SUBJECT NAME: ENGINEERING MATHEMATICS II

(COMMON TO BIO GROUPS)

SUBJECT CODE: SMT1106

COURSE MATERIAL

UNIT-II DIFFERENTIAL CALCULUS

Definition 1. Differentiation

The rate at which a function changes with respect to the independent variable is called the derivative of the function.

(i.e) If y= f(x) be a function, where x and y are real variables which are independent and dependent variables respectively, then the derivative of y with respect to x is $\frac{dy}{dx}$.

Definition 2. Derivative of addition or subtraction of functions

If f(x) and g(x) are two functions of x, then $\frac{d[f(x) \pm g(x)]}{dx} = \frac{d[f(x)]}{dx} \pm \frac{d[g(x)]}{dx}$

Definition 3. Product rule

If $y = uv$, where u and v are functions of x, then $\frac{d[uv]}{dx} = v\frac{d[u]}{dx} + u\frac{d[v]}{dx}$

Definition 4. Quotient rule

If $y = \frac{u}{v}$, where u and v are functions of x, then $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Important Derivatives Formulae

1. $\frac{d}{dx}(c) = 0$ where ‘c’ is any constant.
2. $\frac{d}{dx}(x^n) = nx^{n-1}$.
3. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$.
4. $\frac{d}{dx}(a^x) = a^x \log a$
5. $\frac{d}{dx}(e^x) = e^x$.
6. $\frac{d}{dx}(\sin x) = \cos x$.
7. $\frac{d}{dx}(\cos x) = -\sin x$.
8. $\frac{d}{dx}(\tan x) = \sec^2 x$.

9. $\frac{d}{dx}(\cot x) = -\cos ec^2 x$.

10. $\frac{d}{dx}(\sec x) = \sec x \tan x$.

11. $\frac{d}{dx}(\cos ecx) = -\cos ecx \cot x$.

12. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

13. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$.

14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

15. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$.

16. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{1-x^2}}$.

17. $\frac{d}{dx}(\cos ec^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$

Problems

I. Ordinary Differentiation Problems

1. **Differentiate** $\mathbf{x + \frac{1}{x}}$

Solution Let $y = x + \frac{1}{x}$

Then $\frac{dy}{dx} = \frac{d\left(x+\frac{1}{x}\right)}{dx} \quad = \frac{d(x)}{dx} + \frac{d(x^{-1})}{dx} = 1 - \frac{1}{x^2}$

2. **Differentiate** $\mathbf{3\tan x + 2\cos x - e^x + 5}$

Solution:

Let $y = 3\tan x + 2\cos x - e^x + 5$

Then $\frac{dy}{dx} = \frac{d(3\tan x + 2\cos x - e^x + 5)}{dx} \quad = 3\frac{d(\tan x)}{dx} + 2\frac{d(\cos x)}{dx} - \frac{d(e^x)}{dx} + \frac{d(5)}{dx}$

$= 3sec^2 x - 2\sin x - e^x$

3. **Differentiate** $\mathbf{y = e^{2x}\cos 3x}$

Solution: $\frac{dy}{dx} = \frac{d(e^{2x}\cos 3x)}{dx} \quad = \cos 3x\,\frac{d(e^{2x})}{dx} + e^{2x}\,\frac{d(\cos 3x)}{dx}$

$= 2\cos 3x\, e^{2x} - 3e^{2x}\sin 3x$

4. **Differentiate** $\mathbf{y = e^{\sin x}x^2}$

Solution: $\frac{dy}{dx} = \frac{d(e^{\sin x}x^2)}{dx}$

$= x^2 \frac{d(e^{sinx})}{dx} + e^{sinx}\frac{d(x^2)}{dx}$
$= x^2 e^{sinx}(cosx) + 2xe^{sinx}$

5. **Differentiate y = $\mathbf{x^3e^{-x}tanx}$**
Solution: $\frac{dy}{dx} = \frac{d(x^3e^{-x}tanx)}{dx}$
$= e^{-x}tanx\frac{d(x^3)}{dx} + x^3\ tanx\frac{d(e^{-x})}{dx} + x^3e^{-x}\frac{d(tanx)}{dx}$
$= 3x^2e^{-x}tanx - x^3e^{-x}tanx + x^3e^{-x}sec^2x$

6. **Differentiate y** $= \mathbf{\frac{e^x}{cosx}}$
Solution: $\frac{dy}{dx} = \frac{d\left(\frac{e^x}{cosx}\right)}{dx} = \frac{cosx\ e^x\ - e^x\ (-sinx)}{cos^2x}$
$= \frac{cosx\ e^x\ + e^x\ (sinx)}{cos^2x}$

7. **Differentiate y** $= \mathbf{\frac{ax+b}{cx+d}}$
Solution:$\frac{dy}{dx} = \frac{(cx+d)a-(ax+b)c}{(cx+d)^2}$ (by quotient rule)

8. **Differentiate** $\mathbf{\frac{x^2+2x+3}{\sqrt{x}}}$
Solution: $\frac{dy}{dx} = \frac{\sqrt{x}(2x+2)-(x^2+2x+3)\frac{1}{2}x^{-1/2}}{(\sqrt{x})^2} = \frac{2\sqrt{x}(x+1)-(x^2+2x+3)\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$
$= \frac{2\sqrt{x}\times2\sqrt{x}(x+1)-(x^2+2x+3)}{2\sqrt{x}(\sqrt{x})^2} = \frac{4x(x+1)-(x^2+2x+3)}{2x^{3/2}}$
$= \frac{4x^2+4x-x^2-2x-3}{2x^{3/2}} = \frac{3x^2+2x-3}{2x^{3/2}}$

9. **Differentiate y** $= \mathbf{(3x^2-1)^3}$
Solution: Given y $= (3x^2-1)^3$
Differentiating w.r.to x, we get
$\Rightarrow \frac{dy}{dx} = 3(3x^2-1)^2 6x$
$= 3(9x^4 - 6x^2 + 1) = 27x^4 - 18x^2 + 3$

10. **Differentiate:** $\mathbf{log\left(\frac{1+sinx}{1-sinx}\right)}$
Solution: Let $y = log\left(\frac{1+sinx}{1-sinx}\right)$
$\Rightarrow y = log(1+sinx) - log(1-sinx)$
Differentiate y w.r.to x, we get
$\frac{dy}{dx} = \frac{1}{1+sinx}cosx - \frac{1}{1-sinx}(-cosx)$

$= \frac{(1-sinx)cosx+cosx(1+sinx)}{(1+sinx)(1-sinx)}$

$= \frac{cosx - sinx\ cosx + cosx + cosx\ sinx}{1-sin^2x}$
$= \frac{2\ cosx}{cos^2x} = 2\frac{1}{cosx} = 2\ secx$

II. Differentiation Problems on Logarithmic Functions

1. **Differentiate** $\mathbf{x^{sinx}}$
Solution: Let $y = x^{sinx}$
Taking log on both sides, we get $logy = sinx\ logx$

Now differentiating with respect to x

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log x(\cos x) + \sin x\frac{1}{x}$ (Using product rule)

$\Rightarrow \frac{dy}{dx} = y\left(\log x(\cos x) + \sin x\frac{1}{x}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{y(x\cos x \log x + \sin x)}{x}$

$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\frac{x\cos x \log x + \sin x}{x}\right)$

2. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Solution: Given $x^y = e^{x-y}$

Taking log on both sides, we get $\log x^y = \log e^{x-y}$

$\Rightarrow y\log x = (x-y)\log_e e$

$\Rightarrow y\log x = (x-y)$.........(1)

$\Rightarrow \frac{1}{x}y + \log x\frac{dy}{dx} = 1 - \frac{dy}{dx}$

$\Rightarrow \log x\frac{dy}{dx} + \frac{dy}{dx} = 1 - \frac{y}{x}$

$\Rightarrow \frac{dy}{dx}(\log x + 1) = \frac{x-y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x(1+\log x)}$

$\Rightarrow \frac{dy}{dx} = \frac{y\log x}{x(1+\log x)}$....(2)

Again from (1) $y + y\log x = x$

$\Rightarrow y(1+\log x) = x, \frac{y}{x} = \frac{1}{1+\log x}$

$\Rightarrow \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

3. If y = $x^{x^{x^{\cdots\infty}}}$, then find $\frac{dy}{dx}$

Solution:

Given $y = x^{x^{x^{\cdots\infty}}} = x^y$

Taking log on both sides

$\log y = y\log x$

Differentiating w. r. to x we get

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = y\frac{1}{x} + \log x\frac{dy}{dx}$

$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \left(\frac{1-y\log x}{y}\right)\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\frac{y}{1-y\log x}\right) = \frac{y^2}{x(1-y\log x)}$

4. Differentiate $y = \log\left(\frac{x^2+1}{x^2-1}\right)$

Solution:

$y = \log(x^2+1) - \log(x^2-1)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x^2+1}2x - \frac{1}{x^2-1}2x$

$\Rightarrow \frac{dy}{dx} = 2x\left(\frac{1}{x^2+1} - \frac{1}{x^2-1}\right)$

$\Rightarrow \frac{dy}{dx} = 2x\left(\frac{x^2-1-(x^2+1)}{(x^2+1)(x^2-1)}\right) = 2x\left(\frac{x^2-1-x^2-1)}{x^4-1}\right) = 2x\left(\frac{-2)}{x^4-1}\right) = \frac{-4x}{x^4-1}$

5. **Differentiate $\mathbf{y = e^{3x^2+2x+3}}$**

Solution: $\frac{dy}{dx} = e^{3x^2+2x+3}(6x+2)$

III. Differentiation of Implicit functions

If two variables x and y are connected by the relation f(x, y) = 0 and none of the variable is directly expressed in terms of the other, then the relation is called an implicit function.

Problems

1. **Find $\frac{dy}{dx}$, if $x^3+y^3 = 3axy$**

Solution:

Differentiating w.r.to x, we get

$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$

$\Rightarrow 3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2$

$\Rightarrow \frac{dy}{dx}(3y^2 - 3ax) = 3ay - 3x^2$

$\Rightarrow \frac{dy}{dx} = \frac{(3ay-3x^2)}{3y^2-3ax} = \frac{3(ay-x^2)}{3(y^2-ax)} = \frac{(ay-x^2)}{(y^2-ax)}$

2. **Find $\frac{dy}{dx}$, if $x^2 + y^2 = 16$**

Solution:

Given $x^2 + y^2 = 16$

$\Rightarrow y^2 = 16 - x^2$

$\Rightarrow y = \sqrt{16 - x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(16 - x^2)^{-1/2} \times (-2x)$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{\sqrt{16-x^2}} = -\frac{x}{y}$

3. **Find $\frac{dy}{dx}$, if $x = at^2, y = 2at$**

Solution: Given $x = at^2, y = 2at$

$\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$

Now $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}$

4. **Find $\frac{dy}{dx}$, if $y^2 + x^3 - xy + \cos y = 0$**

Solution:

Given $y^2 + x^3 - xy + \cos y = 0$

$\Rightarrow 2y\frac{dy}{dx} + 3x^2 - \frac{d}{dx}(xy) - \sin y\frac{dy}{dx} = 0$

$\Rightarrow (2y - \sin y)\frac{dy}{dx} + 3x^2 - \left(x\frac{dy}{dx} + y \times 1\right) = 0$

$\Rightarrow (2y - \sin y - x)\frac{dy}{dx} + 3x^2 - y = 0$

$\Rightarrow (2y - \sin y - x)\frac{dy}{dx} = y - 3x^2$

$\Rightarrow \frac{dy}{dx} = \frac{y-3x^2}{2y-\sin y-x}$

IV.Successive Differentiation

The process of differentiating a given function again and again is called as successive differentiation and the results of such differentiation are called successive derivatives.

Notations:

i) $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$,....... n^{th} order derivative: $\frac{d^ny}{dx^n}$

ii) f ′(x) , f ″(x) , f ‴(x),….., nth order derivative: $f^n(x)$

iii) Dy,D^2y,D^3y……., nth order derivative: D^ny

iv) y′ , y″ , y‴ ,……, nth order derivative: $y^{(n)}$

v) y_1,y_2,y_3,……..,nth order derivative: y_n

Problems

1. If y=sin(sinx), prove that $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = 0$

Solution:

Given y=sin(sinx)…………………(1)

Differentiating (1) with respect to x we get,

$\frac{dy}{dx}$ = cos(sinx).cosx………………..(2)

Differentiating (2) with respect to x we get,

$\frac{d^2y}{dx^2}$ =cos(sinx)(-sinx)+cosx(-sin(sinx).cosx) [Product Rule]

$\frac{d^2y}{dx^2}$ = -sinxcos(sinx) – $\cos^2$xsin(sinx)……………..(3)

Therefore,

$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x =$ -sinxcos(sinx) – $\cos^2$xsin(sinx) +(tanx) cos(sinx).cosx +y $\cos^2$x

= - sinxcos(sinx) – y $\cos^2$x + sinxcos(sinx) + y $\cos^2$x

= 0

2.If x = a(cost + tsint), y = a(sint – tcost), find $\frac{d^2y}{dx^2}$

Solution:

$$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t) = at\cos t.$$

$$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t.$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at\sin t}{at\cos t} = \tan t.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{\sec^2 t}{at\cos t} = \frac{1}{at\cos^3 t}.$$

3. If $ax^2+2hxy+by^2=1$ **then prove that** $\dfrac{d^2y}{dx^2}=\dfrac{h^2-ab}{(hx+by)^3}$

Solution:

Given $ax^2+2hxy+by^2=1$ ………………………….(1)

Differentiating (1) partially with respect to x,we get,

$$2ax+2h\left(x\frac{dy}{dx}+y\right)+2by\frac{dy}{dx}=0$$

Then , $\dfrac{dy}{dx}=\dfrac{-(ax+hy)}{(hx+by)}$ ………………………….(2)

Differentiating (2) with respect to x again,we get,

$$\frac{d^2y}{dx^2}=\frac{(hx+by)\left[-a-h\frac{dy}{dx}\right]+(ax+hy)\left[h+b\frac{dy}{dx}\right]}{(hx+by)^2}$$

$$=\frac{(h^2-ab)y-\frac{dy}{dx}(h^2-ab)x}{(hx+by)^2}=\frac{(h^2-ab)\left(y-x\frac{dy}{dx}\right)}{(hx+by)^2}$$

$$=\frac{(h^2-ab)\left(y+x\frac{(ax+hy)}{(hx+by)}\right)}{(hx+by)^2}=\frac{(h^2-ab)(ax^2+by^2+2hxy)}{(hx+by)^3}$$

Thus, $\dfrac{d^2y}{dx^2}=\dfrac{h^2-ab}{(hx+by)^3}$ (from(1))

n^th derivative of some standard functions:

S.No	Y=f(x)	$y_n=\dfrac{d^ny}{dx^n}=D^ny$
1.	e^{mx}	m^ne^{mx}
2.	$(ax+b)^m$	$m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)a^n(ax+b)^{m-n}$
3.	$\dfrac{1}{ax+b}$	$\dfrac{(-1)^n n!a^n}{(ax+b)^{n+1}}$
4.	$\log(ax+b)$	$\dfrac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$
5.	$\sin(ax+b)$	$a^n\sin\left(\dfrac{n\pi}{2}+ax+b\right)$
6.	$\cos(ax+b)$	$a^n\cos\left(\dfrac{n\pi}{2}+ax+b\right)$

4. Find y_n, where $y = \dfrac{3}{(x+1)(2x-1)}$

Solution:

Resolving into partial fractions,

$$y = \frac{2}{2x-1} - \frac{1}{x+1}$$

$$\therefore y_n = \frac{2(-1)^n 2^n n!}{(2x-1)^{n+1}} - \frac{(-1)^n n!}{(x+1)^{n+1}}$$

5. Find the n^{th} derivative of $\log(9x^2-1)$.

Solution:

Let $y = \log(9x^2 - 1) = \log\{(3x+1)(3x-1)\}$

$$= \log(3x+1) + \log(3x-1)$$

Then $y_n = \dfrac{d^n}{dx^n}(\log(3x+1)) + \dfrac{d^n}{dx^n}(\log(3x-1))$

$$\therefore y_n = \frac{(-1)^{n-1}(n-1)!(3)^n}{(3x+1)^n} + \frac{(-1)^{n-1}(n-1)!(3)^n}{(3x-1)^n}$$

6. Find y_n, where $y = e^{7x+5}$

Solution:

Let $y = e^{7x+5}$

Then $y_n = \dfrac{d^n}{dx^n}\left(e^{7x+5}\right) = \dfrac{d^n}{dx^n}\left(e^5 e^{7x}\right) = e^5 \dfrac{d^n}{dx^n}\left(e^{7x}\right)$

$$\Rightarrow y_n = e^5 7^n e^{7x}$$

Leibnitz formula for the n^{th} derivative of a product

If u and v are functions of x,then

$$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \ldots\ldots.. + nc_r u_{n-r} v_r + \ldots\ldots. + uv_n$$

Problems

7. Find the nth differential coefficient of $x^2 \log x$

Solution:

Take u=logx , v = x^2

$$\frac{d^n}{dx^n}\left(x^2 \log x\right) = \frac{d^n}{dx^n}(\log x)x^2 + nc_1 \frac{d^{n-1}}{dx^{n-1}}(\log x)\frac{d}{dx}\left(x^2\right) + nc_2 \frac{d^{n-2}}{dx^{n-2}}(\log x)\frac{d^2}{dx^2}\left(x^2\right)$$

(since all the other terms are zero)

$$= \frac{(-1)^{n-1}(n-1)!x^2}{(x)^n} + \frac{n(-1)^{n-2}(n-2)!(2x)}{(x)^{n-1}} + \frac{n(n-1)(-1)^{n-3}(n-3)!2}{2(x)^{n-2}}$$

$$= \frac{2(-1)^{n-2}(n-3)!}{(x)^{n-2}}$$

8. If y = x^2e^x, show that $y_n = \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y$ **where y_n stands for** $\frac{d^n y}{dx^n}$

Solution:

Take u $=e^x$,v = x^2

$$y_n = \frac{d^n}{dx^n}(x^2e^x) = \frac{d^n}{dx^n}(e^x)x^2 + nc_1\frac{d^{n-1}}{dx^{n-1}}(e^x)\frac{d}{dx}(x^2) + nc_2\frac{d^{n-2}}{dx^{n-2}}(e^x)\frac{d^2}{dx^2}(x^2)$$

(since all the other terms are zero)

$$\therefore y_n = e^x x^2 + 2nxe^x + n(n-1)e^x$$

Now, $y_1 = x^2e^x + 2xe^x$, $y_2 = x^2e^x + 4xe^x + 2e^x$

$$\therefore \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y = \frac{n(n-1)[x^2e^x + 4xe^x + 2e^x]}{2}$$

$$- n(n-2)[x^2e^x + 2xe^x] + \frac{(n-1)(n-2)x^2e^x}{2}$$

$$= x^2e^x\left[\frac{n(n-1)}{2} - n(n-2) + \frac{(n-1)(n-2)}{2}\right] + xe^x[2n(n-1) - 2n(n-2)] + n(n-1)e^x$$

$=y_n$ on simplification.

V. Partial Differentiation

Consider $z = f(x, y)$, here z is a function of two independent variables x and y. z can be differentiated with respect to x or y but when we are differentiating z with respect to x (or y) we must keep the variable y (or x) as a constant.

Notations:

Let z=f(x,y)

First order partial derivatives of f(x, y) with respect to x and y.

$\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$

Second order partial derivatives of f(x, y) with respect to x and y

$\frac{\partial^2 f}{\partial x^2} = f_{xx}$, $\frac{\partial^2 f}{\partial y^2} = f_{yy}$

Second order mixed partial derivatives of f(x, y)

$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$, $\frac{\partial^2 f}{\partial y \partial x} = f_{yx}$

Problems:

1. **If u = $x^3 + y^3 + 3xy$, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$**

   Solution: Given If u = $x^3 + y^3 + 3xy$

   $\frac{\partial u}{\partial x} = 3x^2 + 3y$, $\frac{\partial u}{\partial y} = 3y^2 + 3x$

2. **If u = log ($x^3 + y^3 + z^3 - 3xyz$), show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{(x+y+z)}$**

Solution: u = log $(x^3 + y^3 + z^3 - 3xyz)$

$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} 3x^2 - 3yz$,

$\frac{\partial u}{\partial y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} 3y^2 - 3xz,$

$\frac{\partial u}{\partial z} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} 3z^2 - 3xy$

Now $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2+3y^2+3z^2-3yz-3xz-3xy}{x^3 + y^3 + z^3 - 3xyz}$

$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{3}{x+y+z}$

3. If $f(x,y) = x^2 \sin y + y^2 \cos x$, then find its all first and 2nd order partial derivatives.

Solution: Given $f(x, y) = x^2 \sin y + y^2 \cos x$

$f_x = 2x \sin y - y^2 \sin x;\ f_y = x^2 \cos y + 2y \cos x.$

$f_{xx} = 2 \sin y - y^2 \cos x;\ f_{yy} = -x^2 \sin y + 2 \cos x;$

$f_{xy} = 2x \cos y - 2y \sin x;\ fyx = 2x \cos y - 2y \sin x.$

4. If $f(x,y) = \frac{y}{x} \log x$, then find its all 1st and 2nd order derivatives.

Solution: $f_x = \frac{y}{x}\frac{1}{x} + \log x \left(\frac{-y}{x^2}\right) = \frac{y}{x^2}(1 - \log x), f_y = \frac{\log x}{x}$,

$f_{xx} = \frac{y}{x^2}\left(-\frac{1}{x}\right) - \frac{2y}{x^3}(1 - \log x) = \frac{y}{x^3}\left(-1 - 2(1 - \log x)\right) = \frac{y}{x^3}(\log x - 3)$;

$f_{yy} = 0,\ f_{yx} = \frac{1}{x^2}(1 - \log x);\ f_{xy} = \frac{1}{x}\frac{1}{x} - \frac{1}{x^2}\log x = \frac{1}{x^2}(1 - \log x).$

5. **Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ for $u = \sin(ax + by + cz)$**

Solution:

$\frac{\partial u}{\partial x} = a \cos(ax + by + cz)$

$\frac{\partial u}{\partial x} = b \cos(ax + by + cz)$

$\frac{\partial u}{\partial x} = c \cos(ax + by + cz)$

VI. Euler's Theorem for Homogeneous Functions

A homogenous function of degree *n* of the variables *x*, *y*, *z* is a function in which each term has degree *n*. For example, the function $f(x, y, z) = Ax^3 + By^3 + Cz^3 + Dxy^2 + Exz^2 + Fyz^2 + Gyx^2 + Hzx^2 + Izy^2 + Jxyz$, is a homogeneous function of *x*, *y*, *z*, in which all terms are of degree three.

Note:

A function f(x ,y) of two independent variables x and y is said to be homogeneous in x and y of degree n if $f(tx, ty) = t^n f(x, y)$ for any positive quantity t.

Euler's theorem:

1). If $f(x, y)$ is a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

2). If $f(x, y, z)$ is a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

Result: If z is a homogeneous function of x, y of degree n and z=f(u) then

(i). $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

Problems on Euler's theorem

1. Verify Euler's theorem when $\mathbf{u = x^3 + y^3 + z^3 + 3xyz}$

Solution:

Given $u = x^3 + y^3 + z^3 + 3xyz$

Now $tu = (tx)^3 + (ty)^3 + (tz)^3 + 3txtytz$

$= t^3(x^3 + y^3 + z^3 + 3xyz) = t^3 u$

Therefore u is a homogeneous function of degree 3.

$\frac{\partial u}{\partial x} = 3x^2 + 3yz$

$\frac{\partial u}{\partial y} = 3y^2 + 3xz$

$\frac{\partial u}{\partial z} = 3z^2 + 3xy$

Therefore $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = x(3x^2 + 3yz) + y(3y^2 + 3xz) + z(3z^2 + 3xy)$

$= 3x^3 + 3y^3 + 3z^3 + 9xyz$

$= 3(x^3 + x^3 + 3xy) = 3u$

Hence Euler's theorem is verified.

2. If $\mathbf{u = x\,log\left(\frac{y}{x}\right)}$, then prove that $\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$

Solution:

Given $u = x\log\left(\frac{y}{x}\right)$

u is a homogeneous function of degree 1.

Therefore by Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \times u = u$

3. If $\mathbf{(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{logx - logy}{x^2 + y^2}}$, then prove that $\mathbf{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0}$

Solution:

$f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$

Now $f(tx, ty) = \frac{1}{(tx)^2} + \frac{1}{txty} + \frac{\log tx - \log ty}{(tx)^2 + (ty)^2}$

$= \frac{1}{t^2x^2} + \frac{1}{t^2xy} + \frac{\log\frac{tx}{ty}}{t^2(x^2 + y^2)}$

$= \frac{1}{t^2}\left(\frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}\right)$

$= t^{-2}\left(\frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}\right)$

Therefore f(x, y) is a homogeneous function of degree -2

By Euler's theorem , $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -2f$

$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f = 0$

4. If u = $\mathbf{tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)}$, show that $\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin2u}$

Solution: Given u = $\tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$

$\Rightarrow \tan u = \left(\frac{x^3 + y^3}{x - y}\right)$

Let $z = \tan u = \left(\frac{x^3 + y^3}{x - y}\right)$

And z is a homogeneous function of order 2.

We know that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

Here $f(u) = \tan u$

$\Rightarrow f'(u) = \sec^2 u$

Therefore by the result,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\tan u}{\sec^2 u} = 2\frac{\sin u}{\cos u} \times \cos^2 u$$

$$= 2\sin u \times \cos u = \sin 2u$$

(Or)

By Euler's theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2z$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x\frac{1}{\cos^2 u}\frac{\partial u}{\partial x} + y\frac{1}{\cos^2 u}\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u}$$

$$\Rightarrow x\frac{1}{\cos u}\frac{\partial u}{\partial x} + y\frac{1}{\cos u}\frac{\partial u}{\partial y} = 2\frac{\sin u}{1}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u = \sin 2u.$$

All the best