# UNIT IV

### ANALYSIS & DESIGN OF DIGITAL FILTERS

## Normalized Ideal Low-Pass Filter

$$H(j\omega) = e^{-j\omega} \qquad 0 \leq |\omega| \leq 1$$

$$= 0 \qquad |\omega| > 1$$

## Filter Transfer Function

$$H(j\omega) = |H(j\omega)|\, e^{j\theta(\omega)}$$

$\theta(\omega) = \text{Arg}H(j\omega)$

$\tau(\omega) = -\, d\,\theta(\omega)/d\,\omega = $ group delay

$\omega_c = $ cut off frequency at which $|H(j\omega_c)|^2 = 1/2$

or

$20 \log |H(j\,\omega)|_{\omega = \omega c} = -3db$

$A(\omega) = -10 \log |H(j\,\omega)|^2$ (=attenuation) dB

a

$|H(j\omega)|^2$

1

-1    1    $\omega$

-arg H(j$\omega$)

1

-1    1    $\omega$

-1

b

$|H_1(j\omega)|$

1

$\dfrac{1}{\sqrt{1+\varepsilon^2}}$

$\dfrac{1}{\alpha}$

|← pass band →|←transition band→|← stop band →

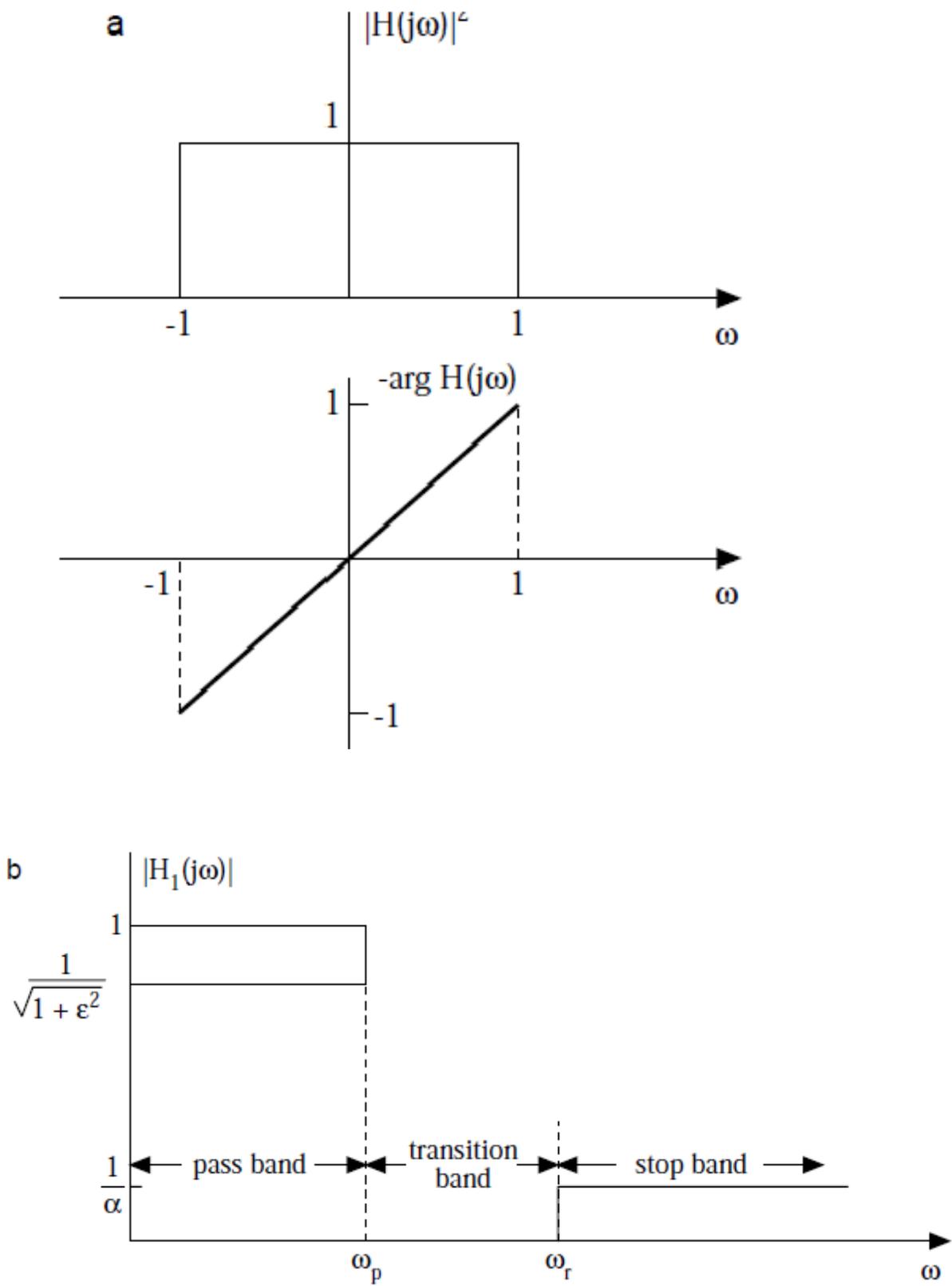$\omega_p$    $\omega_r$    $\omega$

Fig 4.1 Filter transfer function

$|H(j\omega)|=[\ H(j\omega)H^{'}(j\omega)]^{1/2} =[\ H(j\omega)H(-j\omega)]^{1/2} =[\ H(s)H^{'}(s)|_{s=j\omega}]^{1/2}$

$|H(j\omega)|^2 =\{K(\omega^2 = z_1{}^2)(\omega^2+z_2{}^2).....\}/\{(\omega^2+p_1{}^2)(\ \omega^2+p_2{}^2)...\}$

Complex poles and zeroes occur in conjugate pairs. Both the numerator and denominator polynomials of the magnitude squared function are polynomials of $\omega^2$ with real coefficients, and these polynomials are greater than zero for all $\omega$.

## 4.2 Butterworth Approximation

### Definition of Butterworth Low-Pass Filter

$|H(j\omega)|^2 =1/\{1 + (\omega/\omega_c)^{2n}\};\qquad |H(j\omega_c)|^2= 1/2$

$10\log|H(j\omega)^2||_{\omega=\omega c} = -3.01\cong -3.0$ dB

Normalized

$|H(j\omega)|^2 = 1/(1+\omega^{2n});\qquad |H(j1)|^2 = 1/2$

### Properties of Butterworth Approximation

$|H(j0)|^2= 1;\ \ |H(j1)|^2=1/2;\ \ |H(j\infty)|^2=0$

$-10\log|H(j1)|^2= -10\log 0.5=3.01\cong3.0$ dB

$|H(j\omega)|^2$ monotonically decreasing for $\omega\geq0$. Its maximum value is at $\omega=0$. The first $(2n-1)$ derivatives of an nth order low pass Butterworth filter are zero at $\omega=0$ maximally flat magnitude).

The high frequency roll-off of an nth order filter is 20n dB/decade.

$-10\log|H(j\omega)|^2= -\log(1/\ 1+\omega^{2n})\cong -\log(1/\omega^{2n})=10\log\omega^{2n}$

$$=20n\log\omega\ \textbf{dB}$$

**Transfer Function of Butterworth Approximation**

$|H(j\omega)|^2 = H(s)H(-s)|_{s=j\omega} = \{1/\ 1+(\omega/\omega_c)^{2n}\} = 1/(1+(-1)^n(s/\omega_c)^{2n})$

$(1+(-1)^n(s/\omega_c)^{2n})=0$ or $s_k = \omega_c\, e^{j\pi(1-n+2K)/2n}$, K=0,1,2,3....,2n-1

**Stable Function**

Left half plane poles are used

$s_k = \omega_c[-\sin\{(2K+1)\pi/2n)\}+j\cos\{(2K+1)\pi/2n)\}]$, K=0,1,....n-1.

## Transfer Function

$$H(s) = (-1)^n \prod_{K=0}^{n-1} \frac{s_K}{s - s_K}$$

**Butterworth Normalized Low -Pass Filter**

The Butterworth polynomials ($\omega_c =1$) to be used for normalized filters.

| n | Butterworth Polynomials |
|---|---|
| 1 | $s + 1$ |
| 2 | $s^2 + 1.41421s + 1$ |
| 3 | $(s + 1)(s + 1)$ |
| 4 | $(s^2 + 0.76537s + 1)(s^2 + 1.84776s + 1)$ |
| 5 | $(s + 1)(s^2 + 0.61803s + 1)(s^2 + 1.61803s + 1)$ |
| 6 | $(s + 0.51764s + 1)(s^2 + 1.41421s + 1)(s^2 + 1.93185s + 1)$ |

| | |
|---|---|
| 7 | $(s+1)(s^2+0.44504s+1)(s^2+1.24798s+1)(s^2+1.80194s+1)$ |
| 8 | $(s^2+0.39018s+1)(s^2+1.11114s+1)(s^2+1.66294s+1)(s^2+1.96157s+1)$ |
| 9 | $(s+1)(s^2+0.34730s+1)(s^2+s+1)(s^2+1.53209s+1)(s^2+1.87939s+1)$ |
| 10 | $(s^2+0.31287s+1)(s^2+0.90798s+1)(s^2+1.41421s+1)(s^2+1.78201s+1)$ |

**Butterworth Filter specifications**

$A_p$ = maximum passband attenuation

$f_p$ =passband edge frequency

Maximum allowable attenuation in the stopband

$f_r$=stopband edge frequency

$A_p=10\log[1+(\omega_p/\omega_c)^{2n}]$

$A_r= 10\log[1+ (\omega_r/\omega_c)^{2n}]$

$\omega_p=2\Pi f_p$

$\omega_r=2\Pi f_r$

Solve $A_p$ and $A_r$ to find

$(|\log[(10^{0.1Ap}-1)/(10^{0.1Ar}-1)]|)$

n=    ----------------------------------

$(|\log(\omega_p/\omega_r)|)$

k=selectivity parameter= $\omega_p/\omega_r =f_p/f_r <1$

d=discrimination factor=$((10^{0.1Ap}-1)/(10^{0.1Ar}-1))$

**note : a) larger values of k imply steeper roll off,   b) smaller d values imply greater difference between $A_p$ and $A_r$**

$n \geq |\log d|/|\log k|$       (accept next higher integer to non integer  $n$)

$\omega_c= \omega_p / (10^{0.1Ap}-1)^{1/2n}$

$\omega_c = \omega_p / (10^{0.1Ar}-1)^{1/2n}$ = meets stopband attenuation exactly and exceeds the requirement of passband specification shows magnitude –squared characteristics of the Butterworth low-pass
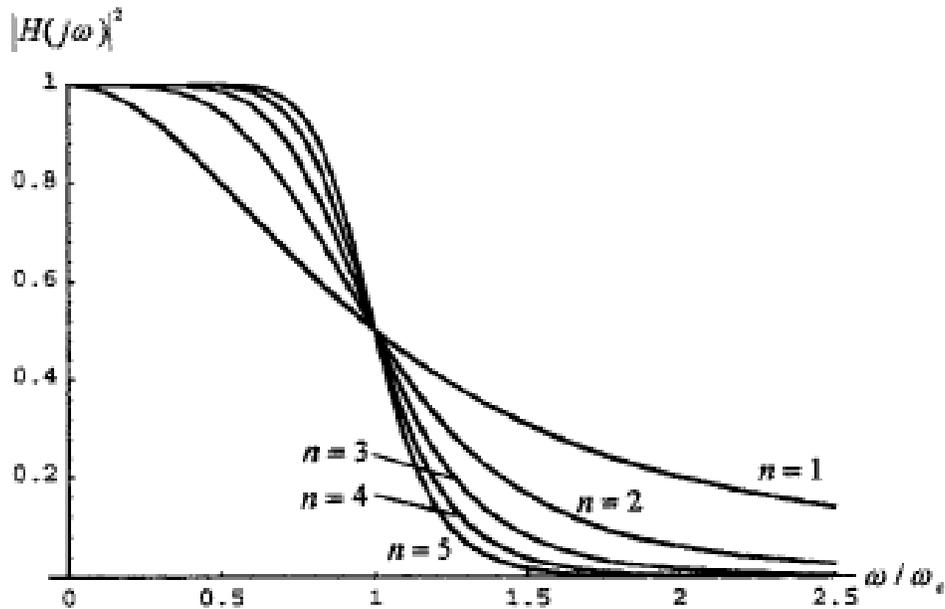


filter

**Fig 4.2** magnitude –squared characteristics of the Butterworth low-pass filter

## Butterworth Filter Design

**Problem 1**:

Filter requirements: a) no more than 1.5 dB deviation from ideal filter at 1300 Hz ;

b) at least 35 dB for frequencies above 6000 Hz.

**Solution:**

$A_p=1.5dB$  $\omega_p= 2\pi x1300$ rads$^{-1}$

$A_p=35dB$  $\omega_r=2\pi x6000$ rads$^{-1}$

$0.1A_r$

$d=\sqrt{(10^{0.1Ap}-1)} /( (\sqrt{(10^{0.1Ar}-1)}) = (\sqrt{(10^{0.15}-1)})/( \sqrt{(10^{3.5}-1)}) = 0.6423/56.2252$

$= 1.1424x10^{-2}$

$n \geq |\log d|/|\log k| = 1.9422/0.6576 = 2.953$          $\rightarrow n = 3$

$s_k = -\sin((2K+1)\pi)/(2n)) + j \cos ((2K+1) \pi)/(2n))$        $K=0,1,2,3\ldots,n-1$

$s_0 = -\sin(\pi/6) + j\cos (\pi/6) = -1/2 + j (\sqrt{3}/2)$

$s_1 = -\sin (3\pi/6) + j\cos(3\pi/6) = -1$

$s_2 = -\sin (5\pi/6) + j\cos(5\pi/6) = = -1/2 - j (\sqrt{3}/2)$

$$H(s) = (-1)^n \prod_{K=0}^{n-1} \frac{S_k}{s - S_k} = -\frac{\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)}{s - \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)} \frac{-1}{s - (-1)} \frac{-\frac{1}{2} - j\frac{\sqrt{3}}{2}}{s - \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)}$$

$= 1/((s+1)(s^2 +s+1)) = $ normalized

$\omega_c = \omega_p(10^{0.1Ap}-1)^{-1/2n} = 2\pi \times 1300(10^{0.15}-1)^{-1/6} = 9416$ rads$^{-1}$

$H(s/\omega_c) = 1/(((s/9461)+1) [(s/9461)^2 +(s/9461) +1])$

## Problem 2:

For the given specification design an analog Butterworth filter.

$$0.9 \leq |H(j\Omega)| \leq 1, \quad \text{for} \quad 0 \leq \Omega \leq 0.2\pi$$
$$|H(j\Omega)| \leq 0.2, \quad \text{for} \quad 0.4\pi \leq \Omega \leq \pi$$

Solution:

$$\Omega_p = 0.2\pi; \quad \Omega_s = 0.4\pi$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.9 \Rightarrow \varepsilon = 0.4843$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.899$$

$$N \geq \frac{\log\left(\dfrac{\lambda}{\varepsilon}\right)}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} \geq 3.34$$

N=4

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

Cut off frequency,

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}} = 0.24\pi$$

Obtain transfer function by substituting s by,

$$\frac{s}{\Omega_c} = \frac{s}{0.24\pi}$$

Finally,

$$H(s) = \frac{0.323}{(s^2 + 0.577s + 0.0576\pi^2)(s^2 + 1.393s + 0.0576\pi^2)}$$

**Problem 3** Design an analog Butterworth filter that has a -2DB pass band attenuation at a frequency of 20 rad/sec an at least -10 db stop band attenuation at 30 rad/sec.

Solution ,

$$N \geq \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} \geq 3.37 \Rightarrow N = 4$$

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p} - 1\right)^{1/2N}} = 21.3868$$

Obtain the transfer function by substituting,

$$\frac{s}{\Omega_c} = \frac{s}{21.3868}$$

$$H(s) = \frac{0.20921 \times 10^6}{(s^2 + 16.3686s + 457.394)(s^2 + 39.5176s + 457.394)}$$

**Problem 4** Design Butterworth filter using impulse invariant method for the following specification

$$0.8 \leq \left| H(e^{j\omega}) \right| \leq 1, \quad 0 \leq \omega \leq 0.2\pi$$

$$\left| H(e^{j\omega}) \right| \leq 0.2, \quad 0.6\pi \leq \omega \leq \pi$$

Solution

$$\frac{1}{\sqrt{1 + \varepsilon^2}} = 0.8 \Rightarrow \varepsilon = 0.75$$

$$\frac{1}{\sqrt{1 + \lambda^2}} = 0.2 \Rightarrow \lambda = 4.899$$

Ws = 0.6 pi rad: w$_p$ =0.2 pi rad . then assume T= 1 sec

$$\frac{\omega_s}{\omega_p} = \frac{\Omega_s T}{\Omega_p T} = \frac{\Omega_s}{\Omega_p} = \frac{0.6\pi}{0.2\pi} = 3$$

$$N \geq \frac{\log\left(\dfrac{\lambda}{\varepsilon}\right)}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} \geq \frac{\log\left(\dfrac{4.899}{0.75}\right)}{\log 3} \geq 1.71$$

For N=2 the transfer function of normalized Butterworth filter is

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Cut off frequency

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{\frac{1}{N}}} = \frac{0.2\pi}{(0.75)^{\frac{1}{N}}} = 0.231\pi$$

$$H_a(s) = H(s)\Big|_{s \to \frac{s}{\Omega_c}}$$

$$= \frac{0.5266}{(s + 0.51 + j0.51)(s + 0.51 - j0.51)}$$

Using impulse invariant method, if

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k},$$

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

$$H(z) = \frac{j0.516}{1 - e^{-0.51 - j0.51} z^{-1}} - \frac{j0.516}{1 - e^{-0.51 + j0.51} z^{-1}}$$

By solving,

$$H(z) = \frac{0.3022}{1 - 1.047 z^{-1} + 0.36 z^{-2}}$$

**Problem 5:** Design Butterworth filter using bilinear transformation

$$0.8 \le \left| H(e^{j\omega}) \right| \le 1, \quad 0 \le \omega \le 0.2\pi$$

$$\left| H(e^{j\omega}) \right| \le 0.2, \quad 0.6\pi \le \omega \le \pi$$

Solution

Ws = 0.6 pi rad:  w$_p$ =0.2 pi rad. then assume T= 1 sec

prewrapping the frequency

$$\Omega_s = 2\tan\frac{\omega_s}{2} = 2.752; \quad \Omega_p = 2\tan\frac{\omega_p}{2} = 0.6498$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8 \Rightarrow \varepsilon = 0.75; \quad \frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.899$$

$$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 4.235$$

$$N \ge \frac{\log\left(\dfrac{\lambda}{\varepsilon}\right)}{\log\left(\dfrac{1}{k}\right)} \ge \frac{\log\left(\dfrac{4.899}{0.75}\right)}{\log(4.235)} \ge 1.3$$

for N=2 the transfer function is

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

cut off frequency

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \frac{0.2\pi}{(0.75)^{1/2}} = 0.75 \text{rad}/\sec$$

$$H_a(s) = H(s)\Big|_{s \to \frac{s}{\Omega_c}}$$

$$= \frac{0.5625}{s^2 + 1.06s + 0.5625}$$

For bilinear transformation.

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{0.5625(1+z^{-1})^2}{4(1-z^{-1})^2 + 2.12(1-z^{-2}) + 0.5625(1+z^{-1})^2}$$

## 4.2 Chebyshev Filter Approximation

Definition of Chebyshev Filters (equi-ripple passband)

$C_0(\omega) = 1$ and $C_1(\omega) = \omega$. $|H(j\omega)|^2 = 1/(1 + \varepsilon\, C_n^2(\omega)) =$ normalized

$C_n(\omega) =$ Chebyshev polynomials $= \cos(n \cos^{-1}\omega)$    $0 \le \omega \le 1$

   $= \cosh(n \cos^{-1}\omega)$   $\omega > 1$

$\varepsilon =$ ripple factor

If we set $u = \cos^{-1}\omega$, then $C_n(\omega) = \cos nu$ and thus

$C_0(\omega) = \cos 0 = 1$, $C_1(\omega) = \cos u = \cos(\cos^{-1}\omega) = \omega$,

$C_2(\omega) = \cos 2u = 2\cos^2 u - 1 = 2\omega^2 - 1$

$C_3(\omega) = \cos 3u = 4\cos^3 u - 3\cos u = 4\omega^3 - 3\omega$. etc.

**Recursive Formula for Chebyshev Polynomials**

From $\cos[(n+1)u] = 2\cos nu - \cos[(n-1)u]$, we get

$C_{n+1}(\omega) = 2\omega C_n(\omega) - C_{n-1}(\omega)$        $n = 0,1,2,3\ldots$

**gives the first ten Chebyshev polynomials**

**Table 12.2 Chebyshev Polynomials $C_n(\omega)$**

------------------------------------------------------------------------------

| $n$ | Chebyshev Polynomials $C_n(\omega)$ |
|---|---|
| 0 | 1 |
| 1 | $\omega$ |

| 2 | $2\omega^2 - 1$ |
|---|---|
| 3 | $4\omega^3 - 3\omega$ |
| 4 | $8\omega^4 - 8\omega^2 + 1$ |
| 5 | $16\omega^5 - 20\omega^3 + 5\omega$ |
| 6 | $32\omega^6 - 48\omega^4 + 18\omega^2 - 1$ |
| 7 | $64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega$ |
| 8 | $128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1$ |
| 9 | $256\omega^9 - 576\omega^7 + 432\omega^5 - 120\omega^3 + 9\omega$ |
| 10 | $512\omega^{10} - 128\omega^8 + 1120\omega^6 - 400\omega^4 + 50\omega^2 - 1$ |

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## Properties of the Chebyshev Polynomials

1. For any $n$

$$0 \leq |C_n(\omega)| \leq 1 \quad \text{for } 0 \leq |\omega| \leq 1$$

$$|C_n(\omega)| > 1 \quad \text{for } |\omega| > 1$$

2. $C_n(1) = 1$ for any $n$.

3. $|C_n(\omega)|$ increases monotonically for $|\omega| > 1$

4. $C_n(\omega)$ is an even (odd) polynomial if $n$ is even (odd).

5. $|C_n(0)| = 0$ for odd $n$.

6. $|C_n(0)| = 1$ for even $n$.


## Chebyshev Magnitude Response Properties

1. $|H(j\omega)|_{\omega=0} = 1$ when $n$ is odd

$$= 1/(\sqrt{(1+\varepsilon^2)}) \text{ when } n \text{ is even}$$

2. Since $C_n(1)=1$ for any $n$

$$|H(j\,1)|= 1/(\sqrt{(1+\varepsilon^2)})\quad\text{for any } n.$$

3. $|H(j\,\omega)|$ decreases exponentially.

## Pole Location of Chebyshev Filters

$$|H(j\,\omega)|^2= 1/(1+\varepsilon\,C_n^2(\omega)) = 1/(1+\varepsilon\,C_n^2(-js))\,|_{s=j\omega}$$

$$s= \sigma +j\,\omega$$

$$\sigma_K= \pm\sin[(2K+1)(\pi/2n)]\sinh[(1/n)\sinh^{-1}(1/\varepsilon)]$$

$$\omega_K= \cos[(2K+1)(\pi/2n)]\cosh[(1/n)\sinh^{-1}(1/\varepsilon)]\quad K=0,1,2,\ldots 2n-1$$

$$(\sigma_{K/\sinh}{}^2 y)+(\omega_{K/\cosh}{}^2 y)=1\quad\text{an ellipse on the } \sigma - \omega \text{ plane.}$$

$$y=(1/n)\sinh^{-1}(1/\varepsilon)$$

## Design Relations of Chebyshev Filters

$$|H(j\,\omega)|^2= 1/(1 +\varepsilon\,C_n^2(\omega/\,\omega_p))$$

$$|H(j\,\omega)|^2= 1/(1 +\varepsilon\,C_n^2(\omega/\,\omega_p))\,|_{\omega=\omega_p} = 1/(1+\varepsilon^2)$$

$$A_p=10\log(1+\varepsilon^2)$$

$$\varepsilon =\sqrt{(10^{0.1A_p}-1)}$$

$$A_r= 10\log[1+\varepsilon^2 C_n^2(\omega_r/\,\omega_p)]$$

$$=10\log[1+\varepsilon^2\cosh^2[n\cosh^{-1}(\omega_r/\,\omega_p)]]$$

$$n\geq(\cosh^{-1}((10^{0.1A_p}-1)/\,\varepsilon))^{1/2})/(\cosh^{-1}(\omega_r/\,\omega_p))$$

$$k=\omega_p/\,\omega_r= f_p/\,f_r,\qquad\qquad d= ((10^{0.1A_p}-1)/(10^{0.1A_r}-1))^{1/2}$$

or

$$n\geq \cosh^{-1}(1/d)/\cosh^{-1}(1/k)$$

**Left-Hand Poles for the Transfer Function**

$s_k = \sin[(2K+1)(\pi/2n)]\sinh[(1/n)\sinh^{-1}(1/\varepsilon)] + j\cos[(2K+1)(\pi/2n)]\cosh[(1/n)\sinh^{-1}(1/\varepsilon)]$

$$H(s) = -\prod_{K=0}^{n-1} (s_k/s - s_k), \quad n \text{ is odd}$$

$$H(s) = (1/\sqrt{(1+\varepsilon^2)})\prod_{K=0}^{n-1} (s_k/s - s_k), \quad n \text{ is odd}$$

For non-normalized transfer function set $s/\omega_p$ in place of s

$|H(j\omega)|^2 = 1/2 = 1/(1+\varepsilon^2 C_n(\omega_c))$ , 3-dB cutoff

$\omega_c = \cosh((1/n)\cosh^{-1}(1/\varepsilon))$

Or

$$n \geq \frac{\cosh^{-1}\left(\dfrac{1}{d}\right)}{\cosh^{-1}\left(\dfrac{1}{k}\right)}$$

**Problem 1**

Filter Requirements: a) ripple not to exceed 2 dB up to $\omega_p$

b) 50 dB rejection above 5 $\omega_p$

**Solution**

$$A_p \leq 2dB \text{ at } \omega = \omega_p$$

$$A_r \geq 50dB \text{ at } \omega = \omega_r = 5\omega_p$$

$$\varepsilon = \left(10^{0.1A_p} - 1\right)^{1/2} = \left(10^{0.2} - 1\right)^{1/2} = 0.765$$

$$k = \frac{\omega_p}{\omega_r} = \frac{\omega_p}{5\omega_p} = 0.2$$

$$d = \frac{0.765}{(10^{0.1A_r} - 1)^{1/2}} = \frac{0.765}{\left(10^5 - 1\right)^{1/2}} = 2.42 \times 10^{-3}$$

$$n \geq \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)} = \frac{\ln\left(1/d + \sqrt{1/d^2 - 1}\right)}{\ln\left(1/k + \sqrt{1/k^2 - 1}\right)} = \frac{2.718}{2.312} = 2.91$$

Accept n=3

$$y = \frac{1}{n}\sinh^{-1}\frac{1}{\varepsilon} = \frac{1}{n}\ln\left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1}\right) = 0.361$$

$$\sinh y = \frac{e^y - e^{-y}}{2} = 0.3689 \quad \cosh y = \frac{e^y + e^{-y}}{2} = 1.0659$$

$$s_0 = \sin\left(\frac{\pi}{6}\right)(0.3689) + j\cos\left(\frac{\pi}{6}\right)(1.0659) = -0.1844 + j0.9231$$

$$s_1 = \sin\left(\frac{\pi}{2}\right)(0.3689) + j\cos\left(\frac{\pi}{2}\right)(1.0659) = -0.3689$$

$$s_2 = -\sin\left(\frac{\pi}{6}\right)(0.3689) + j\cos\left(\frac{5\pi}{6}\right)(1.0659) = -0.1844 - j0.9231 = s_0^*$$

$$H(s) = \frac{0.3289}{(s+0.3689)(s^2 + 0.3689s + 0.8861)}$$

**Problem 2** Design a chebyshev filter for the following specification using 1) bilinear transformation 2) impulse invariant method.

$$0.8 \le \left|H(e^{j\omega})\right| \le 1, \quad 0 \le \omega \le 0.2\pi$$

$$\left|H(e^{j\omega})\right| \le 0.2, \quad 0.6\pi \le \omega \le \pi$$

Solution

Using bilinear transformation method,

$$\Omega_s = 2\tan\frac{\omega_s}{2} = 2.752$$

$$\Omega_p = 2\tan\frac{\omega_p}{2} = 0.6498$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8 \Rightarrow \varepsilon = 0.75$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.899$$

$$N \geq \frac{\cosh^{-1}\left(\dfrac{\lambda}{\varepsilon}\right)}{\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{2.564}{2.122} = 1.208$$

Find a and b:

$$\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2} + 1} = 3$$

$$a = \Omega_p \frac{\left[\mu^{\frac{1}{N}} - \mu^{-\frac{1}{N}}\right]}{2} = 0.3752$$

$$b = \Omega_p \frac{\left[\mu^{\frac{1}{N}} + \mu^{-\frac{1}{N}}\right]}{2} = 0.75$$

To find poles and zeros

$$S_k = a\cos\phi_k + jb\sin\phi_k; \quad k = 1,2,...N$$

$$\text{where, } \phi_k = \frac{\pi}{2} + \left[\frac{2k-1}{2N}\right]\pi; \quad k = 1,2,...N$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4} = 135$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{4} = \frac{5\pi}{4} = 225$$

$$S_1 = -0.2653 + j0.53, \quad S_2 = -0.2653 - j0.53$$

$$H(s) = \frac{0.28}{s^2 + 0.5306s + 0.3516}$$

Using bilinear transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{0.28(1+z^{-1})^2}{5.4128 - 7.298z^{-1} + 3.29z^{-2}}$$

Using impulse invariant method

$$\frac{\omega_s}{\omega_p} = \frac{\Omega_s}{\Omega_p} = \frac{0.6\pi}{0.2\pi} = 3$$

$$N = \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}(1/k)} = \frac{\cosh^{-1}\frac{4.899}{0.75}}{\cosh^{-1}(3)} = 1.45 \approx 2$$

$$\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2} + 1} = 3$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.3627$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.7255$$

$$\phi_1 = 135°, \phi_2 = 225$$

$$s_1 = -0.2564 + j\,0513$$

$$s_2 = -0.2564 - j\,0.513$$

$$H(s) = \frac{0.264}{s^2 + 0.513s + 0.33}$$

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

$$P_1 = -0.257 - j0.515$$

$$P_2 = -0.257 + j0.515$$

$$H(z) = \frac{0.1967 z^{-1}}{1 + 0.5929 z^{-2} - 1.34 z^{-1}}$$

**Properties of FIR filter**

An FIR filter has a number of useful properties which sometimes make it preferable to aninfinite impulse response (IIR) filter. FIR filters:

- Require no feedback. This means that any rounding errors are not compounded by summed iterations. The same relative error occurs in each calculation. This also makes implementation simpler.

- Are inherently stable, since the output is a sum of a finite number of finite multiples of the input values, so can beno greater than $\sum |b_i|$ times the largest value appearing in the input.

- Can easily be designed to be linear phase by making the coefficient sequence symmetric. This property is sometimes desired for phase-sensitive applications, for example data communications, crossover filters, and mastering.

The main disadvantage of FIR filters is that considerably more computation power in a general purpose processor is required compared to an IIR filter with similar sharpness orselectivity, especially when low frequency (relative to the sample rate) cutoffs are needed. However many

digital signal processors provide specialized hardware features to make FIR filters approximately as efficient as IIR for many applications.

**4.3 Impulse invariance method**

By impulse invariance method, the analogue impulse response h(t) is sampled to get the discreet sample response h(n). If the analogue filter has a system function Ha (s) then the system function of the digital filter can be achieved from the sampling theorem as

$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a\left(\left(f - kF_s\right)\right)$$

Or equivalently

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a\left(\left(\omega - 2\pi kF_s\right)\right)$$

The period of the sampling T should be small enough to avoid or minimize the effect of the aliasing.

From the equation

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a\left(\left(\omega - 2\pi kF_s\right)\right)$$

We can generalize

$$H(z)\big|_{z=e^{sT}} = F_s \sum_{k=-\infty}^{\infty} H_a\left(\left(s - j2\pi kF_s\right)\right)$$

Then the mapping is characterized by

$$z = e^{sT}$$

We substitute by $s = \sigma + j\Omega$

$$z = e^{\sigma T} e^{j\Omega T}$$

It is clear that

If $\sigma = 0$. The imaginary axis $(s = j\Omega)$ in the s plane map into the unity circle $(z = e^{j\Omega T})$ in the z-plane.

If $\sigma < 0$, the left hand plane in the s plane map into the inside of the unity circle in the z-plane.

On the assumption that the poles of the analogue filter are distinct, we can write

$$H_{a(s)} = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$

$p_k$ Are the poles of the analogue filter and $c_k$'s are the coefficients of the partial expansion. Consequently

$$h_a(t) = \sum_{k=1}^{N} c_k e^{p_k t}$$

If the ample the analogue impulse response, we get

$$h(n) = h_a(nT) = \sum_{k=1}^{N} c_k e^{p_k nT}$$

So the system function of the IIR filter

$$H_a(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$= \sum_{k=1}^{N} C_k \sum_{n=0}^{\infty} \left( e^{p_k T} z^{-1} \right)^n$$

For $p_k < 0$

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

We observe that the digital filter has poles at

$z_k = e^{p_k T}$    k=1, 2,….N

Example: Convert the analog filter with the system function

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2 + 9}$$

Solution:

$$H_a(s) = \frac{1/2}{s+0.1-j3} + \frac{1/2}{s+0.1+j3}$$

Then the transfer function of the digital filter

$$H(z) = \frac{1/2}{1-e^{-0.1T}e^{j3T}z^{-1}} + \frac{1/2}{1-e^{-0.1T}e^{j3T}z^{-1}}$$

$$H(z) = \frac{1-\left(e^{-0.1T}\cos 3T\right)z^{-1}}{1-\left(2e^{-0.1T}\cos 3T\right)z^{-1}+e^{-0.2T}z^{-2}}$$

## 4.4 Design procedure using impulse invariance

**Determining design parameters**

Using the impulse invariance design procedure, we have noted that the relation between frequency in the continuous – time and discrete time domains is $\omega = \Omega T$, where T is merely a design parameter( and not necessarily sampling frequency). Leaving T as an arbitrary constant for now, we obtain

$$\Omega_p = \omega_p /T = 0.47124/T$$

$$\Omega_s = \omega_s /T = 1.0996/T$$

At the passband edge frequency,

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\dfrac{\Omega_p}{\Omega_c}\right)^{2N}} = (1-\delta_1)^2$$

With a few algebraic manipulations, we obtain

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \frac{1}{(1-\delta_1)^2} - 1 = k_1$$

Similarly, at the stop band edge frequency, we have

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\dfrac{\Omega_s}{\Omega_c}\right)^{2N}} = \delta_2^2 \text{ which produces}$$

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = \frac{1}{\delta_2^2} - 1 = k_2$$

Dividing, we obtain

$$\frac{\left(\Omega_p/\Omega_c\right)^{2N}}{\left(\Omega_s/\Omega_c\right)^{2N}} = \frac{k_1}{k_2} \text{ or}$$

$$N = \frac{1}{2}\frac{\log(k2/k1)}{\log\left(\Omega_s/\Omega_p\right)} = 2.7144$$

Note that since $\dfrac{\Omega_s}{\Omega_p} = \dfrac{\omega_s/T}{\omega_p/T} = \dfrac{\omega_s}{\omega_p}$, the design parameter T has no effect at all on the value of N

that is needed. Furthermore, as expected, the value of N specify increases as either $\delta_1$ and $\delta_2$ decreases in magnitude( which implies decreasing the ripple in the passband and / or stopband) or as the ratio $\Omega_s/\Omega_p$ decreases ( which implies a decrease in the width of the transition band).

Since the number of poles must be an integer, we round upto N=3.

Matching the frequency response exactly at passband produces

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = k_1 \text{ or } \Omega_c = \frac{\Omega_p}{k_1^{1/2N}} = \frac{0.4716}{T}$$

( This value is slightly different from the one I gave in class.) If we were instead to match the frequency response at stopband, we would obtain

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = k_2 \text{ or } \Omega_c = \frac{\Omega_s}{k_2^{1/2N}} = \frac{0.5112}{T}$$

In principle, any value of the critical frequency that satisfies $\dfrac{0.4716}{T} \le \Omega_c \le \dfrac{0.5112}{T}$ would be valid. As discussed in class, we will choose the value that meets the passband spec exactly, $\Omega_c = 0.4716/T$

Because this value leaves the greatest margin for error at the stop band edge. Since the impulse invariance procedure always incurs a certain degree of aliasing, it will be expected that the actual response of the filter at the stop band edge will be larger than designed for. Hence, having the stop band edge response be "Over designed" will leave more room for error to allow for the effects of aliasing.

**Prototype filter design**

As noted above, the poles of a Butterworth filter lie in the left half of the s-plane on a circle of radius $\Omega_c$ in complex conjugate pairs separated by an angle of $\pi/N$ radians. Since there are three poles, this produces a filter with the system function

$$H(s) = \frac{\Omega_c{}^3}{\left(s - \Omega_c e^{j\pi}\right)\left(s - \Omega_c e^{j(\pi + \pi/3)}\right)\left(s - \Omega_c e^{j(\pi - \pi/3)}\right)}$$

Plugging in the value of $\Omega_c = 0.4716/T$, we obtain

$$H(s) = \frac{\Omega_c{}^3}{\left(s - \Omega_c e^{j\pi}\right)\left(s - \Omega_c e^{j(\pi + \pi/3)}\right)\left(s - \Omega_c e^{j(\pi - \pi/3)}\right)}$$

$$H(s) = \frac{(0.4716/T)^3}{(s + 0.4716/T)\left(s - (-0.236 + 0.408 j)/T\right)\left(s - (-0.236 - 0.408 j)/T\right)}$$

Problem 1:

For the analog transfer function

$$H(s) = \frac{2}{(s+1)(s+2)}$$

Determine H(z) using impulse invariance method.
Assume T = 1 sec.

Solution

$$H(s) = \frac{2}{(s+1)(s+2)}$$

Applying partial fractions,

$$H(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)}$$

$$= \frac{2}{(s-(-1))} - \frac{2}{(s-(-2))}$$

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

$$(s - p_k) \rightarrow \left(1 - e^{p_k T} z^{-1}\right)$$

Here , $p_k \rightarrow p_1, p_2$

$$p_1 = -1; \quad p_2 = -2; \quad T = 1\sec$$

$$H(z) = \frac{2}{1 - e^{-1} z^{-1}} - \frac{2}{1 - e^{-2} z^{-1}}$$

$$= \frac{2}{1 - 0.3678 z^{-1}} - \frac{2}{1 - 0.1353 z^{-1}}$$

$$H(z) = \frac{0.465 z^{-1}}{1 - 0.503 z^{-1} + 0.0497 z^{-2}}$$

### 4.5 Bilinear Transformation

Let us consider an analog filter with the system function

$$H(s) = \frac{b}{s + a}$$

The system can be also described by the differential equation

$$\frac{dy(t)}{dx} + ay(t) = bx(t)$$

Starting from the integral

$$y(t) = \int_{t_0}^{t} y'(\tau) d\tau + y(t_0)$$

By the trapezoidal formula at t=nT and $t_0 = nT - T$

$$y(nT) = (T/2)[y'(nT) + y'(nT-T)] + y(nT-T)$$

But as

$$y'(nT) = -ay(nT) + bx(nT)$$

The integral become

$$\left(1 + \frac{aT}{2}\right)y(n) - \left(1 - \frac{aT}{2}\right)y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right]$$

Convert the equation into z domain

$$\left(1 + \frac{aY}{2}\right)Y(z) - \left(1 - \frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}\left[X(z) + z^{-1}X(z)\right]$$

Then the transfer function

$$H(z) = \frac{b}{\dfrac{2}{T}\left(\dfrac{1 - z^{-1}}{1 + z^{-1}}\right) + a}$$

Clearly the mapping from s-plane to z-plane is

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

To investigate the characteristics of the bilinear transformation, let

$$z = re^{j\omega} \quad s = \sigma + j\omega$$

Then the mapping formula can be expressed as

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

$$= \frac{2}{T}\frac{re^{j\omega} - 1}{re^{j\omega} + 1}$$

Or

$$s = \frac{2}{T}\left( \frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega} \right)$$

$$= \sigma + j\Omega$$

If r<1 leads to $\sigma < 0$ or the left hand plane in the s plane maps into the inside of the unity circle in the z-plane .

When r=1, $\sigma = 0$ and

$$\Omega = \frac{2}{T}\frac{\sin\omega}{1 + \cos\omega}$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

**Problem 1**

Convert the analog filter with the system function

$$H(s) = \frac{s + 0.1}{(s + 0.1)^2 + 16}$$

Into a IIR digital filter by means of bilinear transformation. The digital filter is to have a resonance frequency $\omega_r = \pi / 2$

The analog filter has a resonance frequency of $\Omega = 4$. This is to be mapped into $\omega_r = \pi / 2$ by using

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

This leads to T=0.5

Thus the desired mapping is

s=4{(1-z$^{-1}$)/( 1+z$^{-1}$)}

And the resulting digital filter has the system function

$$H(z) = \frac{0.128 + 0.006z^{-1} - 0.122z^{-2}}{1 + 0.0006z^{-1} + 0.975z^{-2}}$$

$$H(z) \approx \frac{0.128 + 0.006z^{-1} - 0.122z^{-2}}{1 + 0.975z^{-2}}$$

with poles at $p_{1,2} = 0.987e^{\pm j\pi/2}$

with zero at $z_{1,2} = -10.95$

**problem 2** Design a single pole low pass filte with 3 dB bandwidth of $0.2\pi$ using the billinear transform to analogue filter

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

The digital filter is specified to have -3dB gain at $\omega_c = 0.2\pi$. In the frequency domain of the analogue filter

$$\Omega_c = \frac{2}{T}\tan\frac{\omega}{2} = 0.65/T$$

Thus the analogue filter has the system function

$$H(s) = \frac{0.65/T}{s + 0.65/T}$$

Now, we apply the bilinear transformation to get

$$H(z) = \frac{0.245(1 + z^{-1})}{1 - 0.509z^{-1}}$$

The frequency response of the digital filter

$$H(\omega) = \frac{0.245(1 + e^{-j\omega})}{1 - 0.509e^{-j\omega}}$$

which applies

$$H(0) = 1 \quad and \quad |H(0.2\pi)| = 0.707$$

**4.6 Design Procedure Using Bilinear Transformation**

**Determining Design parameters**

Here we will convert from continuous time to discrete time form using the bilinear transform.

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

This produces the non-linear relationship between continuous-time frequency and discrete-time frequency

$$\Omega = \frac{2}{T}\tan(\omega/2)$$

Converting the critical frequencies $\omega_p$ and $\omega_s$ to their continuous time counterparts produces

$$\Omega_p = \frac{2}{T}\tan(\omega_p/2) = 0.4802/T$$

$$\Omega_s = \frac{2}{T}\tan(\omega_s/2) = 1.226/T$$

We use again the design equation

$$N = \frac{1}{2}\frac{\log(k_2/k_1)}{\log(\Omega_s/\Omega_p)} = 2.4546$$

Note that although we still need to have 3 poles, the actual fractional value of n smaller because of the greater ratio of stopband to passband edge frequencies provided by the nonlinear frequency warping of the bilinear transform.

We will match the specs exactly in the passband, although there is no specific need to do so when we use the bilinear transform. The equation for the critical frequency then becomes

$$\Omega_c = \frac{\Omega_p}{k_1^{1/2N}} = \frac{0.4806}{T}$$

**Prototype Filter Design**

As before, the transfer function for the Butterworth filter is

$$H(s) = \frac{\Omega_c^3}{(s-\Omega_c e^{j\pi})(s-\Omega_c e^{j(\pi+\pi/3)})(s-\Omega_c e^{j(\pi-\pi/3)})}$$

$$H(s) = \frac{0.110/T^3}{s^3 + \frac{0.9612s^2}{T} + \frac{0.4620s}{T^2} + \frac{0.110}{T^3}}$$

Problem 1:

Apply bilinear transformation to

$$H(s) = \frac{2}{(s+1)(s+2)}$$

with $T = 1\sec$ and find $H(z)$

**Solution :**

Substitute,

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$H(z) = \frac{2}{(s+1)(s+2)}\Bigg|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$H(z) = \frac{2}{\left(2\left[\dfrac{1-z^{-1}}{1+z^{-1}}\right]+1\right)\left(2\left[\dfrac{1-z^{-1}}{1+z^{-1}}\right]+2\right)}$$

$$= \frac{\left(1+z^{-1}\right)^2}{6\left(1-\frac{1}{3}z^{-1}\right)}$$

**Window (Filter) Descriptions**

- $T=1$
- $-\pi \leq \omega \leq \pi$ or $0 \leq \omega \leq 2\pi$
- DFT bin$=2\pi/N$
- Windows are even (about the origin) sequences with an odd number of points.
- The right -most point of the window will be discarded.
- N will be taken to be even, and the total points will be odd, and hence N=2x(total points)=even

      **Rectangle Window**

W (n) =1.0      n=-N/2,.......,-1,0,1,2,N/2

$$W(\omega) = \sum_{n=-N/2}^{N/2} w(n)e^{-j n\omega}$$

To make it realizable shift the sequence by N/2 to the right. Hence we obtain

$$w(n) = 1 \qquad\qquad n = 0,1,\cdots,N-1$$

$$W(\omega) = \sum_{n=0}^{N-1} e^{-j n\omega} = e^{-j\frac{N-1}{2}\omega}\,\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\frac{\omega}{2}}$$
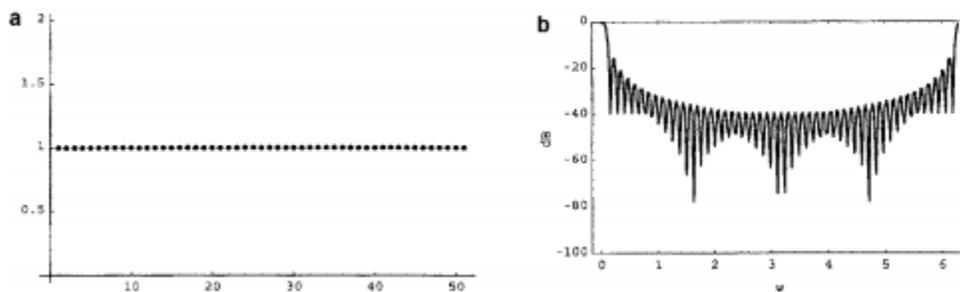


**Fig 4.3**
a> Rectangular window,
b> Amplitude spectrum off rectangular window

## Hann window

$$w(n) = \cos^2\left(\frac{n}{N}\pi\right) = \frac{1}{2}\left[1 + \cos\left(\frac{2n}{N}\pi\right)\right] \qquad n = -\tfrac{N}{2},\cdots,-1,0,1,\cdots,\tfrac{N}{2}$$

$$w(n) = \sin^2\left[\left(\frac{n}{N}\right)\pi\right] = \frac{1}{2}\left[1 - \cos\left[\left(\frac{2n}{N}\right)\pi\right]\right] \qquad n = 0,1,\cdots,N-1$$

DFT of the window is

$$W(\omega) = \frac{1}{2}D(\omega) + \frac{1}{4}\left[D\left(\omega - \frac{2\pi}{N}\right) + D\left(\omega + \frac{2\pi}{N}\right)\right]$$

$$D(\omega) = e^{j\frac{\omega}{2}}\,\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\frac{\omega}{2}} = \text{Dirichlet Kernel} \qquad -\pi \le \omega \le \pi$$
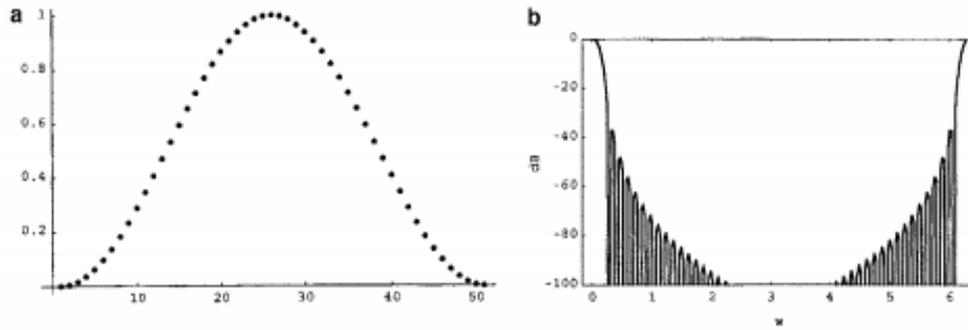
**Fig 4.4**
a> Hann window,
b>Amplitude Spectrum of the hann window.

## 4.7 Hamming Window

$$w(n) = \alpha + (1-\alpha)\cos\frac{2\pi}{N}n \qquad n = -\frac{N}{2},\cdots,-1,0,1,\cdots,\frac{N}{2}$$

$$W(\omega) = \alpha D(\omega) + \frac{1}{2}(1-\alpha)\left[D\left(\omega - \frac{2\pi}{N}\right) + D\left(\omega + \frac{2\pi}{N}\right)\right] \qquad -\pi \leq \omega \leq \pi$$

$D$ = Dirichlet Kernel (see 7.3.5)

$$w(n) = 0.54 + 0.46\cos\frac{2\pi}{N}n \qquad n = -\frac{N}{2},\cdots,-1,0,1,\cdots,\frac{N}{2}$$

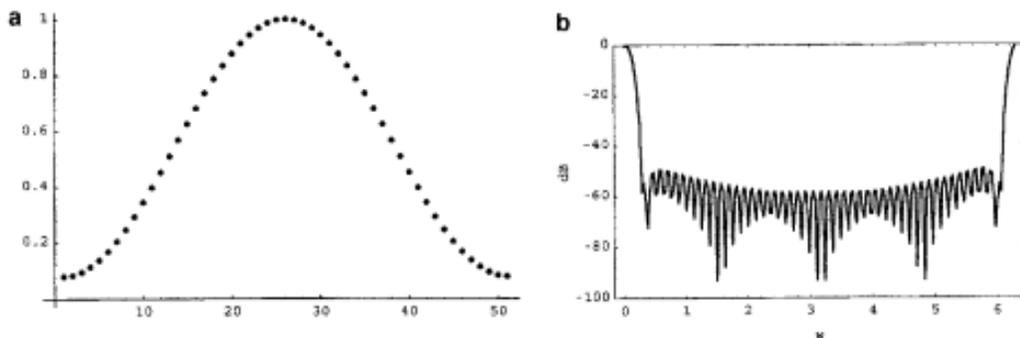$$w(n) = 0.54 - 0.46\cos\frac{2\pi}{N}n \qquad n = 0,1,\cdots,N-1$$

**Fig 4.5**
a> Hamming window
b> Amplitude Spectrum of the hamming window.

**Problem 1**

Design a linear phase FIR low pass filter of order seven with cut-off frequency 1 rad/sec using rectangular window.

The desired frequency response $H_d(\omega)$ for the low pass FIR filter is given by,

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & \text{for } |\omega| < |\omega_c| \\ 0 & \text{otherwise} \end{cases} \quad ...(1)$$

Here M=length of filter=7(given)

$$H_d(\omega) = \begin{cases} e^{-3j\omega} & \text{for } |\omega| < |\omega_c| \\ \cup \end{cases} \quad ....(2)$$

We will obtain $h_{d(n)}$ by taking IFT of equation(2) $\therefore h_{d(n)} = \frac{1}{2\pi}\left[\frac{e^{j(n-3)} - e^{-j(n-3)}}{j(n-3)}\right]$

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H_d(\omega) \cdot e^{j\omega n} d\omega$$

Here given cut-off frequency=$(\omega_c)$=1.Now for the symmetric filter we can write the range of integration $(\omega_c)$, from $h_{d(n)} = \frac{\sin(n-3)}{\pi(n-3)}$ $\omega_{c=-1}$ to $\omega_{c=1}$

$$\therefore h_d(n) = \frac{1}{2\pi}\int_{-1}^{1} e^{-3j\omega} \cdot e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} e^{j\omega(n-3)} d\omega. \quad ...(4)$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-3)}}{j(n-3)}\right]_{-1}^{1}$$

$$\therefore h_{d(n)} = \frac{1}{2\pi} \left[ \frac{e^{j(n-3)} - e^{-j(n-3)}}{j(n-3)} \right]$$

Now we have the trigonometric identity,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$h_{d(n)} = \frac{\sin(n-3)}{\pi(n-3)} \qquad \text{.....for } n \neq 3$$

Now if n=3 then equation (4) becomes,

$$h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} \\ \dfrac{1}{\pi} \end{cases}$$

$$\therefore W_R(n) = \begin{cases} 1 \\ o \end{cases}$$

$$h(n) = \begin{cases} h_d(n) \\ 0 \end{cases}$$

$$h_d(0) = h(0) = \frac{\sin(-3)}{-3\pi} = 0.015 \qquad h_d(n) = \frac{1}{2\pi}\int_{-1}^{1} e^0 \cdot d\omega = \frac{1}{2\pi}[1-(-1)] = \frac{1}{\pi} \ldots(6)$$

Thus we can write,

$$h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} \\ \dfrac{1}{\pi} \end{cases} \text{For } n \neq 3, \text{ for n=3} \qquad\qquad\qquad \ldots(7)$$

Step 3:

Here we have for rectangular window we have to make use of rectangular window of the order 7.We have for rectangular window,

$$\therefore W_R(n) = \begin{cases} 1 \\ o \end{cases} \text{ for } n=0 \text{ to } 6,\text{otherwise} \qquad \text{(for n=0 to M-1)}$$

Now h(n) is , $h(n) = h_d(n) \cdot W_R(n)$

Thus using equation (7) we get,

$$h(n) = \begin{cases} h_d(n) \\ 0 \end{cases} \text{for n=0 to 6;otherwise} \qquad ....(8)$$

| Value of n | Value of $h_d(n) = h(n)$ |
|---|---|
| 0 | $h_d(0) = h(0) = \dfrac{\sin(-3)}{-3\pi} = 0.015$ |
| 1 | $h_d(1) = h(1) = \dfrac{\sin(-2)}{-2\pi} = 0.145$ |
| 2 | $h_d(2) = h(2) = \dfrac{\sin(-1)}{-\pi} = 0.268$ |
| 3 | $h_d(3) = h(3) = \dfrac{1}{\pi} = 0.318$ |
| 4 | $h_d(4) = h(4) = \dfrac{\sin(1)}{\pi} = 0.268$ |
| 5 | $h_d(5) = h(5) = \dfrac{\sin(2)}{2\pi} = 0.145$ |
| 6 | $h_d(6) = h(6) = \dfrac{\sin(3)}{3\pi} = 0.0149$ |

**Problem 2**

Find the magnitude and phase response function of seventh order low pass linear phase FIR filter cut-off frequency 1 rad/sec using hanning window.

This is similar to problem(1);only difference is that here ,we have to use hanning window.

We have to obtained the equation of $h_d(n)$.It is,

$$h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} \\ \dfrac{1}{\pi} \end{cases} \text{for } n \neq 3, \text{for n=3} \qquad .....(1)$$

Now hanning window is defined as,

$$W_h(h) = \frac{1}{2}\left[1 - \cos\frac{2\pi n}{M-1}\right]$$

But M=7 (given)

$$\therefore W_h(n) = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{6}\right)\right] \qquad \text{....(2)}$$

Now the coefficients of FIR filter that means h(n) is obtained by using the equation,

$$h(n) = h_d(n) \cdot W_h(n) \qquad \text{.....(3)}$$

Making use of these equations we can obtain the values of h(n) as shown in the below table.

| Value of n | $h_d(n) = \begin{cases}\dfrac{\sin(n-3)}{\pi(n-3)} & \text{for } n \neq 3 \\ \dfrac{1}{\pi}\end{cases}$ | $W_h(n) = \frac{1}{2}\left[1-\cos\left(\frac{2\pi n}{6}\right)\right]$ | $h(n) = h_d(n) \cdot W_h(n)$ |
|---|---|---|---|
| 0 | 0.015 | 0 | 0 |
| 1 | 0.145 | 0.25 | 0.03625 |
| 2 | 0.268 | 0.75 | 0.201 |
| 3 | 0.318 | 1 | 0.318 |
| 4 | 0.268 | 0.75 | 0.201 |
| 5 | 0.145 | 0.25 | 0.3625 |
| 6 | 0.149 | 0 | 0 |

Magnitude and phase:

We have, $H_d(\omega) = \begin{cases} 1 \cdot e^{-j(\frac{M-1}{2})} & \text{for } |\omega| \leq |\omega_c| \text{,otherwise} \\ 0 \end{cases}$

$$\therefore |H_d(\omega)| = \begin{cases} 1 & \text{for } |\omega| \leq |\omega_c|, \text{ otherwise} \\ 0 \end{cases}$$

And $\angle H_d(\omega) = -\omega(\dfrac{M-1}{2}) = -\omega(\dfrac{7-1}{2}) = -3\omega$

Thus the magnitude and phase angle has been calculated.