x+y-z = 2 and x-y-z = 3

Let the required plane be ax+by+cz+d=0 -----------(1)

This plane passes through (1,0,-2)

a -2c+d = 0 -------------(2)

(1)) is perpendicular to the plane 2x+y-z = 2

As the planes are perpendicular to each other their normal are also perpendicular to each other. By applying the condition for the perpendicularity of two lines we have

2a + b – c = 0 ------------(3)

Similarly as (1) is perpendicular to x – y – z = 3 we have

a – b – c = 0 ------------(4)

Solving (3) and (4) we get

$$\frac{a}{-1-1} = \frac{-b}{-2+1} = \frac{c}{-2-1}$$

a = - 2 , b = 1, c = - 3

Substitute these values in (2)

- 2 – 2(- 3) + d = 0

d = - 4

Substitute d = - 4 in (1)

-2x+y-3z-4 = 0

2x – y +3z +4 = 0

7. Find the equation of the plane through the points (-1,1,1) and (1,-1,1) and perpendicular to the plane x + 2y + 2z = 5

Let the required plane be ax+by+cz+d=0 -----------(1)

This plane passes through the point (-1,1,1)

- a+ b + c + d = 0 ------------(2)

Also (1) passes through the point (1, -1, 1)

a – b + c + d = 0 ------------(3)

(1)is perpendicular to the plane x + 2y + 2z – 5 =0

a + 2b + 2c = 0 --------------(4)

(1) – (3) $\Rightarrow$ - 2a + 2b = 0 ------------------(5)

Solving for (a,b,c) from (4) and (5) we have

$$\frac{a}{0-4} = \frac{-b}{0+4} = \frac{c}{2+4}$$

a = 2 , b = 2, c = - 3

Substitute these values in (2)

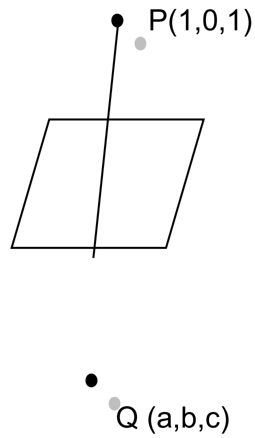
- 2+ 2- 3 + d = 0

d = 3

Substitute d = 3 in (1) $\Rightarrow$ 2x+2y - 3z +3 = 0

8. Find the foot of the perpendicular drawn from the point (1,0,1) on the plane x+2y+3z=6 and also find its image in the plane.

Let the plane be x+2y+3z=6 ----------------- (1)



Let Q(a,b,c) be the image of P. The d.r's of PQ are a-1,b-0,c-1

The line PQ and the normal to the plane are parallel.

Hence $\frac{a-1}{1} = \frac{b}{2} = \frac{c-1}{3} = k$ (say)

a= k+1, b = 2k , c= 3k+1 -------------(2)

Let M is the foot of the perpendicular from the point P to the plane and is also the midpoint of PQ.

$M = (\frac{a+1}{2}, \frac{b}{2}, \frac{c+1}{2})$

M lies on the plane (1)

$$\frac{a+1}{2} + 2\left(\frac{b}{2}\right) + 3\left(\frac{c+1}{2}\right) = 6$$

$$\frac{a}{2} + b + 3\frac{c}{2} + \frac{1}{2} + \frac{3}{2} = 6$$

a+2b+3c = 8 (3)

substituting in (3),

$$k+1 + 2 (2k) +3 (3K+1) = 8 \Rightarrow K = \frac{2}{7}$$

Hence $a = \frac{2}{7} + 1$, $b =, \frac{2}{7} \times 2$ $c = \left(3 \times \frac{2}{7}\right) + 1$

a = 9/7 , b = 4/7 , c = 13/7

$$M = \left(\frac{\frac{9}{7}+1}{2}, \frac{\frac{4}{7}}{2}, \frac{\frac{13}{7}+1}{2} \right)$$

$$= \left(\frac{8}{7}, \frac{2}{7}, \frac{10}{7}\right)$$

The image Q is $\left(\frac{9}{7},\frac{4}{7},\frac{13}{7}\right)$

9. Find the equation of the plane through the line of intersection of the planes x+y+z =1 and 2x+3Y- z +4 = 0 and parallel to the line joining the points (-1,2,-3) and (2,3,4)?

The equation of the required plane will be

$X+y+z -1 + \lambda (2x +3y-z +4) = 0$ (1)

$(1+ 2 \lambda) x + (1+ 3 \lambda) y + (1- \lambda)z +(-1+4 \lambda) = 0$ (2)

(1) Is parallel to the line joining the points (-1,2,-3) and (2,3,4) .

d.r' s of this line are (3,1,7) (3)

The normal to the plane and this line are perpendicular.

From (2) and (3) $(1+2 \lambda)3 + (1+3 \lambda) (1) + (1- \lambda) 7 = 0$

$3+6 \lambda + 1+ 3 \lambda +7 - 7 \lambda = 0$

$2 \lambda +11 = 0$ or $\lambda = -11/2$

Substituting $\lambda = -11/2$ in (1) the required plane will be

(x+y+z-1) -11/2 (2x+3y-z+4) =0

2x+2y+2z-2-22x-33y+11z-44 =0

$-20x-31y+13z-46 = 0$

ie $20x+31y-13z+46 = 0$ is the required plane.

10. Find the equation of the plane passing through the line of intersection 0f the planes $2x-5y+z = 3$ and $x+y+4z = 5$ and parallel to the plane $x+3y+6z = 1$.

The required plane will be

$(2x-5y+z-3)+\lambda(x+y+4z-5) = 0$ (1)

$(2+\lambda)x+(-5+\lambda)y+(1+4\lambda)z+(-3-5\lambda) = 0$ (2)

(2) Is parallel to the plane $x+3y+6z-1=0$

Hence their normals are also parallel

$$\frac{2+\lambda}{1} = \frac{-5+\lambda}{3} = \frac{1+4\lambda}{6}$$

$$\frac{2+\lambda}{1} = \frac{-5+\lambda}{3}$$

$6+3\lambda = -5+\lambda$ o r $2\lambda+11=0$. Hence $\lambda = \frac{-11}{2}$

The required plane will be obtained by substituting λ in (1).

Hence we have $x+3y+6z-7=0$.

The Straight Line:

Different forms of the equation of a straight line in space are

(1)Non- Symmetric form:

we know that two planes in general intersect in a line. Hence a line in space can be represented by two linear equations.

$$a_1x + b_1y+c_1z+d_1=0 \quad and \quad a_2x + b_2y+c_2z+d_2=0$$

(2) Symmetric form:

Equation of the straight line passing through he point (x,y,z) and having D.R'S (l,m,n) is given by

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

(3)Two point form:

Equation of the line joining the points (x_1,y_1,z_1) and (x_2,y_2,z_2) is given by

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

The plane and the Straight line:

Let us consider the plane $ax+by+cz+d=0$ (1)

and the straight line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ (2)

a. The straight line is normal to the plane, if

$$\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$$

b. The straight line is parallel to the plane if $al+bm+cn = 0$

c. The angle between the line and the normal to the plane is 90º - θ where θ is the angle at which the line (2) is inclined to the plane (1), thus

$$\text{Cos } (90^{\circ}-\theta) = \text{Sin } \theta = \frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

Coplanar Straight lines:

Let us consider the lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

$$\frac{x-x_1}{l_2}=\frac{y-y_1}{m_2}=\frac{z-z_1}{n_2}$$

The lines are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Equation of the plane in which the lines lie is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Skew Lines:

Two straight lines which do not lie in the same plane are called skew lines or non planar.Skew lines are neither parallel nor intersecting. Such line has a common perpendicular. The length of the segment of this common perpendicular line is called the shortest distance (S.D) between them. The common perpendicular line itself is called the S.D line.

Shortest distance between Two Skew lines (S.D) :

Let us consider the lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1},$$

$$\frac{x-x_1}{l_2} = \frac{y-y_1}{m_2} = \frac{z-z_1}{n_2}$$

hortest distance S.D $= \dfrac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\sum (l_1m_2 - l_2m_1)^2}}$

Equation of the S.D. between them is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = \begin{bmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{bmatrix}$$

Where, *l, m, n* are D.C 'S of the line of shortest distance

Example : 1

Find the D.C's of the straight line

x= 2y =3z

Solution:

The given line can be written as

$$\frac{x}{1} = \frac{y}{1/2} = \frac{z}{1/3}$$

$\therefore$ the D.R.'s of the line are $\left(1, \frac{1}{2}, \frac{1}{3}\right)$

$\therefore$ the D.C.'s of the line are $\left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$.

<u>Example</u> : 2

Find the equation of the plane that contains the parallel lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$

and $\frac{x-3}{1} = \frac{y+2}{2} = \frac{z+4}{3}$

<u>Solution:</u> The given lines are $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ …(1)

$$\frac{x-3}{1} = \frac{y+2}{2} = \frac{z+4}{3} \qquad \text{…(2)}$$

Any plane containing the line (1) is of the form

a(x-1) + b (y-2) + c (z-3) = 0 …(3)

The lines (1) and (2) are perpendicular to the normal of the plane

D.R.'S of the line (1) are (1,2,3)

D.R.'S of the line (2) are (1,2,3)

D.R.'S of Normal line of the plane are (*a,b,c*)

$\therefore$ by perpendicularity condition

a+2b+3c = 0 …(4)

Since the required plane also has the point (3,-2,-4)

$\therefore$ *a(+3-1) + b(-2-2) + c(-4-3) = o*

2a-4b-7c = 0 …(5)

Solving 4 and 5

$$\frac{a}{-14+12} = \frac{-b}{-6-7} = \frac{c}{-4-4} = k$$

$$\frac{a}{-2} = \frac{b}{13} = \frac{c}{-8} = k$$

$\therefore$ *a = -2k , b = 13k , c = -8k*

$\therefore$ *-2k (x-1) + 13 k (y-2) – 8k (z-3) = 0*

$2x-13y+8z = 0$ is the required plane.

Example : 3

Find the equation of the plane which contains the line $x = \frac{y+3}{2} = \frac{z+5}{3}$ and which is perpendicular to plane $2x+7y-3z=1$

Solution:

The equation of the plane containing the line $x = \frac{y+3}{2} = \frac{z+5}{3}$ …(1)

is given by $a(x-0) + b(y+3) + c(z+5) = 0$ …(2)

The plane (2) is perpendicular to the plane $2x+7y-3z= 1$

$\therefore$ By perpendicularity condition $2a+7b-3c = 0$ …(3)

The line (1) lies in the plane (2)

$\therefore$ Normal to the plane is perpendicular to the line (1)

D.R'S of the line (1) are (1,2,3)

$\therefore$ By perpendicularity condition $a+2b+3c=0$ …(4)

solving (3) and (4)

$$\frac{a}{-9} = \frac{-b}{-3} = \frac{c}{1} = k$$

$a = -9K, \quad b=3K, \quad c=K$

$\therefore$ The required plane is $-9k(x) + 3k(y+3) + k(z+5) = 0$

i.e., $9x-3y-z-14=0$

Example : 4

Find the value of k so that the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$ may be perpendicular to each other.

Solution:

D.R.'S of the given lines are (-3,2k,2) and (3k,1,-5)

Since the lines are perpendicular, $-3(3k) + 2k(1) + 2(-5) = 0$

$$k= \frac{-10}{7}$$

Example : 5

Find the shortest distance and the equation of the line of shortest distance of the lines

$$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}\ \ and\ \ \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$

Solution:

Given lines

AB= $\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}=r_1$ …(1)

CD= $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}=r_2$ …(2)

Any point P on the line AB is ($3r_1+8,\ -16r_1-9,\ 7r_1+10$)

Any point Q on the line CD is ($3r_2+15,\ 8r_2+29,\ -5r_2+5$)

Let PQ be the shortest distance which perpendicular to AB and CD

D.R.'S of the line AB are (3, -16,7)

D.R.'S of the line CD are (3,8,-5)

D.R.'S of the line PQ are

($3r_2-3r_1+7,\ 8r_2+16r_1+38,\ -5r_2-7r_1-5$)

$\therefore$ PQ $\perp^r$ AB

$3(3r_2-3r_1+7)-16(8r_2+16r_1+38)+7(-5r_2-7r_1-5)=0$

$77r_2+157r_1=-311$ …(3)

$\therefore$PQ $\perp^r$ CD

$3(3r_2-3r_1+7)+8(8r_2+16r_1+38)-5(-5r_2-7r_1-5)=0$

$7r_2+11r_1=-25$ …(4)

Solving (3) & (4)

$r_1=-1,\ r_2=-2$

$\therefore$P(5,7,3) & Q (9, 13, 15)

$\therefore$PQ= $\sqrt{(9-5)^2+(13-7)^2+(15-3)^2}$

=14

The equation of the line of the shortest distance is

$$\frac{x-5}{2}=\frac{y-7}{3}=\frac{z-3}{6}$$

Sphere and Circle:

A sphere is the locus of a point, which moves in space such that its distance from a fixed point is constant.

The equation of the sphere whose centre is (a, b, c) and radius is r, is given by

$$(x-a)^2 + (y-b^2) + (z-c)^2 = r^2$$

The standard form of equation to a sphere is given by $x^2+y^2+z^2 +2ux +2vy+2wz + d = 0$

whose centre is $(-u, -v, -w)$ and radius is r = $\sqrt{u^2 + v^2 + w^2 - d}$

The equation of the sphere joining the end point (x_1, y_1, z_1) and (x_2, y_2, z_2) of its diameter is given by $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$

Tangent line and Tangent Plane:

When a straight line intersects a sphere at two coincident points or when it touches a sphere at a point P, the line is called a tangent line of the sphere at P and is perpendicular to the radius of the sphere through P. There are many such tangent lines lie on plane through P, which is perpendicular to the radius of the sphere through P. This plane is called the tangent plane of the sphere at p.

Equation of the tangent plane to the sphere $x^2+y^2+z^2 + 2ux +2vy+2wz + d = 0$

at P (x_1, y_1, z_1) is $xx_1+yy_1+zz_1+u(x+x_1) + v(y+y_1) + w(z+z_1) + d = 0$

Plane Section of a sphere:

The points common to a sphere of radius r and a plane at a distance h from its centre lie on a circle of radius $\sqrt{r^2 - h^2}$ with its centre at the foot of the perpendicular from centre of the sphere to the plane. This circle is a great circle, if the plane passes through the centre of the sphere.

Note:

The intersection of two sphere is also a circle. The circle of intersection in this case, is jointly represented by the spheres S_1 and S_2 as $S_1 – S_2 = 0$

Equation of a sphere through a circle:

Let the given circle be represented jointly by the equations

S : $x^2 + y^2 + z^2\ 2ux + 2vy + 2wz + d = 0$

and $P: ax + by + cz + d = 0$

then $S + \lambda P = 0$ gives the equation of the sphere through the circle S=0,P=0

Also, $S_1 + \lambda S_2 = 0$, gives the equation of the sphere through the circle $S_1=0, S_2=0$

Orthogonal Spheres:

Two orthogonal spheres are such that the tangent planes to the spheres at any common point are at right angles.

If the two spheres

$x^2+y^2+z^2 + 2u_1 x + 2v_1 y + 2w_1 z + d_1 = 0$ and

$x^2+y^2+z^2 + 2u_2 x + 2v_2 y + 2w_2 z + d_2 = 0$ are orthogonal, then

$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2.$

Note:

1. Two spheres touch each other if there is a common point of intersection of the spheres at which the tangent planes of the spheres coincide.
2. Two spheres touch one another externally, if the distance between their centres is the same as the sum of their radii.
3. Two spheres touch one another internally, if the distance between their centres is the same as the difference of their radii.

Example : 1

Find the equation of the sphere having the centre (3,-2,2) and passing through the point (-1,3,4).

Solution :

$$(\text{Radius})^2 = [\text{Distance from } (3,-2,2) \text{ to } (-1,3,4)]^2$$

$$r^2 = (3+1)^2 + (-2-3)^2 + (2-4)^2$$

$$= 45$$

$\therefore$ Equation of the sphere is

$$(x-3)^2 + (y+2)^2 + (z-2)^2 = 45$$

$$\therefore x^2+y^2+z^2-6x+4y-4z-28 = 0$$

Example :2

Find the equation of the sphere which has the line joining the points (2,7,5) and (8,-5,1) as diameter

Solution:

The equation of the sphere is $(x-2)(x-8) + (y-7)(y+5) + (z-5)(z-1) = 0$

i.e., $x^2+ y^2+z^2-10x-2y-6z-14 = 0.$

Example :3

Find the equation of the sphere whose centre is (6,-1,2) and which touches the plane $2x-y+2z=2$

Solution:

Radius of the sphere is the perpendicular distance of (6, -1,2) from the plane

$$\therefore \text{Radius} = \pm \frac{2(6)-(-1)+2(2)-2}{\sqrt{2^2+1^2+2^2}}$$

$=5$

$\therefore$ The equation of the sphere is $(x-6)^2 + (y+1)^2 + (z-2)^2 = 5^2$

$$x^2+y^2+z^2-12x+2y-4z+16 = 0$$

Example : 4

Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z = 3$ and Find also the point of contact.

Solution:

Given that $x^2+y^2+z^2-2x-4y+2z = 0$ …(1)

$2x-2y+z+12 = 0$ …(2)

Centre of the sphere (1) is (1,2,-1)

Radius is $\sqrt{1+4+1+3} = 3$

Length of perpendicular from (1,2,-1) to the plane (2) is

$$= \frac{|2-4-1+12|}{\sqrt{4+4+1}}$$

$=3$

i.e., Length of perpendicular from (1,2,-1) to the plane (2) = radius of the sphere

$\therefore$ The plane (2) touches the sphere (1).

Let P be the point of contact. Then CP is normal to the plane (2)

i.e., Normal line of the plane (2) and the line CP are parallel.

$\therefore$ Their D.R.'S are proportional

$\therefore$ D.R.'S of CP are (2,-2,1)

Also CP passes through C(1,2,-1)

Hence equation of CP is

$$\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1} = r$$

any point on the line CP is $(2r+1, -2r+2, r-1)$ if this point lies on the plane (2), it will be the point of contact. Then

$2(2r+1) - 2(-2r+2) + (r-1) + 12 = 0$ which gives r = -1.

$\therefore$ P the point of contact is (-1,4,-2)

Example :5

Find the equation of tangent planes of the spheres $x^2+y^2+z^2-4x-4y-4z+10 = 0$ which are parallel to the plane $x - z = 0$

Solution:

Let (x_1,y_1,z_1) be the point on the sphere at which the tangent plane is drawn. The equation of the tangent plane at (x_1, y_1, z_1) is $xx_1+yy_1+zz_1 - 2(x+x_1) - 2(y+y_1) - 2(z+z_1) + 10 = 0$

i.e., $(x_1 - 2)x + (y_1 - 2)y + (z_1 - 2)z - 2x_1 - 2y_1 - 2z_1 + 10 = 0$ ...(1)

This plane is parallel to $x - z = 0$

$$\therefore \frac{x_1 - 2}{1} = \frac{y_1 - 2}{0} = \frac{z_1 - 2}{-1} = k \text{ (say)}$$

$\therefore x_1 = k+2, \quad y_1 = 2, \; z_1 = -k + 2$ ...(2)

Since (x_1,y_1,z_1) lies on the sphere, we have

$(k+2)^2 + 2^2 + (-k+2)^2 - 4(k+2) - 8 - 4(-k+2) + 10 = 0$

$\therefore k^2 - 1 = 0$

$\therefore k = \pm 1$

from (2), the points are (3,2,1) and (1,2,3)

$\therefore$ The equation of the tangent planes are $x-z-2 = 0$ and $-x + z - 2 = 0$

Example : 6

Find the centre and radius of the circle given by $x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 = 0$

and $x+2y+2z-20 = 0$

Solution:

Given, $x^2+y^2+z^2-2x-4y-6z-2=0$ ...(1)

$x+2y+2z-20=0$ ...(2)

Centre and Radius of the sphere (1) is

C (1,2,3) and Radius R= $\sqrt{1+4+9+2}$

R = 4

C_2, the centre of the given circle is the foot of the perpendicular from C_1 on the given plane (2)

$\therefore C_1C_2$ is normal to the plane (2) and parallel to the normal line of the plane (2)

$\therefore$ D.R.'S of C_1C_2 are (1,2,2)

$\therefore$ Equation of C_1C_2 are

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{2} = r$$

Any point on the line C_1C_2 is $(r+1, 2r+2,2r+3)$ …(3)

If the co-ordinates in (3) represent C_2, they should satisfy the plane (2)

$\therefore$ C_2 lies on the plane (2)

$\therefore (r+1) +2 (2r +2) + 2(2r +3) = 20$ …(4)

r= 1

$\therefore$ C_2 is (2,4,5)

$\therefore$ length of $C_1C_2 = \sqrt{(2-1)^2+(4-2)^2+(5-3)^2} = 3$

if r is the radius of the circle $r^2 = R^2 - (C_1C_2)^2$

$= 16-9 = 7$

$r = \sqrt{7}$

$\therefore$ centre and radius of circle are (2,4,5) and $\sqrt{7}$.

Example : 7

Find the equation of the sphere which pass through the circle $x^2+y^2+z^2-2x+2y+4z-3 = 0$; $2x+y+z-4 = 0$ and touch the plane $3x+4y-14 = 0$.

Solution:

Given circle is the intersection of the sphere and the plane

$S: x^2+y^2+z^2-2x+2y+4z-3 = 0$ … (1)

$P: 2x+y+z-4 = 0$ …(2)

Then $S+\lambda P = 0$ represents a sphere passing through the circle determined by (1) & (2)

$\therefore S+\lambda P = (x^2+y^2+z^2-2x+2y+4z-3)+ \lambda (2x+y+z-4) = 0$

i.e., $x^2+y^2+z^2 -2x (1-\lambda) + y (2+\lambda) + z (4+\lambda) - (3+4\lambda) = 0$ … (3)

Centre and radius of the sphere (3) are

$$\left(1-\lambda, \frac{-(2+\lambda)}{2}, \frac{-(4+\lambda)}{2}\right)$$

radius = $\sqrt{(1-\lambda)^2+\left(\frac{2+\lambda}{2}\right)^2+\left(\frac{4+\lambda}{2}\right)^2+(3+4\lambda)}$

Since the sphere touches the plane 3x+4y-14 = 0 the perpendicular distance from the centre of the sphere to this plane is equal to the radius of the sphere

$$\therefore \frac{3(1-\lambda)-2(2+\lambda)-14}{\sqrt{3^2+4^2}} = \sqrt{(1-\lambda)^2+\left(\frac{2+\lambda}{2}\right)^2+\left(\frac{4+\lambda}{2}\right)^2+(3+4\lambda)}$$

i.e., $2\lambda^2 - 4\lambda = 0$

$\lambda(\lambda - 2) = 0$

$\lambda = 0, \lambda = 2$

using these values in (3)

$x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0$

$x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$

<u>Example</u> :8 Find the equation of the sphere having the circle $x^2 + y^2 + z^2 + 10y - 4z - 8 = 0$; $x+y+z=3$ as a great circle.

<u>Solution:</u>

The equation of any sphere passing through the given circle is of the form

$x^2 + y^2 + z^2 + 10y - 4z - 8 + \lambda (x+y+z-3) = 0$

i.e., $x^2 + y^2 + z^2 + \lambda x + (\lambda+10) + (\lambda - 4) z - (3\lambda + 8) = 0$ …(1)

If the given circle is a great circle of sphere (1) centre of sphere (1) should be on the plane $x+y+z-3= 0$ in which the given circle lies.

$\therefore$ Centre of sphere (1) is $\left(-\frac{\lambda}{2}, \frac{-(\lambda+10)}{2}, \frac{-(\lambda-4)}{2}\right)$

$$\therefore -\frac{\lambda}{2} - \frac{1}{2}(\lambda+10) - \frac{1}{2}(\lambda-4) = 3$$

$\because \lambda = -4$

$\therefore$ Equation of the required sphere is $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$

<u>Example</u> :9

Find the equation of the sphere through the circle

$x^2 + y^2 + z^2 - x + 9y - 5z - 5 = 0$

$x^2 + y^2 + z^2 + 10y - 4z - 8 = 0$ as a great circle

<u>Solution:</u>

Given

$S_1; x^2 + y^2 + z^2 - x + 9y - 5z - 5 = 0$

$S_2 : x^2 + y^2 + z^2 + 10y - 4z - 8 = 0$ …(1)

$S_1 - S_2 = 0$ represent a plane

Hence $S_1=0$, $S_1 - S_2 = 0$ determines a circle

$\therefore$ $S_1 + \lambda (S_1 - S_2) = 0$ represent the equation of the sphere passing through the circle determines by $S_1 = 0$ and $S_1 - S_2 = 0$

$\therefore (x^2 + y^2 + z^2 - x + 9y - 5z - 5) + \lambda (x+y+z-3) = 0$

$x^2 + y^2 + z^2 + (\lambda-1)x + (9+\lambda) y + (\lambda-5) z - 5 - 3\lambda = 0$ … (1)

Its centre is $\left(\frac{1-\lambda}{2}, -\left(\frac{9+\lambda}{2}\right), \left(\frac{5-\lambda}{2}\right)\right)$

Since the cross section is a great circle this centres lies on the plane $x+y+z - 3 = 0$ (1)

$$\therefore \quad \frac{1-\lambda}{2} - \left(\frac{9+\lambda}{2}\right) + \frac{5-\lambda}{2} - 3 = 0$$

$\therefore$ $\lambda = -3$

$\therefore$ From (1), Required sphere is $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$

Example :10

Prove that the two spheres $x^2 + y^2 + z^2 + 4y - 4z = 0$ and $x^2 + y^2 + z^2 + 10x + 2z + 10 = 0$ touch each other and find the point of contact.

Solution :

Given

$x^2 + y^2 + z^2 - 2x + 4y - 4z = 0$ …(1)

$x^2 + y^2 + z^2 + 10x + 2z + 10 = 0$ …(2)

Centre and radius of sphere (1) is C_1 (1,-2,2) and $r_1 = 3$

Centre and radius of sphere (2) is C_2 (-5,0,-1) and $r_2 = 4$

$C_1C_2 = \sqrt{(-5-1)^2 + (0+2)^2 + (-1-2)^2} = 7$

We see that $C_1C_2 = r_1 + r_2$

$\therefore$ The two sphere touch each other externally. Also the point of contact P divides C_1C_2 internally in the ratio $r_1 : r_2$ i.e., 3:4

Hence P $\left(\frac{-15+4}{3+4}, \frac{0-8}{3+4}, \frac{-3+8}{3+4}\right)$

i.e., P is $\left(\frac{-11}{7}, \frac{-8}{7}, \frac{5}{7}\right)$

Example : 11

Prove that the two sphere $x^2 + y^2 + z^2 + 6y + 2z + 8 = 0$

and $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$ intersect each other orthogonally.

Solution:

From the equation of spheres

$u_1 = 0, \quad v_1 = 3, \quad w_1 = 1, \quad d_1 = 8\ ; \quad u_2 = 3, v_2 = 4, \quad w_2 = 2, \quad d_2 = 20$

$\therefore 2u_1u_2 + 2v_1v_2 + 2w_1w_2 - (d_1+d_2)$

$= 0 + 24 + 4 - (8+20) = 0$

Hence the two spheres intersect orthogonally.