SMTX1011 Applied Numerical Method

(Common to all Engineering Except CSE, IT, and Bio groups)

III Year V Semester (Batch 2010 onwards)

Course Material

Course Objective: The ability to identify, reflect upon, evaluate and apply different types of knowledge and information to form independent judgments. Analytical, logical thinking and conclusions based on quantitative information will be the main objective of learning this subject.

Unit 3: Numerical Solution of Algebraic and Transcendental Equations

Numerical Solutions of Algebraic and Transcendental Equations –Regula Falsi method - Newton Raphson's method - Graffe's Root Squaring method- Simultaneous linear algebraic equations - Gauss –Jordan method-Crout's method - Gauss – Seidal method –Relaxation method.

Regula - Falsi Method

This method is the oldest method for finding the real roots of an algebraic and transcendental equations. consider the equation $f(x)=0$ and let 'a' & 'b' be two values of x such that $f(a)$ and $f(b)$ are of opposite signs. Hence a root of $f(x)=0$ lies between a & b Let it be c. To find c, let us calculate x_n values $(n=1,2,\ldots)$

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

Now we find the sign of $f(x_1)$ and re assign a and b to those values for which the signs of the functional values are different. then

x_2 is calculated using the same formula, this process is repeated until two consecutive values of x_n is the same or equal upto the preferred no. of decimal places. Then that value of x_n is the required root.

order of convergence of this method is 1.618

Problem

Obtain a real root of $x^3 - x - 1 = 0$ correct to three decimal places by regula falsi method

$$f(x) = x^3 - x - 1$$

$f(1) = -1$, $f(2) = 5$ $\Rightarrow$ $a = 1$, $b = 2$

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1\times 5 - 2\times(-1)}{5-(-1)} = 1.1667$$

$f(1.1667) = -0.57867$ $\Rightarrow$ root lies between 1.1667 & 2

$\therefore \quad a = 1.1667, \quad b = 2$

$f(a) = -0.57869, \quad f(b) = 5$

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = 1.25311 \Rightarrow f(x_2) = -0.15757$$

Proceeding in the same way the iteration values are tabulated below.

$x_1 = 1.1667$	Root lies between x_1, b.
$x_2 = 1.25311$	Root lies between x_2, b.
$x_3 = 1.27593$	" x_3, b
$x_4 = 1.30361$	" x_4, b
$x_5 = 1.31569$	" x_5, b
$x_6 = 1.32088$	" x_6, b
$x_7 = 1.32309$	" x_7, b
$x_8 = 1.32403$	" x_8, b
$x_9 = 1.32443$	

Hence the root is 1.324.

Practice Problems

Solve the foll. equations by regula falsi method

1. Find a root between 2 and 3 for $x \log_{10} x = 1.2$

Ans 2.741

2. Find a root between 0 & 1 for $x e^x = 2$.

Ans 0.85260.

Newton's Method (or) Newton-Raphson Method.

To solve $f(x) = 0$, the iterative formula is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

provided $|f(x) \cdot f''(x)| < |f'(x)|^2$

order of convergence of this method is 2

Problem Find the positive root of $f(x) = x^3 - 3x^2 + 7x - 8 = 0$ by Newton-Raphson's method

$$f(x) = x^3 - 3x^2 + 7x - 8$$

$$f'(x) = 3x^2 - 6x + 7$$

$f(0) = -8$, $f(1) = -3$, $f(2) = 2$

$\Rightarrow$ Root lies between 1 and 2.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 3x_n^2 + 7x_n - 8}{3x_n^2 - 6x_n + 7}$$

$$x_{n+1} = \frac{2x_n^3 + 3x_n^2 + 8}{3x_n^2 - 6x_n + 7}$$

put $x_0 = 2$ ($\because f(2)$ is closer to zero than $f(1)$).

$$x_1 = \frac{2 \cdot 2^3 + 3 \cdot 2^2 + 8}{3 \cdot 2^2 - 6 \cdot 2 + 7} = 1.71428$$

proceeding the same way we the iterative values

$x_2 = 1.67422$

$x_3 = 1.67416$

$\therefore$ The root is 1.674

Problems for Practice

1. Find a +ve roots of $f(x) = \cos x - xe^x = 0$

   Ans 0.51776

2. Find a +ve root of $x \tan x = 1.28$

   Ans 0.93826.

GRAFFE'S ROOT SQUARING METHOD

Solve $x^3 - 2x^2 - 5x + 6 = 0$ by graffe's root squaring method.

The co-efficients of the successive root squaring are tabulated below

	m	2^m	co-efficients			
			a_0	a_1	a_2	a_3
Given equation	0	1	1	-2	-5	6
			a_0^2	a_1^2	a_2^2	a_3^2
			1	4	25	36
				$-2a_0a_2$	$-2a_1a_3$	
				10	24	
First Squaring	1	2	1	14	49	36
			1	196	2401	1296
				-98	-1008	
Second Squaring	2	4	1	98	1393	1296
			1	9604	1940449	1679616
				-2786	-254016	
Third squaring	3	8	1	6818	1686433	1679616
			B_0	B_1	B_2	B_3

$$|R_1| = |\alpha_1|^8 = \left|\frac{B_1}{B_0}\right| = \frac{6818}{1} \Rightarrow |\alpha_1| = 3.0144$$

$$|R_2| = |\alpha_2|^8 = \left|\frac{B_2}{B_1}\right| = \frac{1686433}{6818} \Rightarrow |\alpha_2| = 1.9914$$

$$|R_3| = |\alpha_3|^8 = \left|\frac{B_3}{B_2}\right| = \frac{1679616}{1686433} \Rightarrow |\alpha_3| = 0.9960$$

To find the exact roots the corresponding values are substituted in $f(x)$ with both the signs (+ve and -ve). The roots are 3.0144, -1.9914, 0.9960

Problems for Practice

Solve the foll. equations by Graeffe's method

1. $x^3 - 6x^2 + 11x - 6 = 0.$

   Ans 3, 2, 1

2. $x^3 - x^2 - 17x - 15 = 0$

   Ans 5, -3, -1

3. $x^3 - 2x + 2 = 0$

   Ans $1.7693, \; -0.8847 \pm 0.5897i$

(II) Gauss - Jordon Method

To solve the system of equation represented in matrix form $AX = B$, the coefficient matrix is reduced to a diagonal matrix by means linear transformations. thereby the values of the variable are found one after the other.

Example:- Solve : $x+2y+3z=6$, $2x+4y+z=7$, $3x+2y+9z=14$ by Gauss - Jordon method.

Matrix form of the equation is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 14 \end{pmatrix}$$

Augmented Matrix is

$$(A,B) = \begin{pmatrix} 1 & 2 & 3 & 6 \\ 2 & 4 & 1 & 7 \\ 3 & 2 & 9 & 14 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & -5 & -5 \\ 0 & -4 & 0 & -4 \end{pmatrix} \quad \begin{matrix} R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & -4 & 0 & -4 \\ 0 & 0 & -5 & -5 \end{pmatrix} \quad R_2 \rightleftharpoons R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & -4 & 0 & -4 \\ 0 & 0 & -5 & -5 \end{pmatrix} \quad R_1 \leftrightarrow R_1 + \tfrac{1}{2} R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -4 & 0 & -4 \\ 0 & 0 & -5 & -5 \end{pmatrix} \quad R_1 \leftrightarrow R_1 + 3/5\, R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{matrix} R_2 \leftrightarrow R_2/-4 \\ R_3 \leftrightarrow R_3/-5 \end{matrix} \quad \Rightarrow \boxed{x=1,\ y=1,\ z=1}$$

Problems for Practice

Solve the following system of equation by Gauss - Jordon Method

1. $2x+3y-z=3$, $x+y+3z=-2$, $x+y+z=0$

   Ans $x=1$, $y=0$, $z=-1$

2. $2x+y+4z=12$, $8x-3y+2z=20$, $4x+11y-z=33$

   Ans $x=3$, $y=2$, $z=1$

Solve $x+y+2z=7,\ 3x+2y+4z=13,\ 4x+3y+2z=8$ using Crout method.

Solution

Given $\begin{pmatrix} 1 & 1 & 2 \\ 3 & 2 & 4 \\ 4 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 13 \\ 8 \end{pmatrix}$

The Augmented matrix $=[A,B] = \begin{bmatrix} 1 & 1 & 2 & 7 \\ 3 & 2 & 4 & 13 \\ 4 & 3 & 2 & 8 \end{bmatrix}$

Let the Required Derived matrix.

$$(D,M) = \begin{bmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{bmatrix}$$

Step 1 : Elements of 1^{st} column.

$$DM = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot \end{bmatrix}$$

Step 2 Elements of 1^{st} Row

$u_{12} = \dfrac{a_{12}}{l_{11}} = \dfrac{1}{1} = 1$

$u_{13} = \dfrac{a_{13}}{l_{11}} = \dfrac{2}{1} = 2$

$y_1 = \dfrac{b_1}{l_{11}} = \dfrac{7}{1} = 7$

$$DM = \begin{bmatrix} 1 & 1 & 2 & 7 \\ 3 & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot \end{bmatrix}$$

Step 3 ; Elements of 2^{nd} column.

$l_{22} = a_{22} - u_{12}l_{21} = 2-3(1) = -1$

$l_{32} = a_{32} - u_{31}u_{12} = 3-4(1) = -1$

$$D\cdot M = \begin{bmatrix} 1 & 1 & 2 & 7 \\ 3 & -1 & \cdot & \cdot \\ 4 & -1 & \cdot & \cdot \end{bmatrix}$$

Step 4: Elements of 2^{nd} Row.

$$u_{23} = \frac{1}{l_{22}}\left[a_{23} - u_{13}l_{31}\right] = \frac{1}{-1}\left[4 - 3(2)\right] = 2.$$

$$y_2 = \frac{1}{l_{22}}\left[b_2 - l_{21}y_1\right] = \frac{1}{-1}\left[13 - 3(7)\right] = 8$$

$$DM = \begin{bmatrix} 1 & 1 & 2 & 7 \\ 3 & -1 & 2 & 8 \\ 4 & -1 & - & - \end{bmatrix}$$

Step 5 Element of 3^{rd} column.

$$l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 2 - 4(2) - (-1)2 = -4$$

$$DM = \begin{bmatrix} 1 & 1 & 2 & 7 \\ 3 & -1 & 2 & 8 \\ 4 & -1 & -4 & \cdot \end{bmatrix}$$

Step 6 Elements of 3^{rd} Row

$$y_3 = \frac{1}{l_{33}}\left[b_3 - l_{31}y_1 - l_{32}y_2\right] = \frac{1}{-4}\left[8 - 4(7) - (-1)8\right] = 3$$

$$D.M = \begin{bmatrix} 1 & 1 & 2 & 7 \\ 3 & -1 & 2 & 8 \\ 4 & -1 & -4 & 3 \end{bmatrix}$$

The solution is got from $UX = Y$

ie $$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ \\ \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 3 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} x + y + 2z &= 7 \\ y + 2z &= 8 \\ z &= 3 \end{aligned}$$

Hence $x = -1, \quad y = 2, \quad z = 3.$

solve by crout method, the following

(1) $x+y+z=3$, $2x-y+3z=16$, $3x+y-z=-3$.

Ans: $x=1$, $y=-2$, $z=4$

(2) $10x+y+z=12$, $2x+10y+z=13$, $2x+2y+10z=14$

Ans: $x=1$, $y=1$, $z=1$

(1)

Solve the following system of equations by Gauss-Seidel method.

$4x+2y+z=14$, $x+5y-z=10$, $x+y+8z=20$.

Here the diagonal elements are dominant, hence we apply Gauss-Seidel method.

Let the initial values of $y=0$ and $z=0$.

I Iteration

$x^{(1)} = \frac{1}{4}[14-2y-z] = \frac{14}{4} = 3.5$

$y^{(1)} = \frac{1}{5}[10-x+z] = 1.3$.

$z^{(1)} = \frac{1}{8}[20-x-y] = \frac{1}{8}(20-3.5-1.3) = 1.9$.

II Iteration:

$x^{(2)} = \frac{1}{4}[14-2y^{(1)}-z^{(1)}] = 2.375$

$y^{(2)} = \frac{1}{5}[10-x^{(2)}-z^{(1)}] = 1.905$

$z^{(3)} = \frac{1}{8}[20-x^{(2)}-y^{(2)}] = 1.965$

III Iteration

$x^{(3)} = \frac{1}{4}[14-y^{(2)}-z^{(2)}] = 2.05625$

$y^{(3)} = \frac{1}{15}[10-x^{(3)}-z^{(2)}] = 1.98175$

$z^{(3)} = \frac{1}{8}[20-x^{(3)}-y^{(3)}] = 1.99525$

IV iteration;

$x^{(4)} = \frac{1}{4}[14-y^{(3)}-z^{(3)}] = 2.010312$

$y^{(4)} = \frac{1}{5}[10-x^{(4)}-z^{(3)}] = 1.996988$

$z^{(4)} = \frac{1}{8}[20-x^{(4)}-y^{(4)}] = 1.999087$

(2)

continuing the iteration process, the values are tabulated as follows.

Iteration	$x = \frac{1}{4}(14-2y-z)$	$y = \frac{1}{5}[10-x+z]$	$z = \frac{1}{8}[20-x-y]$
0		0	0
1	3.5	1.3	1.9
2	2.375	1.905	1.965
3	2.05625	1.98175	1.99525
4	2.0103125	1.996988	1.999087
5	2.001734	1.99947	1.999849
6	2.00030	1.99991	1.99997

The values of solution correct upto 4 decimal places are $x = 2.0000$, $y = 1.9999$, $z = 1.9999$

—o—

solve the following system of equations using Gauss-Seidel method.

1) $10x - 5y - 2z = 3$, $4x - 10y + 3z = -3$, $x + 6y + 10z = -3$

Ans $x = 0.342$, $y = 0.285$, $z = -0.505$

2) $x + y + 54z = 110$, $27x + 6y - z = 85$, $6x + 15y + 2z = 72$

soln $x = 2.425$, $y = 3.573$, $z = 1.926$.

Relaxation Method :

consider the system of equation

$a_1 x + b_1 y + c_1 z = d_1$, $a_2 x + b_2 y + c_2 z = d_2$,

$a_3 x + b_3 y + c_3 z = d_3$.

We define the residuals r_1, r_2, r_3 by the relation

$r_1 = a_1 x + b_1 y + c_1 z - d_1$, $r_2 = a_2 x + b_2 y + c_2 z - d_2$

and $r_3 = a_3 x + b_3 y + c_3 z - d_3$

The Operation table

	x	y	z	r_1	r_2	r_3
R_1	1	0	0	a_1	a_2	a_3
R_2	0	1	0	b_1	b_2	b_3
R_3	0	0	1	c_1	c_2	c_3

Solve the following system of equation using relaxation method, $12x + y + z = 31$, $2x + 8y - z = 24$, $3x + 4y + 10z = 58$

Soln Since the coefficient matrix $A = \begin{bmatrix} 12 & 1 & 1 \\ 2 & 8 & -1 \\ 3 & 4 & 10 \end{bmatrix}$ is diagonally dominant, we apply relaxation method.

The residuals r_1, r_2, r_3 is given by

$r_1 = 12x + y + z - 31$, $r_2 = 2x + 8y - z - 24$, $r_3 = 3x + 4y + 10z - 58$

Operation Table.

	x	y	z	r_1	r_2	r_3
R_1	1	0	0	12	2	3
R_2	0	1	0	1	8	4
R_3	0	0	1	1	-1	10

Relaxation Procedure is as given below

Operation	x	y	z	r_1	r_2	r_3
initial value	0	0	0	-31	-24	-58
$6R_3$	0	0	6	-25	-30	2
$4R_2$	0	4	0	-21	2	18
$2R_1$	2	0	0	3	6	24
$-2R_3$	0	0	-2	1	8	4
$-R_2$	0	-1	0	0	0	0
	2	3	4	0	0	0

Since All the residuals are Zero, the solution is $x=2$, $y=3$, $z=4$

2) Solve the following system of equation $9x-y+2z=9$, $x+10y-2z=15$, $2x-2y-13z=-17$ using Relaxation method.

Soln Since the coefficient matrix $A=\begin{bmatrix} 9 & -1 & 2 \\ 1 & 10 & -2 \\ 2 & -2 & -13 \end{bmatrix}$ is diagonally dominant, we apply Relaxation method.

Operation Table.

	x	y	z	r_1	r_2	r_3
R_1	1	0	0	9	1	2
R_2	0	1	0	-1	10	-2
R_3	0	0	1	2	-2	-13

Relaxation Procedure.

Operation	x	y	z	r_1	r_2	r_3
Initial value.	0	0	0	-9	-15	17
$1R_3$	0	0	1	-7	-17	4
$2R_2$	0	2	0	-9	3	0
$1R_1$	1	0	0	0	4	2
	1	2	1	0	4	2
	10	20	10	0	40	20
$-4R_2$	0	-4	0	4	0	28
$2R_3$	0	0	2	8	-4	2
$-R_1$	-1	0	0	-1	-5	0
	9	16	12	-1	-5	0
	90	160	120	-10	-50	0
$5R_2$	0	5	0	-15	0	-10
$2R_1$	2	0	0	3	2	-6
	92	165	120	3	2	-6
	920	1650	1200	30	20	-60
$-5R_3$	0	0	-5	20	30	5
$-3R_2$	0	-3	0	23	0	11
$-3R_1$	-3	0	0	-4	-3	5
	917	1647	1195	-4	-3	5

$\Rightarrow 10^3 x = 917, \quad 10^3 y = 1647, \quad 10^3 z = 1195$

$\Rightarrow x = 0.917, \quad y = 1.647, \quad z = 1.195$ (upto 3 decimal places)

Convergence of Relaxation method:

- If the method should converge, the diagonal elements of the coefficient matrix A, should be diagonally dominant.
- ie this method successfull only if

$|a_1| \geq |b_1| + |c_1|$, $|b_2| \geq |a_2| + |c_2|$ and

$|c_3| \geq |a_3| + |b_3|$.

Solve the following equation, using relaxation method

1) $10x - 2y - 2z = 6$, $-x + 10y - 2z = 7$,

$-x - y + 10z = 8$

Ans $x = 1$, $y = 1$, $z = 1$

2) $10x - 2y + z = 12$, $x + 9y - z = 10$, $2x - y + 11z = 20$

<u>Ans</u> $x = 1.26$, $y = 1.16$, $z = 1.69$.