Subject Name: Engineering Mathematics II

(Common to Bio groups)

Subject code: SMT1106

Course Material

UNIT 1 MATRICES

RANK OF A MATRIX

Let A be any matrix of order mxn.The determinants of the sub square matrices of A are called the minors of A. If all the minors of order **(r+1)** are zero but there is at least one non zero minor of order **r,** then **r** is called the rank of A and is written as R(A). For an mxn matrix,

- If m is less than n then the maximum rank of the matrix is m
- If m is greater than n then the maximum rank of the matrix is n.

The rank of a matrix would be zero only if the matrix had no non-zero elements. If a matrix had even one non-zero element, its minimum rank would be one.

Example

1. Find the rank of A = $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix}$

$|A|$ = 1(20-12) -2(5-4) +3(6-8) = 0

Hence R(A) < 3.

Let the second order minor $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2 \neq 0$

R(A)=2.

2. Find the Rank of B = $\begin{pmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix}$ $R_2 = R_2 - 2R_1$, $R_3 = R_3 - 3R_1$, $R_4 = R_4 - 6R_1$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix}$$ $R_2 = 1/5R_2$, $R_3 = R_3$, $R_4 = R_4$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 33/5 & 22/5 \end{pmatrix}$$ $R_3 = R_3 - 4R_2$, $R_4 = R_4 - 9R_2$

$$= \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & 3/5 & 7/5 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ $R_4 = R_4 - R_3$

The number of Nonzero Rows is 3.Hence R(B)=3.

3. Find the Rank of the Matrix A = $\begin{pmatrix} 2 & -2 & 1 \\ 1 & 4 & -1 \\ 4 & 6 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 2 & -2 & 1 \\ 4 & 6 & -3 \end{pmatrix}$$ $R_1 = R_2$, $R_2 = R_1$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & -10 & 1 \end{pmatrix}$$ $R_2 = R_2 - 2R_1$, $R_3 = R_3 - 4R_1$

$$= \begin{pmatrix} 1 & 4 & -1 \\ 0 & -10 & 3 \\ 0 & 0 & -2 \end{pmatrix}$$ $R_3 = R_3 - R_2$

The number of Nonzero Rows is 3. Hence R(A)=3.

4. Find the Rank of the Matrix A = $\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix}$$ $R_3 = R_3 - R_2$

The number of Nonzero Rows is 3. Hence R(A)=3.

5. Find the Rank of the Matrix B = $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

A possible minor of least order is $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{pmatrix}$ whose determinant is non zero.

Hence it is possible to find a nonzero minor of order 3.

Hence R(B)=3.

CONSISTENCY OF LINEAR ALGEBRAIC EQUATION

A general set of m linear equations and n unknowns,

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = c_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = c_m$$

can be rewritten in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_m \end{bmatrix}$$

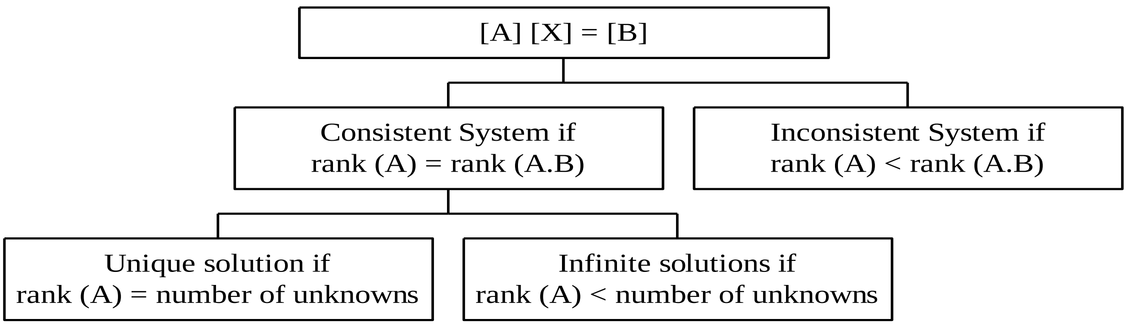
Denoting the matrices by A, X, and C, the system of equation is, AX = C where A is called the coefficient matrix, C is called the right hand side vector and X is called the solution vector. Sometimes AX=C systems of equations are written in the augmented form. That is

$$[\mathrm{A}:\mathrm{C}] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} & \vdots c_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & \vdots c_2 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & \vdots c_m \end{bmatrix}$$

Rouche'sTheorem

1. A system of equations AX =C is **consistent** if the rank of A is equal to the rank of the augmented matrix $(\mathrm{A}:\mathrm{C})$. If in addition, the rank of the coefficient matrix A is same as the number of unknowns, then the solution is unique; if the rank of the coefficient matrix A is less than the number of unknowns, then infinite solutions exist.

2. A system of equations AX=C is **inconsistent** if the rank of A is not equal to the rank of the augmented matrix $(\mathrm{A}:\mathrm{C})$.



Problems

1. **Check whether the following system of equations**

$25x_1 + 5x_2 + x_3 = 106.8$

$64x_1 + 8x_2 + x_3 = 177.2$

$89x_1 + 13x_2 + 2x_3 = 280$ is consistent or inconsistent.

Solution

The augmented matrix is

$$[A:B] = \begin{bmatrix} 25 & 5 & 1 & :106.8 \\ 64 & 8 & 1 & :177.2 \\ 89 & 13 & 2 & :280.0 \end{bmatrix}$$

To find the rank of the augmented matrix consider a square sub matrix of order 3×3 as

$\begin{bmatrix} 5 & 1 & 106.8 \\ 8 & 1 & 177.2 \\ 13 & 2 & 280.0 \end{bmatrix}$ whose determinant is 12. Hence R $[A:B]$ is 3.

So the rank of the augmented matrix is 3 but the rank of the coefficient matrix $[A]$ is 2

as the Determinant of A is zero. Hence R$[A:B] \neq$ R [A].Hence the system is inconsistent.

2. Check the consistency of the system of linear equations and discuss the nature of the solution?

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + x_2 - 2x_3 &= 1 \\ 4x_1 - 3x_2 - x_3 &= 3 \\ 2x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}$$

Solution

The augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}$$

$[A:B]$ is reduced by elementary row transformations to an upper triangular matrix

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $R_2 = R_2 - 3R_1$, $R_3 = R_3 - 4R_1$, $R_4 = R_4 - 2R_1$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $R_2 = R_2 / -5$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $R_3 = R_3 + 11R_1$

Here R$[A:B]$=R[A] =3.Hence the system is consistent. Also R[A] is equal to the number of unknowns. Hence the system has an unique solution.

3. Check whether the following system of equations is a consistent system of equations. Is the solution unique or does it have infinite solutions

$$x_1 + 2x_2 - 3x_3 - 4x_4 = 6$$
$$x_1 + 3x_2 + x_3 - 2x_4 = 4$$
$$2x_1 + 5x_2 - 2x_3 - 5x_4 = 10$$

Solution

The given system has the augmented matrix given by

$$[A:B] = \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 1 & 3 & 1 & -2 & 4 \\ 2 & 5 & -2 & -5 & 10 \end{bmatrix}$$

$[A:B]$ is reduced by elementary row transformations to an upper triangular matrix

$$= \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 1 & 4 & 3 & -2 \end{bmatrix} \quad R_2 = R_2 - R_1,\ R_3 = R_3 - 2R_1$$

$$= \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad R_3 = R_3 - R_2$$

A and $[A:B]$ are each of rank r = 3, the given system is consistent but R[A] is not equal to the number of unknowns. Hence the system does not has a unique solution.

4. Check whether the following system of equations

3x− 2y + 3z = 8

x +3 y+6z = −3

2 x + 6y+12z = −6

is a consistent system of equations and hence solve them.

Solution

Let the augmented matrix of the system be

$$[A:B] = \begin{bmatrix} 3 & -2 & 3 & 8 \\ 1 & 3 & 6 & -3 \\ 2 & 6 & 12 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 6 & -3 \\ 3 & -2 & 3 & 8 \\ 2 & 6 & 12 & -6 \end{bmatrix} \quad R_1 = R_2\ \ R_2 = R_1$$

$$= \begin{bmatrix} 1 & 3 & 6 & -3 \\ 0 & 11 & 15 & -17 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 = R_2 - 3R_1,\ R_3 = R_3 - 2R_1$$

R[A:B] = R[A] = 2**.**Therefore the system is consistent and posses solution but rank is not equal to the number of unknowns which is 3.Hence the system has infinite solution. From the upper triangular matrix we have the reduced system of equations given by

x +3 y+6z = −3 ; 11y+15z = -17 .

By assuming a value for y we have one set of values for z and x.For example when y=3, z = $-10/3$ and x = 8.Similarly by choosing a value for z the corresponding y and x can be calculated. Hence the system has infinite number of solutions.

5. Check whether the following system of equations

X+y + z = 6

3 x −2 y+4z = 9

x − y −z = 0

Is a consistent system of equations and hence solve them.

<u>Solution</u>

Let the augmented matrix of the system be

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & -2 & 4 & 9 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -5 & 1 & -9 \\ 0 & -2 & -2 & -6 \end{bmatrix} \quad R_2=R_2-3R_1 ,\ R_3 = R_3 - R_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/5 & 9/5 \\ 0 & -2 & -2 & -6 \end{bmatrix} \quad R_2 = R_2/-5$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/5 & 9/5 \\ 0 & 0 & -12/5 & -12/5 \end{bmatrix} \quad R_3 = R_3+2R_2$$

Hence R[A B] =R[A] =3 which is equal to the number of unknowns. Hence the system is consistent with unique solution. Now the system of equations takes the form

x+y + z = 6**;** y-z/5 =9/5; $-12/5$z = $-12/5$.

Hence z =1. Substituting z = 1 in y-z/5 =9/5 we have y-1/5 = 9/5 or y = 1/5+9/5 = 10/5.

Hence y =2. Substituting the values of y,z in x+y + z = 6 we have x= 3. Hence the system has the unique solution as x= 3 , y =2 , z =1.

<u>CHARACTERISTIC EQUATION</u>

The equation $|A-\lambda I| = 0$ is called the characteristic equation of the matrix A

Note:

1. Solving $|A-\lambda I| = 0$, we get n roots for λ and these roots are called characteristic roots or eigen values or latent values of the matrix A
2. Corresponding to each value of λ, the equation AX = λX has a non-zero solution vector X

If X_r be the non-zero vector satisfying AX = λX, when$\lambda = \lambda_r$, X_r is said to be the latent vector or eigen vector of a matrix A corresponding to λ_r

Working rule to find characteristic equation:

For a 3 x 3 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$

Method 2:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where S_1 = sum of the main diagonal elements, S_2 = sum of the minors of the main diagonal elements, S_3 = Determinant of A = $|A|$

For a 2 x 2 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$

Method 2:

Its characteristic equation can be written as $\lambda^2 - S_1\lambda + S_2 = 0$ where S_1 = sum of the main diagonal elements, S_2 = Determinant of A = $|A|$

1. **Find the characteristic equation of** $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

Solution: Its characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$,

where S_1 = sum of the main diagonal elements = 8 + 7 + 3 = 18,

S_2 = sum of the minors of the main diagonal elements=45

S_3 = Determinant of A = $|A|$=0

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$.

2. **Find the characteristic equation of** $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

Solution: Let A = $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic equationl of A is $\lambda^2 - S_1\lambda + S_2 . S_1 = sum of the main diagonal elements$ = 3 + 2 = 5 and $S_2 = Determinant of A = |A|$ = 3(2) – 1(-1) = 7

Therefore, the characteristic equation is $\lambda^2 - 5\lambda + 7$ =0.

EIGEN VALUES AND EIGEN VECTORS OF A REAL MATRIX

Working rule to find Eigen values and Eigen vectors:

1. Find the characteristic equation $|A - \lambda I| = 0$
2. Solve the characteristic equation to get characteristic roots. They are called Eigen values
3. To find the Eigen vectors, solve $[A - \lambda I]X = 0$ for different values of λ

Note:

1. Corresponding to n distinct Eigen values, we get n independent Eigen vectors
2. If 2 or more Eigen values are equal, it may or may not be possible to get linearly independent Eigen vectors corresponding to the repeated Eigen values
3. If X_i is a solution for an Eigen value λ_i, then cX_i is also a solution, where c is an arbitrary constant. Thus, the Eigen vector corresponding to an Eigen value is not unique but may be any one of the vectors cX_i

Problems

1. **Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} \end{pmatrix}$**

Solution: Let A = $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = sum of the main diagonal elements = 1 - 1 = 0$,

$S_2 = Determinant of A = |A|$ = 1(-1) – 1(3) = - 4

Therefore, the characteristic equation is $\lambda^2 - 4 = 0$ i.e., $\lambda^2 = 4$ or $\lambda = \pm 2$

Therefore, the eigen values are 2, -2

To find the eigen vectors:

$$[A - \lambda I]X = 0$$

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$--------------- **(1)**

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 \\ 3 & -1-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **[From (1)]**

i.e., $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $3x_1 + x_2 = 0$, $3x_1 + x_2 = 0$

i.e., we get only one equation $3x_1 + x_2 = 0 \Rightarrow 3x_1 = -x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{-3}$

Therefore $X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Case 2: If $\lambda = 2$, $\begin{bmatrix} 1-(2) & 1 \\ 3 & -1-(2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **[From (1)]**

i.e., $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $-x_1 + x_2 = 0 \Rightarrow x_1 - x_2 = 0$

$3x_1 - 3x_2 = 0 \Rightarrow x_1 - x_2 = 0$

i.e., we get only one equation $x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$

Hence, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2.Find the eigen values and eigen vectors of $\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2 + 1 - 3 = 0$,

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -7 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = -5 + (-6) + (-2) = -5 - 6 - 2 = -13$

$S_3 = Determinant of A = |A|$ = 2 (-5)-2 (-6)-7(2) = -10 + 12 – 14 = -12

Therefore, the characteristic equation of A is $\lambda^3 - 13\lambda + 12 = 0$

3	1	0	-13	12
	0	3	9	– 12
	1	3	-4	0

$$(\lambda-3)(\lambda^2+3\lambda-4)=0 \Rightarrow \lambda=3\,,\lambda=\frac{-3\pm\sqrt{3^2-4(1)(-4)}}{2(1)}=\frac{-3\ \pm\sqrt{25}}{2}=\frac{-3\pm 5}{2}$$

$$=\frac{-3+5}{2}\,,\frac{-3-5}{2}=1,-4$$

Therefore, the eigen values are 3, 1, and -4

<u>To find the eigen vectors</u>: **Let** $[A-\lambda I]X=0$

$$\begin{bmatrix} 2-\lambda & 2 & -7 \\ 2 & 1-\lambda & 2 \\ 0 & 1 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda=1$, $\begin{bmatrix} 2-1 & 2 & -7 \\ 2 & 1-1 & 2 \\ 0 & 1 & -3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 2 & -7 \\ 2 & 0 & 2 \\ 0 & 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1+2x_2-7x_3=0$ --------- (1)

$2x_1+0x_2+2x_3=0$ ------- (2)

$0x_1+x_2-4x_3=0$ -------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

x_1 x_2 x_3

2 -7 1 2

0 2 2 0

$$\frac{x_1}{4}=\frac{x_2}{-16}=\frac{x_3}{-4} \Rightarrow \frac{x_1}{1}=\frac{x_2}{-4}=\frac{x_3}{-1}$$

Therefore, $X_1=\begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}$

Case 2: If $\lambda=3$, $\begin{bmatrix} 2-3 & 2 & -7 \\ 2 & 1-3 & 2 \\ 0 & 1 & -3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -1 & 2 & -7 \\ 2 & -2 & 2 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -x_1+2x_2-7x_3=0$ -------- (1)

$2x_1 - 2x_2 + 2x_3 = 0$ -------- (2)

$0x_1 + x_2 - 6x_3 = 0$ -------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

	x_1		x_2		x_3	
2		-7		-1		2
-2		2		2		-2

$$\frac{x_1}{-10} = \frac{x_2}{-12} = \frac{x_3}{-2} \Rightarrow \frac{x_1}{5} = \frac{x_2}{6} = \frac{x_3}{1}$$

$$\Rightarrow \frac{x_1}{-10} = \frac{x_2}{-12} = \frac{x_3}{-2} \Rightarrow \frac{x_1}{5} = \frac{x_2}{6} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$

Case 3: If $\boldsymbol{\lambda = -4}$**,** $\begin{bmatrix} 6 & 2 & -7 \\ 2 & 5 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 6x_1 + 2x_2 - 7x_3 = 0$ -------- (1)

$2x_1 + 5x_2 + 2x_3 = 0$ -------- (2)

$0x_1 + x_2 + x_3 = 0$ -------- (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

	x_1		x_2		x_3	
2		-7		6		2
5		2		2		5

$$\Rightarrow \frac{x_1}{39} = \frac{x_2}{-26} = \frac{x_3}{26} \Rightarrow \frac{x_1}{3} = \frac{x_2}{-2} = \frac{x_3}{2}$$

Therefore, $X_3 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$

3.Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{bmatrix}$.

<u>Solution</u>: Let A = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

<u>To find the characteristic equation:</u>

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements = 0 + 0 + 0 = 0,$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} =$

$-1 - 1 - 1 = -3$

$S_3 = Determinant\ of\ A = |A|$ = 0 -1(-1)+ 1(1) = 0 + 1 + 1 = 2

Therefore, the characteristic equation of A is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

-1	1	0	-3	-2
	0	-1	1	2
	1	-1	-2	0

$(\lambda - (-1))(\lambda^2 - \lambda - 2) = 0 \Rightarrow \lambda = -1,$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \frac{1+3}{2}, \frac{1-3}{2} = 2, -1$$

Therefore, the eigen values are 2, -1, and -1

<u>To find the eigen vectors:</u>

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = 2$, $\begin{bmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + x_3 = 0$ --------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------- (2)

$x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get

x_1 x_2 x_3

1 1 -2 1

-2 1 1 -2

$\Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = -1$, $\begin{bmatrix} 0-(-1) & 1 & 1 \\ 1 & 0-(-1) & 1 \\ 1 & 1 & 0-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + x_2 + x_3 = 0$ ---------- (1)

$x_1 + x_2 + x_3 = 0$ ---------------- (2)

$x_1 + x_2 + x_3 = 0$ ------------ (3). All the three equations are one and the same.

Therefore, $x_1 + x_2 + x_3 = 0$. Put $x_1 = 0 \Rightarrow x_2 + x_3 = 0 \Rightarrow x_3 = -x_2 \Rightarrow \frac{x_2}{1} = \frac{x_3}{-1}$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Since the given matrix is symmetric and the eigen values are repeated, let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$. X_3 Is orthogonal to X_1 and X_2 .

$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l + m + n = 0$ ------------ (1)

$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow 0l + m - n = 0$-------- (2)

Solving (1) and (2) by method of cross-multiplication, we get,

l m n

1 1 1 1

1 -1 0 1

$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$. Therefore, $X_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Thus, for the repeated eigen value $\lambda = -1$, there corresponds two linearly independent eigen vectors X_2 *and* X_3.

4.Find the eigen values and eigen vectors of $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sumofthemaindiagonalelements = 2 + 1 - 1 = 2,$

$S_2 = Sumoftheminorsofthemaindiagonalelements = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} =$ $-4 - 4 + 4 = \ -4$

,

$S_3 = DeterminantofA = \ |A|$ = 2(-4)+2(-2)+2(2) = - 8 – 4 + 4 = - 8

Therefore, the characteristic equation of A is $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

2	1	-2	-4	8
	0	2	0	−8
	1	0	-4	0

$(\lambda - 2)(\lambda^2 - 4) = 0 \Rightarrow \lambda = 2, \qquad \lambda = \ 2, -2$

Therefore, the eigen values are 2, 2, and -2

A is a non-symmetric matrix with repeated eigen values

To find the eigen vectors:

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = -2$, $\begin{bmatrix} 2-(-2) & -2 & 2 \\ 1 & 1-(-2) & 1 \\ 1 & 3 & -1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0$ --------- (1)

$x_1 + 3x_2 + x_3 = 0$ ------------- (2)

$x_1 + 3x_2 + x_3 = 0$ ------------ (3) . Equations (2) and (3) are one and the same.

Considering equations (1) and (2) and using method of cross-multiplication, we get,

x_1 x_2 x_3

-1 1 2 -1

3 1 1 3

$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7} \Rightarrow \frac{x_1}{4} = \frac{x_2}{1} = \frac{x_3}{-7}$

Therefore, $X_1 = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$

Case 2: If $\lambda = 2$, $\begin{bmatrix} 2-2 & -2 & 2 \\ 1 & 1-2 & 1 \\ 1 & 3 & -1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 0x_1 - 2x_2 + 2x_3 = 0$---------- (1)

$x_1 - x_2 + x_3 = 0$---------------- (2)

$x_1 + 3x_2 - 3x_3 = 0$------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

x_1 x_2 x_3

-2 2 0 -2

-1 1 1 -1

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{2} = \frac{x_3}{2} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

We get one eigen vector corresponding to the repeated root $\lambda_2 = \lambda_3 = 2$

5.Find the eigen values and eigen vectors of $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ which is a symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 1 + 5 + 1 = 7,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4 - 8 + 4 = 0$

$S_3 = Determinant of A = |A|$ = 1(4)-1(-2)+3(-14) = 4 + 2-42 = - 36

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$

-2	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

$(\lambda - (-2))(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow \lambda = -2,$

$$\lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(18)}}{2(1)} = \frac{9 \pm \sqrt{81 - 72}}{2} = \frac{9 \pm 3}{2} = \frac{9+3}{2}, \frac{9-3}{2} = 6, 3$$

Therefore, the eigen values are -2, 3, and 6

To find the eigen vectors:

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 & 3 \\ 1 & 5-(-2) & 1 \\ 3 & 1 & 1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 + 7x_2 + x_3 = 0$ ------------- (2)

$3x_1 + x_2 + 3x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

x_1 x_2 x_3

1 3 3 1

7 1 1 7

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}.$$

Therefore, $X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Case 2: If $\lambda = 3$, $\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 + 2x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

x_1 x_2 x_3

1 3 -2 1

2 1 1 2

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case 3: If $\lambda = 6$, $\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -5x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 - x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 5x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

x_1 x_2 x_3

1 3 -5 1

-1 1 1 -1

$\Rightarrow \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS:

Property 1:

(i) The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal (or) The sum of the eigen values of a matrix is equal to the trace of the matrix

(ii) Product of the eigen values is equal to the determinant of the matrix

Property 2:

A square matrix A and its transpose A^T have the same eigen values (or) A square matrix A and its transpose A^T have the same characteristic values

Property 4:

If λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$, $(\lambda \neq 0)$ is the eigen value of A^{-1}

Property 5:

If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value

Property 6:

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$(m being a positive integer)

Property 7:

The eigen values of a real symmetric matrix are real numbers

Property 8:

The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal

Property 9:

Similar matrices have same eigen values

Property 10:

If a real symmetric matrix of order 2 has equal eigen values, then the matrix is a scalar matrix

Property 11:

The eigen vector X of a matrix A is not unique.

Property 12:

If $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct eigen values of a n x n matrix, then the corresponding eigen vectors $X_1, X_2, \dots, X_n$ form a linearly independent set

Property 13:

If two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the equal roots

Property 14:

Two eigen vectors X_1 *and* X_2 are called orthogonal vectors if $X_1^T X_2 = 0$

Property 15:

Eigen vectors of a symmetric matrix corresponding to different eigen values are orthogonal

Property 16:

If A and B are n x n matrices and B is a non-singular matrix then A and $B^{-1}AB$ have same eigen values

Problems:

1. **Find the sum and product of the eigen values of the matrix** $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

   Solution: Sum of the eigen values = Sum of the main diagonal elements = -3.

   Product of the eigen values = $|A|$ = -1 (1 – 1) -1(-1 – 1) + 1(1- (-1)) = 2 + 2 = 4

2. **Two of the eigen values of** $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **are 2 and 8. Find the third eigen value**

Solution: We know that sum of the eigen values = Sum of the main diagonal elements

= 6+3+3 = 12

Given $\lambda_1 = 2, \lambda_2 = 8, \lambda_3 = ?$

Therefore, $\lambda_1 + \lambda_2 + \lambda_3 = 12 \Rightarrow 2 + 8 + \lambda_3 = 12 \Rightarrow \lambda_3 = 2$

Therefore, the third eigen value = 2

3. **If 3 and 15 are the two eigen values of A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, find |A|, without expanding the determinant**

Solution: Given $\lambda_1 = 3 \text{ and } \lambda_2 = 15, \lambda_3 = ?$

We know that sum of the eigen values = Sum of the main diagonal elements

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$

$\Rightarrow 3 + 15 + \lambda_3 = 18 \Rightarrow \lambda_3 = 0$

We know that the product of the eigen values $= |A|$

$\Rightarrow (3)(15)(0) = |A| \qquad \Rightarrow |A| = 0$

4. **If 2, 2, 3 are the eigen values of A = $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$, find the eigen values of A^T**

Solution: By the property "A square matrix A and its transpose A^T have the same eigen values", the eigen values of A^T *are* 2,2,3

5. **Two of the eigen values of A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the eigen values of A^{-1}**

Solution: Sum of the eigen values = Sum of the main diagonal elements = 3 +5+3 = 11

Given 3,6 are two eigen values of A. Let the third eigen value be k.

Then, 3 + 6 + k = 11 $\Rightarrow k = 2$.Therefore, the eigen values of A are 3, 6, 2

By the property "If the eigen values of A are $\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

CAYLEY-HAMILTON THEOREM

Statement: Every square matrix satisfies its own characteristic equation

Uses of Cayley-Hamilton theorem:

(1) To calculate the positive integral powers of A

(2) To calculate the inverse of a square matrix A

Problems:

1. **Show that the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ satisfies its own characteristic equation**

<u>**Solution**</u>:Let A =$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 = 2$$

$$S_2 = |A| = 1 - (-4) = 5$$

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove $A^2 - 2A + 5I = 0$

$$A^2 = A(A) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$$

$$A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Therefore, the given matrix satisfies its own characteristic equation.

2. **Verify Cayley-Hamilton theorem for the matrix A = $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and hence find its inverse.**

Solution: The characteristic polynomial of A is $p(\lambda) = \lambda^2 - \lambda - 1$.

$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A^2 - A - I = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$A^2 - A - I = 0$,

Multiplying by A^{-1} we get $A - I - A^{-1} = 0$,

$A^{-1} = A - I$

$$A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

3. **Verify Cayley-Hamilton theorem for the matrix A = $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$ and hence find is inverse.**

Solution: The characteristic polynomial of A is $p(\lambda) = \lambda^3 - 2\lambda^2 - 5\lambda + 6$.

$$A^2 = \begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix}, A^3 = \begin{pmatrix} 11 & -3 & 22 \\ 29 & 4 & 17 \\ 16 & 3 & 5 \end{pmatrix}$$

To verify $A^3 - 2A^2 - 5A + 6I = 0$ -------------- (1)

$A^3 - 2A^2 - 5A + 6I =$

$$\begin{pmatrix} 11 & -3 & 22 \\ 29 & 4 & 17 \\ 16 & 3 & 5 \end{pmatrix} - 2\begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix} - 5\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiply equation (1) by A^{-1}

We get $A^2 - 2A - 5I + 6A^{-1} = 0$

$6A^{-1} = 5I + 2A - A^2$

$$6A^{-1} = 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} - \begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{pmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{pmatrix}$$

4. **Verify Cayley-Hamilton theorem for the matrix A =** $\begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$ **and hence find its inverse and A^4.**

Solution: The characteristic polynomial of A is $p(\lambda) = \lambda^3 - 4\lambda^2 - 3\lambda + 18 = 0$.

$$A^2 = \begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix}, A^3 = \begin{pmatrix} 65 & 27 & 19 \\ 140 & 27 & 70 \\ -146 & -54 & -46 \end{pmatrix}$$

To verify $A^3 - 4A^2 - 3A + 18I = 0$ -------------- (1)

A ^{3} – 4A ^{2} – 3A + 18 I =

$$\begin{pmatrix} 65 & 27 & 19 \\ 140 & 27 & 70 \\ -146 & -54 & -46 \end{pmatrix} - 4\begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix} - 3\begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix} + 18\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiply equation (1) by A $^{-1}$

We get A ^{2} – 4A – 3 I + 18 A $^{-1}$ = 0

18 A $^{-1}$ = 3 I + 4 A – A^2

$$18A^{-1} = 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 4\begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix} - \begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix}$$

$$= \begin{pmatrix} -32 & -2 & -19 \\ 60 & 6 & 30 \\ 46 & 4 & 29 \end{pmatrix}$$

$$A^{-1} = \frac{1}{18}\begin{pmatrix} -32 & -2 & -19 \\ 60 & 6 & 30 \\ 46 & 4 & 29 \end{pmatrix}$$

Multiply equation (1) by A

We get A ^{4} – 4A^3 – 3 A^2 + 18 A = 0

A ^{4} = 4A3 + 3 A^2 - 18 A

$$A^4 = 4\begin{pmatrix} 65 & 27 & 19 \\ 140 & 27 & 70 \\ -146 & -54 & -46 \end{pmatrix} + 3\begin{pmatrix} 23 & 6 & 7 \\ 20 & 9 & 10 \\ -38 & -12 & -10 \end{pmatrix} - 18\begin{pmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 383 & 108 & 151 \\ 260 & 81 & 130 \\ -734 & -216 & -286 \end{pmatrix}$$

5. Verify Cayley-Hamilton theorem, find A^4 and A^{-1} when A = $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

<u>Solution</u>: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 2 + 2 + 2 = 6$

$S_2 = Sum\, of\, the\, minirs\, of\, the\, main\, diagonal\, elements = 3 + 2 + 3 = 8$

$S_3 = \left| A \right| = 2(4-1) + 1(-2+1) + 2(1-2) = 2(3) - 1 - 2 = 3$

Therefore, the characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove that: $A^3 - 6A^2 + 8A - 3I = 0$------------- (1)

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A^2(A) = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$A^3 - 6A^2 + 8A - 3I$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

To find A^4:

$(1) \Rightarrow A^3 - 6A^2 + 8A - 3I = 0 \Rightarrow A^3 = 6A^2 - 8A + 3I$ ------------- (2)

Multiply by A on both sides, $A^4 = 6A^3 - 8A^2 + 3A = 6(6A^2 - 8A + 3I) - 8A^2 + 3A$

Therefore, $A^4 = 36A^2 - 48A + 18I - 8A^2 + 3A = 28A^2 - 45A + 18I$

Hence, $A^4 = 28 \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} = \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}:

Multiplying (1) by A^{-1},$A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$$\Rightarrow 3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$