SUBJECT NAME: ENGINEERING MATHEMATICS II

(COMMON TO BIO GROUPS)

SUBJECT CODE: SMT1106

COURSE MATERIAL

UNIT- IV VECTOR CALCULUS

Definitions

Scalars

The quantities which have only magnitude and are not related to any direction in space are called scalars. Examples of scalars are (i) mass of a particle (ii) pressure in the atmosphere (iii) temperature of a heated body (iv) speed of a train.

Vectors

The quantities which have both magnitude and direction are called vectors.

Examples of vectors are (i) the gravitational force on a particle in space (ii) the velocity at any point in a moving fluid.

Scalar point function

If to each point p(x,y,z) of a region R in space there corresponds a unique scalar f(p) then f is called a scalar point function.

Example

The temperature distribution in a heated body, density of a body and potential due to a gravity.

Vector point function

If to each point p(x,y,z) of a region R in space there corresponds a unique vector $\vec{f}(p)$ then $\vec{f}$ is called a vector point function.

Example

The velocity of a moving fluid, gravitational force.

Scalar and vector fields

When a point function is defined at every point of space or a portion of space, then we say that a field is defined. The field is termed as a scalar field or vector field as the point function is a scalar point function or a vector point function respectively.

Vector Differential Operator (∇)

The vector differential operator Del, denoted by ∇ is defined as

$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

Gradient of a scalar point function

Let $\phi(x,y,z)$ be a scalar point function defined in a region R of space. Then the vector point function given by $\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi$

$= \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$ is defined as the gradient of ϕ and denoted by

gradϕ

Directional Derivative (D.D)

The directional derivative of a scalar point function ϕ at point (x,y,z) in the direction of a vector $\vec{a}$ is given by D.D = $\nabla\phi.\frac{\vec{a}}{|\vec{a}|}$ (or) D.D = $\nabla\phi\,.\,\hat{a}$

The unit normal vector

The unit vector normal to the surface $\phi(x,y,z)$ = c is given by $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Angle between two surfaces

Angle between the surfaces $\phi_1(x,y,z) = c_1$ and $\phi_2(x,y,z) = c_2$ is given by $\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

Problems

1) Find $\nabla\phi$ if $\phi(x,y,z) = xy - y^2 z$ at the point (1,1,1)

Solution:

$$\nabla\phi = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(xy - y^2 z)$$

$$= \vec{i}\frac{\partial}{\partial x}(xy - y^2 z) + \vec{j}\frac{\partial}{\partial y}(xy - y^2 z) + \vec{k}\frac{\partial}{\partial z}(xy - y^2 z)$$

$$= y\vec{i} + (x - 2yz)\vec{j} - y^2\vec{k}$$

$$\therefore \nabla\phi == y\vec{i} + (x - 2yz)\vec{j} - y^2\vec{k}\,.$$

At (1,1,1), $\nabla\phi = \vec{i}(1) + \vec{j}(1 - (2)(1)(1)) - \vec{k}(1)^2$

$$= \vec{i} - \vec{j} - \vec{k}$$

2) Find $\nabla\phi$ if $\phi(x,y,z) = x^2 y + 2xz^2 - 8$ at the point (1,0,1)

Solution:

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(x^2 y + 2xz^2 - 8)$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2 y + 2xz^2 - 8) + \vec{j}\frac{\partial}{\partial y}(x^2 y + 2xz^2 - 8) + \vec{k}\frac{\partial}{\partial z}(x^2 y + 2xz^2 - 8)$$

$$= (2xy + 2z^2)\vec{i} + (x^2)\vec{j} + 4xz\vec{k}$$

At (1,0,1), $\nabla\phi = \vec{i}(2(1)(0) + 2(1^2)) + \vec{j}(1^2) + \vec{k}4(1)(1)$

$$= 2\vec{i} + \vec{j} + 4\vec{k}$$

3) Find the unit normal vector to the surface $\phi(x,y,z) = x^2 yz^3$

at the point (1,1,1)

Solution:

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(x^2 yz^3) = \vec{i}\frac{\partial}{\partial x}(x^2 yz^3) + \vec{j}\frac{\partial}{\partial y}(x^2 yz^3) + \vec{k}\frac{\partial}{\partial z}(x^2 yz^3)$$

$= 2xyz^3\vec{i} + x^2 z^3 \vec{j} + 3x^2 yz^2 \vec{k}$

At(1,1,1), $\nabla\phi = \vec{i}\,2(1)(1)(1) + \vec{j}(1^2)(1^3) + \vec{k}\,3(1^2)(1)(1^2)$

$= 2\vec{i} + \vec{j} + 3\vec{k}$

$|\nabla\phi| = \sqrt{2^2 + 1^2 + 3^2}$

$= \sqrt{14}$

Unit normal to the surface is $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{2\vec{i} + \vec{j} + 3\vec{k}}{\sqrt{14}}$$

4) Find the unit normal vector to the surface $\phi(x, y, z) = x^2 + y^2 - z$

at the point (1,-1,-2)

Solution:

$\nabla\phi = (\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z})\phi$

$\nabla\phi = (\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z})(x^2 + y^2 - z)$

$= \vec{i}\dfrac{\partial}{\partial x}(x^2 + y^2 - z) + \vec{j}\dfrac{\partial}{\partial y}(x^2 + y^2 - z) + \vec{k}\dfrac{\partial}{\partial z}(x^2 + y^2 - z)$

$= 2x\vec{i} + 2y\vec{j} - \vec{k}$

At (1,-1,-2), $\nabla\phi = \vec{i}\,2(1) + \vec{j}\,2(-1) - \vec{k}$

$= 2\vec{i} - 2\vec{j} - \vec{k}$

$|\nabla\phi| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$

Unit normal to the surface is $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{2\vec{i} - 2\vec{j} - \vec{k}}{3}$$

5) Find the angle between the surfaces xyz and x^3yz at the point (1,1,-2)

Solution:

Given the surface $\phi_1(x,y,z) = xyz$

$$\nabla\phi_1 = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi_1$$

$$\nabla\phi_1 = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(xyz)$$

$$= \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\frac{\partial}{\partial z}(xyz)$$

$$= yz\vec{i} + xz\vec{j} + xy\vec{k}$$

At(1,1,-2), $\nabla\phi_1 = \vec{i}(1)(-2) + \vec{j}(1)(-2) + (1)(1)\vec{k} = -2\vec{i} - 2\vec{j} + \vec{k}$

$$|\nabla\phi_1| = \sqrt{(-2)^2 + (-2)^2 + 1^2}$$

= 3

Given the surface $\phi_2(x,y,z) = x^3yz$

$$\nabla\phi_2 = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi_2$$

$$\nabla\phi_2 = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(x^3yz)$$

$$= \vec{i}\frac{\partial}{\partial x}(x^3yz) + \vec{j}\frac{\partial}{\partial y}(x^3yz) + \vec{k}\frac{\partial}{\partial z}(x^3yz)$$

$$= 3x^2yz\vec{i} + x^3z\vec{j} + x^3y\vec{k}$$

At (1,1,-2), $\nabla\phi_2 = \vec{i}\,3(1^2)(1)(-2) + \vec{j}(1^3)(-2) + (1^3)(1)\vec{k} = -6\vec{i} - 2\vec{j} + \vec{k}$

$$|\nabla\phi_2| = \sqrt{(-6)^2 + (-2)^2 + 1^2} = \sqrt{41}$$

Angle between the surfaces is given by $\cos\theta = \dfrac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$$= \frac{(-2\vec{i} - 2\vec{j} + \vec{k}).(-6\vec{i} - 2\vec{j} + \vec{k})}{3\sqrt{41}}$$

$$= \frac{12+4+1}{3\sqrt{41}} = \frac{17}{3\sqrt{41}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{17}{3\sqrt{41}}\right)$$

6) Find the angle between the normal to the surface xy - z² at the point (1,4,-2) and (1,2,3)

Solution:

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(xy - z^2) = \vec{i}\frac{\partial}{\partial x}(xy - z^2) + \vec{j}\frac{\partial}{\partial y}(xy - z^2) + \vec{k}\frac{\partial}{\partial z}(xy - z^2)$$

$$= y\vec{i} + x\vec{j} - 2z\vec{k}$$

At (1,4,-2), $\nabla\phi_1 = \vec{i}(4) + \vec{j}(1) - 2(-2)\vec{k}$

$= 4\vec{i} + \vec{j} + 4\vec{k}$

$$|\nabla\phi| = \sqrt{4^2 + 1^2 + 4^2}$$

$= \sqrt{33}$

At (1,2,3), $\nabla\phi_2 = \vec{i}(2) + \vec{j}(1) - 2(3)\vec{k}$

$= 2\vec{i} + \vec{j} - 6\vec{k}$

$$|\nabla\phi| = \sqrt{2^2 + 1^2 + (-6)^2}$$

$= \sqrt{41}$

Angle between the surfaces is given by $\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$$= \frac{(4\vec{i} + \vec{j} + 4\vec{k}).(2\vec{i} + \vec{j} - 6\vec{k})}{\sqrt{33}\sqrt{41}}$$

$$= \frac{8+1-24}{\sqrt{33}\sqrt{41}} = \frac{-15}{\sqrt{33}\sqrt{41}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-15}{\sqrt{33}\sqrt{41}}\right)$$

7) Find the directional derivative of $\phi(x,y,z) = xy^2 + yz^3$ at the point (2,-1,1) in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$

Solution:

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(xy^2 + yz^3) = \vec{i}\frac{\partial}{\partial x}(xy^2 + yz^3) + \vec{j}\frac{\partial}{\partial y}(xy^2 + yz^3) + \vec{k}\frac{\partial}{\partial z}(xy^2 + yz^3)$$

$$= y^2\vec{i} + (2xy + z^3)\vec{j} + 3yz^2\vec{k}$$

At (2,-1,1), $\nabla\phi = \vec{i}(-1^2) + \vec{j}(2(2)(-1) + 1^3) + 3(-1)(1^2)\vec{k}$

$= \vec{i} - 3\vec{j} - 3\vec{k}$

To find the directional derivative of ϕ in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$ find the unit vector along the direction

$$\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k} \Rightarrow |\vec{a}| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

Directional derivative of ϕ in the direction $\vec{a}$ at the point (2,-1,1) = $\nabla\phi.\frac{\vec{a}}{|\vec{a}|}$

$$= (\vec{i} - 3\vec{j} - 3\vec{k}).\frac{(\vec{i} + 2\vec{j} + 2\vec{k})}{3}$$

$$= \frac{1-6-6}{3} = \frac{-11}{3} \text{ units.}$$

8) Find the directional derivative of $\phi(x,y,z) = xyz + yz^2$ at the point (1,1,1) in the direction of $\vec{i} + \vec{j} + \vec{k}$

Solution:

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\phi$$

$$\nabla\phi = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})(xyz + yz^2) = \vec{i}\frac{\partial}{\partial x}(xyz + yz^2) + \vec{j}\frac{\partial}{\partial y}(xyz + yz^2) + \vec{k}\frac{\partial}{\partial z}(xyz + yz^2)$$

$$= yz\vec{i} + (xz + z^2)\vec{j} + (xy + 2yz)\vec{k}$$

At (1,1,1), $\nabla\phi = \vec{i}(1)(1) + \vec{j}((1)(1) + 1^2) + ((1)(1) + 2(1)(1))\vec{k}$

$$= \vec{i} + 2\vec{j} + 3\vec{k}$$

To find the directional derivative of ϕ in the direction of the vector $\vec{i} + \vec{j} + \vec{k}$ find the unit vector along the direction

$$\vec{a} = \vec{i} + \vec{j} + \vec{k} \Rightarrow |\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Directional derivative of ϕ in the direction $\vec{a}$ at the point (1,1,1) = $\nabla\phi.\frac{\vec{a}}{|\vec{a}|}$

$$= (\vec{i} + 2\vec{j} + 3\vec{k}).\frac{(\vec{i} + \vec{j} + \vec{k})}{\sqrt{3}}$$

$$= \frac{1 + 2 + 3}{\sqrt{3}} = \frac{6}{\sqrt{3}} \text{ units.}$$

Divergence of a differentiable vector point function $\vec{F}$

The divergence of a differentiable vector point function $\vec{F}$ is denoted by div $\vec{F}$ and is defined by

div $\vec{F}$ = $\nabla \cdot \vec{F} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \cdot \vec{F}$

$$= (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \cdot (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}) \quad \vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl of a vector point function

The curl of a differentiable vector point function $\vec{F}$ is denoted by curl $\vec{F}$ and is defined by

Curl $\vec{F} = \nabla \times \vec{F} = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}) \times \vec{F}$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

Curl $\vec{F}$ = $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

Vector Identities

Let ϕ be a scalar point function and $\vec{U}$ and $\vec{V}$ be vector point functions. Then

(1) $\nabla \cdot (\vec{U} \pm \vec{V}) = \nabla \cdot \vec{U} \pm \nabla \cdot \vec{V}$
(2) $\nabla \times (\vec{U} \pm \vec{V}) = \nabla \times \vec{U} \pm \nabla \times \vec{V}$
(3) $\nabla \cdot (\phi\vec{U}) = \nabla\phi \cdot \vec{U} + \phi\nabla \cdot \vec{U}$
(4) $\nabla \times (\phi\vec{U}) = \nabla\phi \times \vec{U} + \phi\nabla \times \vec{U}$
(5) $\nabla \cdot (\vec{U} \times \vec{V}) = \vec{V} \cdot (\nabla \times \vec{U}) - \vec{U} \cdot (\nabla \times \vec{V})$
(6) $\nabla \times (\vec{U} \times \vec{V}) = (\nabla \cdot \vec{V})\vec{U} - (\nabla \cdot \vec{U})\vec{V} + \vec{U}(\vec{V} \cdot \nabla) - \vec{V}(\vec{U} \cdot \nabla)$
(7) $\nabla(\vec{U} \cdot \vec{V}) = (\nabla \cdot \vec{V})\vec{U} + (\nabla \cdot \vec{U})\vec{V} + \vec{U} \times (\nabla \times \vec{V}) - (\nabla \times \vec{U}) \times \vec{V}$

Solenoidal and Irrotational vectors

A vector point function is solenoidal if div $\vec{F}$ = 0 and it is irrotational if curl $\vec{F}$ = 0.

Note:

If $\vec{F}$ is irrotational, then there exists a scalar function called Scalar Potential ϕ such that

$\vec{F} = \nabla\phi$

Problems

1) Find div $\vec{r}$ and curl $\vec{r}$ if $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Solution:

div $\vec{r} = \nabla \cdot \vec{r} = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}) \cdot \vec{r}$

$$=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot(x\vec{i}+y\vec{j}+z\vec{k})$$

$$=\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}=3.$$

Curl $\vec{r}$ = $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \vec{i}(0-0)-\vec{j}(0-0)+\vec{k}(0-0)=0.$$

2) Find the divergence and curl of the vector $\vec{V}=xyz\vec{i}+3xy^2\vec{j}+(xz^2-y^2z)\vec{k}$ at the point (1,-1,1)

Solution:

Given $\vec{V}=xyz\vec{i}+3xy^2\vec{j}+(xz^2-y^2z)\vec{k}$

div$\vec{V}$ = $\nabla\cdot\vec{V}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot\vec{V}$

$$=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot(xyz\vec{i}+3xy^2\vec{j}+(xz^2-y^2z)\vec{k})$$

$$=\frac{\partial(xyz)}{\partial x}+\frac{\partial(3xy^2)}{\partial y}+\frac{\partial(xz^2-y^2z)}{\partial z}$$

= $yz+6xy+2xz-y^2$

At (1,-1,1), $\nabla\cdot\vec{V}=(-1).1+6(1)(-1)+2(1)(1)-(-1)^2$

= -1-6+2-1 = -6.

Curl $\vec{V}$ = $\nabla\times\vec{V}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\times\vec{V}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3xy^2 & xz^2-y^2z \end{vmatrix}$$

$= \vec{i}\,(\frac{\partial}{\partial y}(xz^2 - y^2 z) - \frac{\partial}{\partial z}(3xy^2)) - \vec{j}(\frac{\partial}{\partial x}(xz^2 - y^2 z) - \frac{\partial}{\partial z}(xyz)) + \vec{k}(\frac{\partial}{\partial x}(3xy^2) - \frac{\partial}{\partial y}(xyz))$.

$= \vec{i}\,(-2yz) - \vec{j}(z^2 - yx) + \vec{k}(3y^2 - xz)$.

At (1,-1,1), $\nabla \times \vec{V} = \vec{i}\,(-2(-1)(1)) - \vec{j}(1^2 - (-1)(1)) + \vec{k}((3(-1)^2 - 1(1))$

$= 2\vec{i} - 2\vec{j} + 2\vec{k}$

3) Find the constants a, b, c so that $\vec{F} = (x + 2y + ax)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ **is irrotational**.

Solution:

Given $\nabla \times \vec{F} = 0$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x + 2y + az) & (bx - 3y - z) & (4x + cy + 2z) \end{vmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \vec{i}\,(\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z)) - \vec{j}(\frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + az) + \\ \vec{k}(\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + az) \end{bmatrix} = 0 .$$

$\Rightarrow \vec{i}\,(c + 1) - \vec{j}(4 - a) + \vec{k}(b - 2) = 0$.

c+1 = 0, 4-a = 0, b-2 = 0

Hence c = -1, a = 4, b = 2.

4) Prove that $\vec{F} = (2x + yz)\vec{i} + (4y + zx)\vec{j} - (6z - xy)\vec{k}$ **is both solenoidal**

and irrotational.

Solution:

$\nabla \cdot \vec{F} = (\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}) \cdot \vec{V}$

$$=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot((2x+yz)\vec{i}+(4y+zx)\vec{j}-(6z-xy)\vec{k})$$

$$=\frac{\partial(2x+yz)}{\partial x}+\frac{\partial(4y+zx)}{\partial y}-\frac{\partial(6z-xy)}{\partial z}$$

= 2+4-6 = 0 for all points (x,y,z)

$\therefore \vec{F}$ is solenoidal vector.

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x+yz & 4y+zx & -(6z-xy)\end{vmatrix}$$

$$=\begin{bmatrix}\vec{i}(\frac{\partial}{\partial y}(-(6z-xy))-\frac{\partial}{\partial z}(4y+zx))-\vec{j}(\frac{\partial}{\partial x}(-(6z-xy))-\frac{\partial}{\partial z}(2x+yz)+\\ \vec{k}(\frac{\partial}{\partial x}(4y+zx)-\frac{\partial}{\partial y}(2x+yz)\end{bmatrix}$$

$\Rightarrow \vec{i}(x-x)-\vec{j}(y-y)+\vec{k}(z-z)=0$ for all points (x,y,z)

$\therefore \vec{F}$ is irrotational vector.

5) Prove that $\vec{F}=(y^2-z^2+3yz-2x)\vec{i}+(3xz+2xy)\vec{j}+(3xy-2xz+2z)\vec{k}$ **is both solenoidal and irrotational and find its scalar potential**.

Solution:

$$\nabla\cdot\vec{F}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot\vec{F}$$

$$=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot((y^2-z^2+3yz-2x)\vec{i}+(3xz+2xy)\vec{j}+(3xy-2xz+2z)\vec{k})$$

$$=\frac{\partial(y^2-z^2+3yz-2x)}{\partial x}+\frac{\partial(3xz+2xy)}{\partial y}+\frac{\partial(3xy-2xz+2z)}{\partial z}$$

= -2+2x-2x+2 = 0 for all points (x,y,z)

$\therefore \vec{F}$ is solenoidal vector.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xz + 2xy) & (3xy - 2xz + 2z) \end{vmatrix}$$

$$= \begin{bmatrix} \vec{i}\left(\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right) - \vec{j}\left(\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x) + \right. \\ \vec{k}\left(\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right) \end{bmatrix}$$

$\Rightarrow \vec{i}(3x - 3x) - \vec{j}(3y - 2z + 2z - 3y) + \vec{k}(3z + 2y - 2y - 3z) = 0$ for all points (x,y,z)

$\therefore \vec{F}$ is irrotational vector.

Since $\vec{F}$ is irrotational, $\vec{F} = \nabla\phi$

$$\Rightarrow (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k} = \vec{i}\,\frac{\partial \phi}{\partial x} + \vec{j}\,\frac{\partial \phi}{\partial y} + \vec{k}\,\frac{\partial \phi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$, we get

$$\frac{\partial \phi}{\partial x} = y^2 - z^2 + 3yz - 2x \quad \text{...(1)}$$

$$\frac{\partial \phi}{\partial y} = 3xz + 2xy \quad \text{..(2)}$$

$$\frac{\partial \phi}{\partial z} = 3xy - 2xz + 2z \quad \text{...(3)}$$

Integrating (1) with respect to ‘x’ treating ‘y’ and ‘z’ as constants, we get

$$\phi = xy^2 - xz^2 + 3xyz - 2\frac{x^2}{2} + f(y,z) \quad \text{..(4)}$$

Integrating (2) with respect to ‘y’ treating ‘x’ and ‘z’ as constants, we get

$$\phi = 3xyz + 2\frac{xy^2}{2} + f(x,z) \quad \text{...(5)}$$

Integrating (3) with respect to ‘z’ treating ‘x’ and ‘y’ as constants, we get

$$\phi = 3xyz - 2x\frac{z^2}{2} + 2\frac{z^2}{2} + f(x,y) \quad \text{...(6)}$$

Hence from equations (4), (5), (6), we get

$$\phi = 3xyz + xy^2 - xz^2 - x^2 + z^2 + c$$

6) Prove that $\vec{F} = 3x^2y^2\vec{i} + (2x^3y + \cos z)\vec{j} - y\sin z\vec{k}$ is irrotational and find its scalar potential.

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y^2 & 2x^3y + \cos z & -y\sin z \end{vmatrix}$$

$$= \begin{bmatrix} \vec{i}\left(\frac{\partial}{\partial y}(-y\sin z) - \frac{\partial}{\partial z}(2x^3y + \cos z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(-y\sin z) - \frac{\partial}{\partial z}(3x^2y^2)\right) \\ \vec{k}\left(\frac{\partial}{\partial x}(2x^3y + \cos z) - \frac{\partial}{\partial y}(3x^2y^2)\right) \end{bmatrix}$$

$\Rightarrow \vec{i}(-\sin z - (-\sin z)) - \vec{j}(0 - 0) + \vec{k}(6x^2y - 6x^2y) = 0$ for all points (x,y,z)

$\therefore \vec{F}$ is irrotational vector.

Since $\vec{F}$ is irrotational, $\vec{F} = \nabla\phi$

$$3x^2y^2\vec{i} + (2x^3y + \cos z)\vec{j} - y\sin z\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$, we get

$$\frac{\partial\phi}{\partial x} = 3x^2y^2 \quad \text{..(1)}$$

$$\frac{\partial\phi}{\partial y} = 2x^3y + \cos z \quad \text{...(2)}$$

$$\frac{\partial\phi}{\partial z} = -y\sin z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ldots\ldots(3)$$

Integrating (1) with respect to 'x' treating 'y' and 'z' as constants, we get

$$\phi = 3\frac{x^3y^2}{3} + f(y,z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Integrating (2) with respect to 'y' treating 'x' and 'z' as constants, we get

$$\phi = 2\frac{x^3y^2}{2} + y\cos z + f(x,z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ldots\ldots(5)$$

Integrating (3) with respect to 'z' treating 'x' and 'y' as constants, we get

$$\phi = y\cos z + f(x,y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(6)$$

Hence from equations (4), (5), (6), we get

$$\phi = x^3y^2 + y\cos z + c$$