SMTX1011 Applied Numerical Method

(Common to all Engineering Except CSE, IT, and Bio groups)

III Year V Semester (Batch 2010 onwards)

Course Material

Course Objective: The ability to identify, reflect upon, evaluate and apply different types of knowledge and information to form independent judgments. Analytical, logical thinking and conclusions based on quantitative information will be the main objective of learning this subject.

Unit 4: Numerical Solution of Ordinary Differential Equations

Numerical Solutions of Ordinary Differential Equations –Taylor's Series- Modified Euler's method – Runge –Kutta Method of fourth order – Predictor – corrector methods –Milne's method – Adam's Bash forth method

UNIT - IV

Numerical solution of ordinary Differential Equations

Introduction

Numerical methods for differential equations are of great importance to the engineers and physicists because practical problems often lead to differential equations that cannot be solved by analytical methods. Here, we discuss various methods for finding to any desired degree of accuracy the numerical solution of any ordinary differential equation with given initial conditions.

Suppose the first order differential equation

$$\frac{dy}{dx} = f(x,y) \text{ ——— } (1)$$

is given with the initial condition $y(x_0) = y_0$. If we can obtain a formula for the solution, we may evaluate it numerically, either directly or by the use of tables. If that formula is too complicated or if no formula for the solution is available, we may apply step-by-step method. In this method, we start from $y(x_0) = y_0$ and proceed with the approximate value of y_1 of the solution of (1) at $x = x_1 = x_0 + h$. In the second step, we compute an approximate value y_2 of the solution at $x = x_2 = x_0 + 2h$ etc. Here h is a fixed number called step size.

There are several methods for solving differential equations numerically and the most important are

(i) Taylor series method

(ii) Modified Euler's method.

(iii) Runge-kutta Method

(iv) Milne's predictor-corrector Method

(v) Adam's Bashforth Method.

Taylor's series Method

If $y = f(x)$ then its Taylor series about the point $x = x_0$ is,

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \frac{(x-x_0)^3}{3!}f'''(x_0) + \cdots$$

This formula can also be written as

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

where x_0 and y_0 denote the initial values of x and y.

Using the notation $y_1 = y(x_0+h)$, $y_2 = y(x_0+2h)$, $y_3 = y(x_0+3h)$ etc., we have by Taylor's formula,

$$y(x_0+h) = y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$y(x_0+2h) = y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots$$

$$y(x_0+3h) = y_3 = y_2 + hy_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \cdots$$

In general, $y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \cdots$

Note

(i) If we increase the terms in Taylor formula then we get more accurate answer.

(ii) y_0' means the value of y' at $x = x_0$ and $y = y_0$

y_0'' means the value of y'' at $x = x_0$ and $y = y_0$ and so on.

similarly y_1' means the value of y' at $x = x_1$ and $y = y_1$ and so on.

problems

1) Using Taylor's method compute $y(1.1)$ correct to 4 decimal places if $y(x)$ satisfies $y' = x+y$, $y(1) = 0$

Solution:

Given $y' = x+y$ and $x_0 = 0$, $y_0 = 0$.

②

using Taylor's series,

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \quad \text{——— (1)}$$

Take $h = 0.1$

$y' = x+y \qquad y_0' = x_0+y_0 = 1$

$y'' = 1+y' \qquad y_0'' = 1+y_0' = 2$

$y''' = y'' \qquad y_0''' = y_0'' = 2$

$y^{iv} = y''' \qquad y_0^{iv} = 2$

$y^{v} = y^{iv} \qquad y_0^{v} = 2$ etc

$$(1) \Rightarrow y_1 = y(1.1) = 0 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(2) + \frac{(0.1)^5}{120}(2) + \cdots$$

$$= 0.1 + 0.01 + 0.00033 + 0.0000083 + 0.000000166 + \cdots$$

$$\boxed{y(1.1) = 0.1103384\overline{7}}$$

2) Using Taylor method, compute $y(0.2)$ and $y(0.4)$ correct to 4 decimal places given $\frac{dy}{dx} = 1-2xy$ and $y(0)=0$.

Solution:

Given $y' = 1-2xy \qquad x_0 = 0 \quad y_0 = 0 \quad h = 0.2$

$x_1 = 0.2 \quad y_1 = ?$

$x_2 = 0.4 \quad y_2 = ?$

$y' = 1-2xy \qquad y_0' = 1-2x_0y_0 = 1$

$y'' = -2(xy'+y) \qquad y_0'' = 0$

$y''' = -2(xy''+2y') \qquad y_0''' = -4$

$y^{iv} = -2(xy'''+3y'') \qquad y_0^{iv} = 0$

$y^{v} = -2(xy^{iv}+4y''') \qquad y_0^{v} = 32$

By Taylor's series,

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$y_1 = y(0.2) = 0 + \frac{0.2}{1}(1) + \frac{(0.2)^2}{2}(0) + \frac{(0.2)^3}{6}(-4) + \frac{(0.2)^4}{24}(0) + \frac{(0.2)^5}{120}(32) + \cdots$$

$$= 0.2 - 0.0053333 + 0.000085333$$

$$\boxed{y(0.2) = 0.19475\,2003}$$

Now $y_1' = 1 - 2x_1 y_1$

$= 1 - 2(0.2)(0.19752003)$

$= 0.9220992$

$y_1'' = -2(x_1 y_1' + y_1)$

$= -2[(0.2)(0.9220992) + 0.19752003]$

$= -0.7583686$

$y_1''' = -2[x_1 y_1'' + 2y_1']$

$= -2[(0.2)(-0.758343686) + 2(0.9220992)]$

$= -3.38505933$

$y_1^{IV} = 5.90408585$

using Taylor's series

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$$

$$y_2 = y(0.4) = 0.19752003 + (0.2)(0.9220992) + \frac{(0.2)^2}{2}(-0.758343686) + \frac{(0.2)^3}{6}(-3.38505933) + \frac{(0.2)^4}{24}(5.90408585)$$

$$\boxed{y(0.4) = 0.354883723}$$

Taylors series method for simultaneous first order differential equations (3)

The simultaneous first order differential equations of the type $\frac{dy}{dx} = f(x,y,z)$, $\frac{dz}{dx} = g(x,y,z)$ with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by using Taylor method.

problems

1) Solve $\frac{dy}{dx} = z - x$, $\frac{dz}{dx} = y + x$ with $y(0) = 1$, $z(0) = 1$, by taking $h = 0.1$ to get $y(0.1)$ and $z(0.1)$. Here y and z are dependent variables and x is independent.

Solution:

Given $y' = z - x$ and $z' = x + y$

$x_0 = 0 \quad y_0 = 1$ | $x_0 = 0 \quad z_0 = 1 \quad h = 0.1$

$x_1 = 0.1 \quad y_1 = ?$ | $x_1 = 0.1 \quad z_1 = ?$

$y' = z - x$ | $z' = x + y$

$y'' = z' - 1$ | $z'' = 1 + y'$

$y''' = z''$ etc. | $z''' = y''$ etc.

By Taylor series, for y_1 and z_1 we have

$$y_1 = y(0.1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \quad \text{---} (1)$$

and

$$z_1 = z(0.1) = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots \quad \text{---} (2)$$

$y_0 = 1$

$y_0' = z_0 - x_0 = 1 - 0 = 1$

$y_0'' = z_0' - 1 = 1 - 1 = 0$

$y_0''' = z_0'' = 2$

$z_0 = 1$

$z_0' = x_0 + y_0 = 0 + 1 = 1$

$z_0'' = 1 + y_0' = 1 + 1 = 2$

$z_0''' = y_0'' = 0$

$z_0^{iv} = y_0''' = 2$

Substituting in (1) & (2), we get

$$y_1 = y(0.1) = 1 + (0.1) + \frac{(0.01)}{2}(0) + \frac{(0.001)}{6}(2) + \cdots$$

$$= 1 + 0.1 + 0.000333 + \cdots$$

$$= 1.1003 \text{ (correct to 4 decimals)}$$

$$\boxed{y(0.1) = 1.1003}$$

$$z_1 = z(0.1) = 1 + (0.1)(1) + \frac{0.01}{2}(2) + \frac{0.001}{6}(0) + \frac{0.0001}{24}(2) + \cdots$$

$$= 1 + 0.1 + 0.01 + 0.0000083 + \cdots$$

$= 1.1100$ correct to 4 decimal places)

$$\therefore \boxed{y(0.1) = 1.1003 \text{ and } z(0.1) = 1.11}$$

2) Using Taylor series method, find approximate values of y and z corresponding to $x = 0.1$ given that $y(0) = 2$, $z(0) = 1$ and $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y^2$.

<u>Solution:</u>

Given $y' = x + z$, $z' = x - y^2$

$x_0 = 0 \quad y_0 = 2 \quad z_0 = 1 \quad h = 0.1$

$x_0 = 0 \quad y_0 = 2$

$x_1 = 0.1 \quad y_1 = ?$

$z_1 = ?$

$y' = x + z \qquad z' = x - y^2$

$y'' = 1 + z' \qquad z'' = 1 - 2yy'$

$y''' = z'' \qquad z''' = -2(yy'' + y'^2)$

<u>To find $y(0.1)$ and $z(0.1)$</u>

Taylor series for y_1 is

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^2}{3!}y_0''' + \cdots \quad \text{———} (1)$$

$y_0' = x_0 + z_0 = 1$

$y_0'' = 1 + z_0' = -3$

$y_0''' = z_0'' = -3$

$z_0' = x_0 - y_0^2 = -4$

$z_0'' = 1 - 2y_0 y_0' = -3$

$z_0''' = -2(y_0 y_0'' + y_0'^2)$

$= 10$

$$(1) \Rightarrow y_1 = 2 + (0.1)(1) + \frac{(0.1)^2}{2!}(-3) + \frac{(0.1)^3}{3!}(-3)$$

$$= 2.1 - 0.015 - 0.0005$$

$$\boxed{y(0.1) = 2.0845}$$

Taylor series for z_1 is

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots$$

(4)

$$z_1 = 1 + (0.1)(-4) + \frac{(0.1)^2}{2!}(-3) + \frac{(0.1)^3}{3!}(10) + \cdots$$

$$= 1 - 0.4 - 0.015 + 0.001666$$

$$\boxed{z(0.1) = 0.5866}$$

Taylorseries method for second order differential equation.

Any differential eqn of the second or higher order can be solved by reducing it to a lower order differential equation. A second order differential eqn can be reduced to a first order differential equation by putting $y' = z$ and then the latter one canbe solved as usual.

Suppose $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = f(x)$ —— (1) is the given differential equation together with the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$ where y_0, y_0' are known values.

put $y' = z$ —— (2)

$y'' = z'$

$(1) \Rightarrow z' + Pz + Qy = f(x)$

$z' = f(x) - Pz - Qy$

$= F(x, y, z)$ —— (3)

Thus we get two first order equations.

Let us solve the second order differential equation (1) under the initial condition $y(x_0) = y_0$ —— (4) and $y'(x_0) = y_0'$ —— (5)

Now from (2) the initial condition (5) becomes $z(x_0) = z_0$.

$\therefore$ to solve the differential equation (1) it is enough if we solve the first two order differential equation (4) and (5) under the conditions $y(x_0) = y_0$ and $z(x_0) = z_0$.

Now Taylor series for (3) is

$$z_1 = z_0 + \frac{h}{1!} z_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \cdots$$

where h is the stepsize and the values of z_0', z_0'', z_0''' can be determined from (3) at the point (x_0, y_0)

Taylor series for (2) is

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$= y_0 + h z_0 + \frac{h^2}{2!} z_0' + \frac{h^3}{3!} z_0'' + \cdots$$

(since $y' = z$, $y'' = z'$, $y''' = z''$ and so on)

similarly we determine the next higher value of z and y (ie) z_2 and y_2 as given below.

$$z_2 = z_1 + \frac{h}{1!} z_1' + \frac{h^2}{2!} z_1'' + \frac{h^3}{3!} z_1''' + \cdots$$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

$$= y_1 + \frac{h}{1!} z_1 + \frac{h^2}{2!} z_1' + \frac{h^3}{3!} z_1'' + \cdots$$

Here the values $z_1', z_1'', z_1''', \ldots$ etc. can be determined at (x_1, y_1) as before. proceeding in this way we can calculate the remaining values of y's

Problems

1) Solve $y'' = y + xy'$ given $y(0) = 1$, $y'(0) = 0$ and calculate $y(0.1)$

Solution: Given $x_0 = 0$, $y_0 = 1$ $y_0' = 0$

$y'' = y + xy'$

$y''' = y' + y' + xy'' = 2y' + xy''$

$y^{iv} = 2y'' + y'' + xy''' = 3y'' + xy'''$

$y^{v} = 4y''' + xy^{iv}$

$y^{vi} = 5y^{iv} + xy^{v}$

$y_0'' = y_0 + x_0 y_0' = 1$

$y_0''' = 2y_0' + x_0 y_0'' = 0$

$y_0^{iv} = 3y_0'' + x_0 y_0''' = 3$

$y_0^{v} = 0$

$y_0^{vi} = 15$

(5)

$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$

$= 1 + 0 + \frac{(0.1)^2}{2}(1) + 0 + \frac{(0.1)^4}{24}(3) + \cdots$

$= 1 + 0.005 + 0.0000125$

$\boxed{y_1 = 1.0050125}$

2) Find the value of $y(1.1)$ and $z(1.1)$ from $\frac{d^2y}{dx^2} + y^2\frac{dy}{dx} = x^3$, $y(1) = 1$, $y'(1) = 1$ by using Taylor series method.

Solution:

Given $y'' + y^2 y' = x^3$ —— (1)

put $y' = z$ —— (2)

$\therefore y'' = z'$ —— (3)

$\therefore$ (1) $\Rightarrow z' + y^2 z = x^3$

$z' = x^3 - y^2 z$ —— (4)

Given, $y(1) = 1$ and $y'(1) = 1$

$\Rightarrow y(1) = 1$ and $z(1) = 1$

(ie) $x_0 = 1$ $y_0 = 1$ $z_0 = 1$

(2) $\Rightarrow$ $y' = z$

$y'' = z'$

$y''' = z''$

(4) $\Rightarrow$ $z' = x^3 - y^2 z$

$z'' = 3x^2 - y^2 z' - z\,2yy'$

$= 3x^2 - y^2 z' - 2z^2 y$

$z''' = 6x - y^2 z'' - z'\,2yy' - 2(z^2 y' + y\,2zz')$

$= 6x - z'' y^2 - 2z'yz - 2(z^3 - 4zz'y)$

using Taylor series,

$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$ —— (5)

$y_0' = z_0 = 1$

$y_0'' = z_0' = (x_0^3 - y_0^2 z_0) = 0$

$y_0''' = z_0'' = (3x_0^2 - y_0^2 z_0' - 2z_0^2 y_0) = 1$

$$(3) \Rightarrow y_1 = 1 + 0.1 + \frac{(0.1)^2}{2}(0) + \frac{(0.1)^3}{6}(1)$$

$$\boxed{y(1.1) = 1.1002}$$

using Taylor series

$$z_1 = z_0 + \frac{h}{1!} z_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \cdots \quad (6)$$

$$z_0' = x_0^3 - y_0^3 z_0 = 0$$

$$z_0'' = 3x_0^2 - y_0^2 z_0' - 2z_0' y_0 = 1$$

$$z_0''' = [6x_0 - z_0'' y_0^2 - 2z_0' y_0 z_0 - 2(z_0^3 - 4z_0 z_0' y_0)]$$

$$= 3$$

$$(6) \Rightarrow z_1 = 1 + (0.1)0 + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(3)$$

$$= 1 + 0.005 + 0.0005$$

$$\boxed{z(1.1) = 1.0055}$$

6

Modified Euler's Method

The equation $y_{n+1} = y_n + h\left[f\left(x_n+\frac{h}{2},\ y_n+\frac{h}{2}f(x_n,y_n)\right)\right]$, $n=0,1,2,\ldots$ is called Modified Euler's formula.

Problems

1) Compute y at $x=0.25$ by Modified Euler's method given $y' = 2xy$, $y(0)=1$

Solution: Given $f(x,y) = 2xy$, $x_0 = 0$, $y_0 = 1$, $h = 0.25$, $x_1 = 0.25$

Using Modified Euler's method,

$$y_1 = y_0 + h\left[f\left(x_0+\frac{h}{2},\ y_0+\frac{h}{2}f(x_0,y_0)\right)\right] \quad \text{—— (1)}$$

$$f(x_0,y_0) = f(0,1) = 2(0)(1) = 0$$

$$y_1 = 1+(0.25)\,f(0.125,\ 1)$$

$$= 1+(0.25)[2\times 0.125\times 1]$$

$$\boxed{y(0.25) = 1.0625}$$

2) Solve $\frac{dy}{dx} = 1-y$ given $y(0)=0$ using Modified Euler's method at $x=0.1$ and 0.2 and compare your results with the exact solutions.

Solution: Given $\frac{dy}{dx} = 1-y = f(x,y)$, $f(x,y) = 1-y$

$x_0 = 0$, $y_0 = 0$, $h = 0.1$

$x_1 = 0.1$, $y_1 = ?$

$x_2 = 0.2$, $y_2 = ?$

By Modified Euler's method,

To find y_1: $y_1 = y_0 + h\,f\left(x_0+\frac{h}{2},\ y_0+\frac{h}{2}f(x_0,y_0)\right)$

$f(x_0,y_0) = 1-y_0 = 1$

$$x_0+\frac{h}{2} = 0+\frac{0.1}{2} = 0.05$$

$$y_0+\frac{h}{2}f(x_0,y_0) = 0+\frac{0.1}{2}(1) = 0.05$$

$$\therefore y_1 = 0+0.1\,f(0.05,\ 0.05)$$

$$= 0.1\,(1-0.05)$$

$$\boxed{y_1 = y(0.1) = 0.095}$$

To find y_2

$$y_2 = y_1 + h f\left(x_1 + \frac{h}{2},\ y_1 + \frac{h}{2} f(x_1, y_1)\right)$$

$$= 0.095 + 0.1\, f(0.15, 0.14025)$$

$$= 0.095 + (0.1)(1 - 0.14025)$$

$$\boxed{y_2 = y(0.2) = 0.18098}$$

$f(x_1, y_1) = 1 - y_1 = 0.905$

$x_1 + \frac{h}{2} = 0.1 + \frac{0.1}{2} = 0.15$

$y_1 + \frac{h}{2} f(x_1, y_1) = 0.095 + \frac{0.1}{2}(0.905)$

$= 0.14025$

Exact solution

$$\frac{dy}{dx} = 1 - y \Rightarrow \int \frac{dy}{1-y} = \int dx$$

$$-\log(1-y) = x + c$$

$$\log(1-y) = -x - c$$

$$1 - y = e^{-x} A \qquad \text{where } A = e^{-c}$$

At $x = 0$, $y = 0$

$\therefore A = 1$

$$\therefore \boxed{y = 1 - e^{-x}}$$

using this exact solution

$$y(0.1) = 1 - e^{-0.1}$$
$$= 0.09516258$$

$$y(0.2) = 1 - e^{-0.2}$$
$$= 0.181269247$$

$$y(0.3) = 1 - e^{-0.3}$$
$$= 0.259181779$$

x	Modified Euler	Exact solution
0.1	0.095	0.09516
0.2	0.18098	0.18127

Fourth order Runge-Kutta method

This method is most commonly used in practice.

Working Rule

To solve $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

calculate $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$k_4 = hf(x_0 + h, y_0 + k_3)$

and $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Now $\boxed{y_1 = y_0 + \Delta y}$

Now starting from (x_1, y_1) and repeating the process, we get (x_2, y_2) etc.

Problems

1) By applying the fourth order Runge-Kutta method find $y(0.2)$ from $y' = y - x$, $y(0) = 2$ taking $h = 0.1$

Solution:

Given $y' = y - x$ $x_0 = 0$ $y_0 = 2$ $h = 0.1$

(i.e) $f(x,y) = y - x$ $x_1 = 0.1$ $y_1 = ?$

$x_2 = 0.2$ $y_2 = ?$

To find $y_1 = y(0.1)$

$y_1 = y_0 + \Delta y$

where $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0) = 0.1(y_0 - x_0) = 0.1(2 - 0) = 0.2$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1f(0.05, 2.1) = 0.1(2.1 - 0.05) = 0.205$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1f(0.05, 2.1025) = 0.1(2.1025 - 0.05) = 0.205$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 2.20525) = 0.1(2.20525 - 0.1)$

$= 0.210525$

$\therefore \Delta y = \frac{1}{6}(0.2 + 2(0.205) + 2(0.20525) + 0.210525]$

$= 0.20517$

$\therefore y(0.1) = y_1 = y_0 + \Delta y = 2 + 0.20517 = 2.20517$

$\boxed{y(0.1) = 2.20517}$

To find $y_2 = y(0.2)$

$$y_2 = y_1 + \Delta y \quad \text{where} \quad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = h f(x_1, y_1) = 0.1 f(0.1, 2.20517)$
$= 0.1(2.20517 - 1)$
$= 0.2105$

$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1 f(0.15, 2.31042)$
$= 0.1(2.31042 - 0.15)$
$= 0.21604$

$k_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.1 f(0.15, 2.31319)$
$= 0.1(2.31319 - 0.15)$
$= 0.21632$

$k_4 = h f(x_1 + h, y_1 + k_3) = 0.1 f(0.2, 2.42149)$
$= 0.1(2.42149 - 0.2)$
$= 0.22214$

$\therefore \Delta y = \frac{1}{6}[0.2105 + 2(0.21604) + 2(0.21632) + 0.22214]$
$= 0.21622$

$\therefore y_2 = y_1 + \Delta y$
$= 2.20517 + 0.21622$

$$\boxed{y_2 = 2.42139}$$

2) Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ given $y(0) = 1$ at $x = 0.2, 0.4$

Solution:

Given $y' = f(x,y) = \frac{y^2 - x^2}{y^2 + x^2}$

$x_0 = 0, \quad h = 0.2 \quad y_0 = 1$

$x_1 = 0.2 \quad y_1 = ?$

$x_2 = 0.4 \quad y_2 = ?$

To find y_1

$f(x_0,y_0) = f(0,1) = \frac{1-0}{1+0} = 1$

$k_1 = hf(x_0,y_0)$

$= 0.2 \times 1 = 0.2$

$k_2 = hf(x_0+\frac{h}{2}, y_0+\frac{k_1}{2})$

$= 0.2 f(0.1, 1.1)$

$= 0.2 \left(\frac{(1.1)^2-(0.1)^2}{(1.1)^2+(0.1)^2} \right)$

$k_2 = 0.1967213$

$k_3 = hf(x_0+\frac{h}{2}, y_0+\frac{k_2}{2})$

$= 0.2 f(0.1, 1.0983606)$

$= 0.2 \left(\frac{(1.0983606)^2-(0.01)^2}{(1.0983606)^2+0.01^2} \right)$

$= 0.1967$

$k_4 = hf(x_0+h, y_0+k_3)$

$= 0.2 f(0.2, 1.1967)$

$= 0.2 \left(\frac{(1.1967)^2-(0.2)^2}{(1.1967)^2+(0.2)^2} \right)$

$= 0.1891$

$\therefore \Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$= \frac{1}{6}[0.2+2(0.19672)+2(1.1967)+0.1891]$

$\Delta y = 0.19598$

$\therefore y_1 = y_0 + \Delta y$

$= 1 + 0.19598$

$\boxed{y_1 = 1.19598}$

To find y_2

$k = hf(x_1,y_1)$

$= 0.2 f(0.2, 1.19598)$

$= 0.2 \left(\frac{(1.19598)^2-(0.2)^2}{(1.19598)^2+(0.2)^2} \right) = 0.1891$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$

$= 0.2\left(f(0.3, 1.29055)\right)$

$= 0.2\left(\frac{(1.29055)^2 - (0.3)^2}{(1.29055)^2 + (0.3)^2}\right)$

$= 0.17949$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$

$= 0.2\, f(0.3, 1.28572)$

$= 0.2\left(\frac{(1.28572)^2 - (0.\ )^2}{(1.28572)^2 + (0.\ )^2}\right)$

$= 0.1793$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$= 0.2\, f(0.4, 1.37528)$

$= 0.2\left(\frac{(1.37528)^2 - (0.4)^2}{(1.37528)^2 + (0.4)^2}\right)$

$= 0.1687$

$\therefore \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.1891 + 2(0.1795) + 2(0.1793) + 0.1687)$

$= 0.1792$

$\therefore y_2 = y_1 + \Delta y$

$= 1.19598 + 0.1792$

$\boxed{y(0.4) = 1.3751}$

Runge-Kutta method for simultaneous differential Equations (9)

consider the equation $\frac{dy}{dx} = f(x,y,z)$, $\frac{dz}{dx} = g(x,y,z)$

$y(x_0) = y_0$, $z(x_0) = z_0$.

To solve this system of differential equations at an interval of h, the values of y_1 and z_1 computed by using the following formulae.

$k_1 = hf(x_0, y_0, z_0)$ $\quad l_1 = hg(x_0, y_0, z_0)$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$ $\quad l_2 = hg(x_0 + h/2, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$ $\quad l_3 = hg(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$ $\quad l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$

Now $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ and $\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$\therefore y_1 = y_0 + \Delta y$ and $z_1 = z_0 + \Delta z$

Having got (x_1, y_1, z_1) we get (x_2, y_2, z_2) by repeating the above algorithm once again starting from (x_1, y_1, z_1)

problems

1) Find $y(0.1)$, $z(0.1)$ from the system of equations $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y^2$ given $y(0) = 2$, $z(0) = 1$ using Runge-Kutta method of fourth order.

Solution: Now $\frac{dy}{dx} = x + z$; $\frac{dz}{dx} = x - y^2$

$\therefore f_1(x, y, z) = x + z$, $f_2(x, y, z) = x - y^2$

$x_0 = 0$, $y_0 = 2$, $z_0 = 1$ $\quad h = 0.1$

we use $k_1 = hf_1(x_0, y_0, z_0)$ $\quad l_1 = hf_2(x_0, y_0, z_0)$

$k_2 = hf_1(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$ $\quad l_2 = hf_2(x_0 + h/2, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

$k_3 = hf_1(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$ $\quad l_3 = hf_2(x_0 + h/2, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$ $\quad l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3)$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ $\quad \Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

Now

$K_1 = (0.1) f(0, 2, 1)$
$= (0.1)(0+1)$
$= 0.1$

$K_2 = (0.1) f(0.05, 2.05, 0.8)$
$= 0.1(0.05 + 0.8)$
$= 0.085$

$K_3 = (0.1) f(0.05, 2.0425, 0.79288)$
$= 0.1(0.05 + 0.79288)$
$= 0.084288$

$K_4 = 0.1 f(0.1, 2.084288, 0.5878)$
$= 0.1(0.1 + 0.5878)$
$= 0.06878$

$l_1 = 0.1 f_2(0, 2, 1)$
$= 0.1(0 - 2^2)$
$= -0.4$

$l_2 = 0.1 f_2(0.05, 2.05, 0.8)$
$= 0.1(0.05 - (2.05)^2)$
$= -0.41525$

$l_3 = (0.1) f(0.05, 2.0425, 0.79285)$
$= 0.1(0.05 - (2.0425)^2)$
$= -0.41122$

$l_4 = 0.1 f(0.1, 2.084238, 0.5878)$
$= 0.1(0.1 - (2.084238)^2)$
$= -0.4244$

$y_1 = 2 + \frac{1}{6}(0.1 + 2(0.085 + 0.084238) + 0.06878)$
$= 2.0845$

$z_1 = 1 + \frac{1}{6}[-0.4 - (0.41525 + 0.41122) \times 2 - 0.4244]$
$= 0.5868$

$\boxed{y(0.1) = 2.0845}$ $\boxed{z(0.1) = 0.5868}$

2) Using Runge Kutta method, find the solution of the system $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y$, $y = 0, z = 1$ when $x = 0$ at interval of $h = 0.1$ $x = 0$ to $x = 0.1$

Solution:

Given $f(x, y, z) = x + z$ $g(x, y, z) = x - y$, $x_0 = 0$, $y_0 = 0$, $z_0 = 1$, $h = 0.1$

To find $y(0.1)$ and $z(0.1)$

$K_1 = h f(x_0, y_0, z_0)$
$= h(x_0 + z_0)$
$= 0.1(0+1)$
$= 0.1$

$l_1 = h g(x_0, y_0, z_0)$
$= h(x_0 - y_0)$
$= 0.1(0-0)$
$= 0$

$$k_2 = hf(x_0 + h/2,\ y_0 + k_1/2,\ z_0 + l_1/2)$$

$$= h[(x_0 + h/2) + z_0 + l_1/2]$$

$$= 0.1[(0 + \tfrac{0.1}{2}) + (1 + \tfrac{0}{2})]$$

$$= 0.105$$

$$k_3 = hf(x_0 + h/2,\ y_0 + \tfrac{k_2}{2},\ z_0 + l_2/2)$$

$$= h((x_0 + h/2) + (z_0 + l_2/2))$$

$$= 0.1[(0 + \tfrac{0.1}{2}) + (1 + \tfrac{0}{2})]$$

$$= 0.105$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$= h[(x_0 + h) + (z_0 + l_3)]$$

$$= 0.1[(0 + 0.1) + (1 - 0.00026)]$$

$$= 0.1099$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.1 + 2(0.105) + 2(0.105) + 0.1099]$$

$$= 0.1050$$

$$y_1 = y_0 + \Delta y$$

$$= 0 + 0.1050$$

$$\boxed{y_1 = 0.1050}$$

$$l_2 = hg(x_0 + h/2,\ y_0 + k_1/2,\ z_0 + l_1/2)$$

$$= h[(x_0 + h/2) - (y_0 + k_1/2)]$$

$$= 0.1[(0 + \tfrac{0.1}{2}) - (0 + \tfrac{0.1}{2})]$$

$$= 0$$

$$l_3 = hg(x_0 + h/2,\ y_0 + k_2/2,\ z_0 + l_2/2)$$

$$= h[(x_0 + h/2) - (y_0 + k_2/2)]$$

$$= 0.1[(0 + \tfrac{0.1}{2}) - (0 + \tfrac{0.105}{2})]$$

$$= -0.00026$$

$$l_4 = hg(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$= h[(x_0 + h) - (y_0 + k_3)]$$

$$= 0.1((0 + 0.1) - (0 + 0.105))$$

$$= -0.0005$$

$$\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= \frac{1}{6}(0 + 0 + 2(-0.00026) + 0.0005)$$

$$= -0.00017$$

$$z_1 = z_0 + \Delta z$$

$$= 1 - 0.00017$$

$$\boxed{z_1 = 0.9998}$$

6

Runge-kutta method for Second order differential equation.

To Solve $y'' = f(x, y, y')$, given $y(x_0) = y_0$

$y'(x_0) = y_0'$.

Now set $y' = z$ and $y'' = z'$.

Hence, differential equation reduces to

$$\frac{dy}{dx} = y' = z.$$

and $\frac{dz}{dx} = z' = y'' = f(x, y, y') = f(x, y, z)$.

$\therefore \frac{dy}{dx} = z$ and $\frac{dz}{dx} = f(x, y, z)$ are

Simultaneous equations where $f_1(x, y, z) = z$

given, also $y(0)$ and $z(0)$ are given.

Example: Given $y'' + xy' + y = 0$; $y(0) = 1$,

$y'(0) = 0$. Find the value of $y(0.1)$ by using

Runge-kutta method of fourth order.

Solution: $y'' = -xy' - y$, $y(0) = 1$, $y'(0) = 0$, $h = 0.1$,

$y_0 = 1$, $x_0 = 0$, $y_1 = y(0.1)$.

Setting $y' = z$.

the equation becomes,

$$y'' = z' = -xz - y.$$

$$\therefore \frac{dy}{dx} = z = f_1(x, y, z).$$

$\frac{dz}{dx} = -xy - z = f_2(x, y, z).$

given $y_0 = 1$, $z_0 = y_0' = 0$

By algorithm,

$k_1 = hf_1(x_0, y_0, z_0) = (0.1) f_1(0,1,0) = 0$

$l_1 = hf_2(x_0, y_0, z_0) = (0.1) f_2(0,1,0) = -0.1$

$k_2 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$

$= (0.1) f_1(0.05, 1, -0.05) = -0.005$

$l_2 = (0.1) f_2(0.05, 1, -0.05) = -0.09975$

$k_3 = h f_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$

$= (0.1) f_1(0.05, 0.9975, -0.0499] = -0.0049$

$l_3 = hf_2(0.05, 0.9975, -0.049) = -0.09950.$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= (0.1) f_1(0.1, 0.99511, -0.0995)$

$= -0.0995$

$l_4 = hf_2(0.1, 0.99511, -0.0995)$

$= (0.1)[-(0.1)(-0.0995) + 0.99511]$

$= -0.0985$

$y_1 = y_0 + \Delta y$

$= 1 + \frac{1}{6}[0 + 2(-0.005) + 2(-0.00499) - 0.00995]$

$= 0.9950$

$y(0.1) = 0.9950.$

Millne's Predictor Corrector formulae:

Suppose our aim is to solve

$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ numerically.

General Millne's Predictor formulae:

Knowing 4 consecutive values of y namely, $y_{n-3}, y_{n-2}, y_{n-1}$ and y_n we calculate y_{n+1} using Predictor formula,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

when $n=3$,

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right].$$

where h is a suitable accepted spacing.

General Millne's Corrector formulae:

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

when $n=3$

$$y_{4,c} = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right].$$

Example 1. Find $y(2)$ if $y(x)$ is the solution of $\frac{dy}{dx} = \frac{1}{2}(x+y)$ given $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$ and $y(1.5) = 4.968$

Solution:

Here $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$

$x_4 = 2.0$, $h = 0.5$, $y_0 = 2$, $y_1 = 2.636$,

$y_2 = 3.595$, $y_3 = 4.968$.

$$f(x,y) = y' = \frac{1}{2}(x+y) \quad \text{.....}(1)$$

By Milne's Predictor formula

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$$\therefore y_{4,P} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right] \text{....}(2)$$

From (1),

$$y'_1 = \frac{1}{2}(x_1+y_1) = \frac{1}{2}[0.5 + 2.636] = 1.5680$$

$$y'_2 = \frac{1}{2}[x_2+y_2] = \frac{1}{2}[1 + 3.595] = 2.2975$$

$$y'_3 = \frac{1}{2}[x_3+y_3] = \frac{1}{2}[1.5 + 4.968] = 3.2340$$

By (2)

$$y_{4,P} = 2 + \frac{4(0.5)}{3}\left[2(1.5680) - (2.2975) + 2(3.2340)\right]$$

$$= 6.8710$$

Using Millne's Corrector formula,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$

i.e) $$y_{4,c} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4] \cdots (3)$$

$$y'_4 = \frac{1}{2}[x_4 + y_4] = \frac{1}{2}[2 + 6.8710] = 4.4355$$

Using (3) we get

$$y_{4,c} = 3.595 + \frac{0.5}{3}[2.2975 + 4(3.2340) + 4.4355]$$
$$= 6.8732.$$

$\therefore$ Corrected value of y at $x = 2$ is 6.8732.

Note: Suppose y_1, y_2, y_3 values are not given use any of the previous methods (ie) Taylor Series method, Euler's method, R.K method) to get the values.

Example 2: Determine the value of $y(0.4)$ using Millne's method given $y' = xy + y^2$, $y(0) = 1$, Use Taylor Series to get the values of $y(0.1)$ $y(0.2)$ & $y(0.3)$

Solution:

Here $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$

$x_4 = 0.4$ and $y_0 = 1$.

$y' = xy + y^2$; $y_0' = x_0 y_0 + y_0^2 = 1$

$y'' = xy' + y + 2yy'$; $y_0'' = x_0 y_0' + y_0 + 2y_0 y_0' = 3$

$y''' = xy'' + y' + y' + 2yy'' + 2y'^2$.

$y''' = xy'' + 2y' + 2yy'' + 2y'^2$; $y_0''' = x_0 y_0'' + 2y_0' + 2y_0 y_0'' + 2y_0'^2$

$= 10$.

$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$

$= 1 + (0.1)(1) + \frac{(0.01)}{2}(3) + \frac{(0.001)}{6}(10) + \cdots$

$= 1 + 0.1 + 0.015 + 0.001666$

$y(0.1) = 1.1167$

$y_1' = x_1 y_1 + y_1^2 = 1.3587$

$y_1'' = x_1 y_1' + y_1 + 2y_1 y_1' = (0.1)(1.3587) + 1.1167 + 2(1.1167)(1.3587)$

$= 4.2871$

$y_1''' = x_1 y_1'' + 2y_1' + 2y_1 y_1'' + 2y_1'^2$

$= 16.4131$

$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$

$= 1.1167 + \frac{0.1}{1}(1.3587) + \frac{0.01}{2}(4.2871) + \frac{(0.001)}{6}(16.4131) + \cdots$

$y(0.2) = 1.2767$

$y_2' = x_2 y_2 + y_2^2 = 1.8853$

$y_2'' = x_2 y_2' + y_2 + 2y_2 y_2' = 6.4677$

$y_3''' = x_2 y_2'' + 2y_2' + 2y_2 y_2'' + y_2'^2 = 28.6825$

$\therefore y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \ldots$

$= 1.2767 + (0.1)(1.8853) + \frac{(0.01)}{2}(6.4677) + \frac{(0.001)}{6}(28.68)$

$y(0.3) = 1.5023$

By Millne's Predictor formula,

$y_{4,P} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3]$

$= 1 + \frac{4(0.1)}{3}[2(1.35827) - 1.8853 + 2(2.7076)]$

$= 1.83297$

Now $y_{4,p}' = x_4 y_4 + y_4^2 = (0.4)(1.83297) + (1.83297)^2$

$= 4.09296$

Using Millne's corrector formula

$y_{4,C} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_{4,p}']$

$= 1.2767 + \frac{0.1}{3}[1.8853 + 4(2.7076) + 4.09296]$

$= 1.83698$

EXERCISE

1. Using Millne's method, find $y(0.2)$ given $\frac{dy}{dx} = (0.2)x + (0.1)y$, $y(0) = 2$, $y(0.05) = 2.0103$, $y(0.1) = 2.0211$, $y(0.15) = 2.0323$.

2. Find $y(0.8)$ given $y' = y - x^2$, $y(0) = 1$, $y(0.2) = 1.1218$, $y(0.4) = 1.46820$, $y(0.6) = 1.73790$

3. Using R.K. method of fourth order find y. Using R.K. method of fourth order find y at $x = 0.1, 0.2, 0.3$ given $y' = xy + y^2$, $y(0) = 1$. Continue your work to get $y(0.4)$ by Millne's method

4. Solve $y' = \frac{1}{2}(1+x)y^2$, $y(0) = 1$ By Euler's method at $x = 0.2, 0.4, 0.6$, and hence find $y(0.8)$ and $y(1)$ by millne's method.

5. Solve $y' = x - y^2$, $y(0) = 1$ to obtain $y(0.4)$ by Millne's method. Obtain the data you require by any method of your liking.

4

Adam - Bashforth (or Adam's) Predictor Corrector Method:

General Adam's Predictor formula is

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

when $n = 3$

$$y_{4,p} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

Corrector formula is

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}'\right]$$

when $n = 3$

$$y_{4,c} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

Example 1: Solve and get $y(2)$ given $\frac{dy}{dx} = \frac{1}{2}(x+y)$ $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$ by Adam's method.

Solution:

$f(x,y) = y' = \frac{1}{2}(x+y)$; $y_0' = \frac{1}{2}(x_0+y_0) = 1$

$y_1' = \frac{1}{2}(x_1+y_1) = \frac{1}{2}[0.5 + 2.636] = 1.5680$

$y_2' = \frac{1}{2}(x_2+y_2) = \frac{1}{2}[1 + 3.595] = 2.2975$

$y_3' = \frac{1}{2}(x_3+y_3) = \frac{1}{2}[1.5 + 4.968] = 3.2340$

By Adam's Predictor formula

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

$$\therefore y_{4,p} = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0']$$

$$= 4.968 + \frac{0.5}{24}[55(3.2340) - 59(2.2975) + 37(1.5680) - 9(1)]$$

$$y_{4,p} = 6.8708$$

$$y_4' = \frac{1}{2}(x_4 + y_4) = \frac{1}{2}[2 + 6.8708] = 4.4354.$$

By Corrector,

$$y_{4,c} = y_3 + \frac{h}{24}[9y_4' + 19y_3' - 5y_2' + y_1']$$

$$= 4.968 + \frac{0.5}{24}[9(4.4354) + 19(3.234) - 5(2.2975) + 1.5680]$$

$$= 6.8731.$$

Example 2: Find $y(0.1), y(0.2), y(0.3)$ from $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$, by using R.K method and hence obtain $y(0.4)$ using Adam's method.

Solution: $f(x,y) = xy + y^2$, $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$, $y_0 = 1$.

$$k_1 = hf(x_0, y_0) = (0.1), f(0.1) = (0.1)1 = 0.1$$

$$k_2 = hf\left[0.05, y_0 + \frac{k_1}{2}\right] = (0.1)f(0.05, 1.05)$$

$$= 0.1155$$

$$k_3 = hf\left(0.05, y_0 + \frac{k_2}{2}\right) = (0.1)f(0.05, 1.0578)$$

$$= 0.1172$$

$k_4 = hf(x_0+h, y_0+k_3) = (0.1) f(0.1, 1.1172)$

$= 0.13598$

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1.1169$

To find y_2:

$k_1 = hf(x_1, y_1) = 0.1359$

$k_2 = hf[x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}] = (0.1) f(0.15, 1.1849)$

$= 0.1582$

$k_3 = hf[0.15, y_1 + \frac{k_2}{2}] = (0.1) f(0.15, 1.196)$

$= 0.16098.$

$k_4 = (0.1) f(0.2, 1.2779) = 0.1889$

$\therefore y_2 = 1.1169 + \frac{1}{6}[0.1359 + 2(0.1582 + 0.16098) + 0.1889]$

$= 1.2774.$

To find y_3:

$k_1 = hf(x_2, y_2) = (0.1) f(0.2, 1.2774) = 0.1887$

$k_2 = hf[x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}] = (0.1) f(0.25, 1.3718)$

$= 0.2225$

$k_3 = hf[x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}]$

$= 0.2274$

$k_4 = hf[x_3, y_2 + \frac{k_3}{2}]$

$= 0.2716$

$y_3 = 1.2774 + \frac{1}{6}[0.1887 + 2(0.2225) + 2(0.2274) + 0.2716]$

$= 1.5041.$

By Adam's Predictor formula,

$$y_{4,P} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$y_0' = x_0 y_0 + y_0^2 = 1$$

$$y_1' = x_2 y_2 + y_1^2 = 1.3592.$$

$$y_2' = x_2 y_2 + y_2^2 = 1.8872$$

$$y_3' = x_3 y_3 + y_3^2 = 2.7135$$

$$y_{4,P} = 1.5041 + \frac{0.1}{2}\left[55(2.7135) - 59(1.8872) + 37(1.392) - 9(1)\right]$$

$$= 1.8341.$$

$$y_4' = x_4 y_4 + y_4^2 = (0.4)(1.8341) + \left[(1.8341)^2\right]$$

$$= 4.0976.$$

$$y_{4,c} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$= 1.5041 + \frac{0.1}{24}\left[9(4.0976) + 19(2.7135) - 5(1.8872) + 1.3592\right]$$

$$y(0.4) = 1.8389$$