UNIT III
DFT AND FFT

3.1 Frequency-domain representation of finite-length sequences:

Discrete Fourier Transform (DFT):

The discrete Fourier transform of a finite-length sequence x(n) is defined as

$$X(k) = \sum_{k=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad 0 \le k \le N-1$$

X(k) is periodic with period N i.e., X(k+N) = X(k).

Inverse Discrete Fourier Transform (IDFT):

The inverse discrete Fourier transform of X(k) is defined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \quad 0 \le n \le N-1$$

For notation purpose discrete Fourier transform and inverse Fourier transform can be represented by

$$X(k) = DFT\,[x(n)]$$
$$x(n) = IDFT\,[X(k)]$$

Formula:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi\frac{kn}{N}}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi\frac{kn}{N}}$$

Where K and n are in the range of 0 ,1,2……N-1

For example, if N=4

K= 0,1,2,3
N=0,1,2,3

Alternative Formula:

$$X(k) = \sum_{n=0}^{N-1} x(n) W^{kn} \quad \longleftarrow W = e^{-j\frac{2\pi}{N}}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W^{-kn}.$$

3.2 Properties of DFT:

Periodicity property:

If X(k) is the N-point DFT of x(n), then

X(k+N)=X(k)

Linearity property:

If $X_1(k)$=DFT[$x_1(n)$] & $X_2(k)$=DFT[$x_2(n)$], then

DFT[$a_1x_1(n)+a_2x_2(n)$]=$a_1X_1(k)+a_2X_2(k)$

Convolution property:

If $X_1(k)$ = DFT[$x_1(n)$] & $X_2(k)$ = DFT[$x_2(n)$], then

DFT[$x_1(n)$ Ⓝ $x_2(n)$] = $X_1(k)X_2(k)$

Where Ⓝ indicates N-point circular convolution.

Multiplication property:

If $X_1(k)$ = DFT[$x_1(n)$] & $X_2(k)$ = DFT[$x_2(n)$], then

DFT[$x_1(n)x_2(n)$] = (1/N)[$X_1(k)$ Ⓝ $X_2(k)$]

Where Ⓝ Indicates N-point circular convolution.

Time reversal property:

If X(k) is the N-point DFT of x(n), then

DFT[x(N–n)] = X(N–k)

Time shift property:

If X(k) is the N-point DFT of x(n), then

$$DFT[x(n-m)] = e^{-j2\pi mk/N}X(k)$$

Symmetry properties:

If $x(n)=x_R(n)+jx_I(n)$ is N-point complex sequence and $X(k)=X_R(k)+jX_I(k)$ is the N-point DFT of x(n) where $x_R(n)$ & $x_I(n)$ are the real & imaginary parts of x(n) and $X_R(k)$ & $X_I(k)$ are the those of X(k), then

(i) $DFT[x^*(n)]=X^*(N-k)$
(ii) $DFT[x^*(N-n)]=X^*(k)$
(iii) $DFT[x_R(n)]=(1/2)[X(k)+X^*(N-k)]$
(iv) $DFT[x_I(n)]=(1/2j)[X(k)-X^*(N-k)]$
(v) $DFT[x_{ce}(n)]=X_R(k)$ where $x_{ce}(n)=(1/2)[x(n)+x^*(N-n)]$
(vi) $DFT[x_{co}(n)]=jX_I(k)$ where $x_{co}(n)=(1/2)[x(n)-x^*(N-n)]$

If x(n) is real, then

(i) If x(n) is real, then
  a. $X(k)=X^*(N-k)$
  b. $X_R(k)=X_R(N-k)$
(ii) If x(n) is real, then
  a) $X(k)=X^*(N-k)$
  b) $X_R(k)=X_R(N-k)$
  c) $X_I(k)= -X_I(N-k)$
  d) $|X(k)|=|X(N-k)|$
  e) $|X(k)|=|X(N-k)|$
  f) $\angle X(k)= -\angle X(N-k)$
(i) $DFT[x_{ce}(n)]=X_R(k)$ where $x_{ce}(n)=(1/2)[x(n)+x(N-n)]$
(ii) $DFT[x_{co}(n)]=jX_I(k)$ where $x_{co}(n)=(1/2)[x(n)-x(N-n)]$

Problem 1:

Find the DFT of a sequence x(n)= {1,1,0,0} and find the IDFT of Y(K)= {1,0,1,0}

Let us assume $N = L = 4$.

We have $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} \quad k = 0, 1, \ldots, N-1$

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 1 + 0 + 0 = 2$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\pi n/2} = x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$$

$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$

$$= 1 - j$$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1 + \cos\pi - j\sin\pi$$

$$= 1 - 1 = 0$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j3n\pi/2} = x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2}$$

$$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$

$$= 1 + j$$

$$X(k) = \{2, 1-j, 0, 1+j\}$$

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)e^{j2\pi nk/N} \quad n = 0, 1, \ldots N-1$$

$$y(0) = \frac{1}{4}\sum_{k=0}^{3} Y(k) \quad n = 0, 1, 2, 3$$

$$= \frac{1}{4}\left[y(0) + y(1) + y(2) + y(3)\right]$$

$$= \frac{1}{4}[1 + 0 + 1 + 0]$$

$$= 0.5$$

$$y(1) = \frac{1}{N}\sum_{k=0}^{3} Y(k)e^{j\pi k/2}$$

$$y(1) = \frac{1}{4}\left[Y(0) + Y(1)e^{j\pi/2} + Y(2)e^{j\pi} + Y(3)e^{j3\pi/2}\right]$$

$$= \frac{1}{4}[1 + 0 + \cos\pi + j\sin\pi + 0]$$

$$= \frac{1}{4}[1 + 0 - 1 + 0] = 0$$

$$y(2) = \frac{1}{4}\left[Y(0) + Y(1)e^{j\pi} + Y(2)e^{j2\pi} + Y(3)e^{j3\pi}\right]$$

$$= \frac{1}{4}[1 + 0 + \cos 2\pi + j\sin 2\pi + 0]$$

$$= \frac{1}{4}[1 + 0 + 1 + 0] = 0.5$$

$$y(3) = \frac{1}{4}\left[Y(0) + Y(1)e^{j3\pi/2} + Y(2)e^{j3\pi} + Y(3)e^{j9\pi/2}\right]$$

$$= \frac{1}{4}[1 + 0 + \cos 3\pi + j\sin 3\pi + 0]$$

$$= \frac{1}{4}[1 + 0 + (-1) + 0] = 0$$

$$y(n) = \{0.5, 0, 0.5, 0\}$$

Problem 2:

Find the DFT of a sequence

$$x(n) = 1 \quad \text{for } 0 \le n \le 2$$
$$= 0 \quad \text{otherwise}$$

For (i) N=4 (ii) N=8.Plot | X(K) | and ∟X(K)

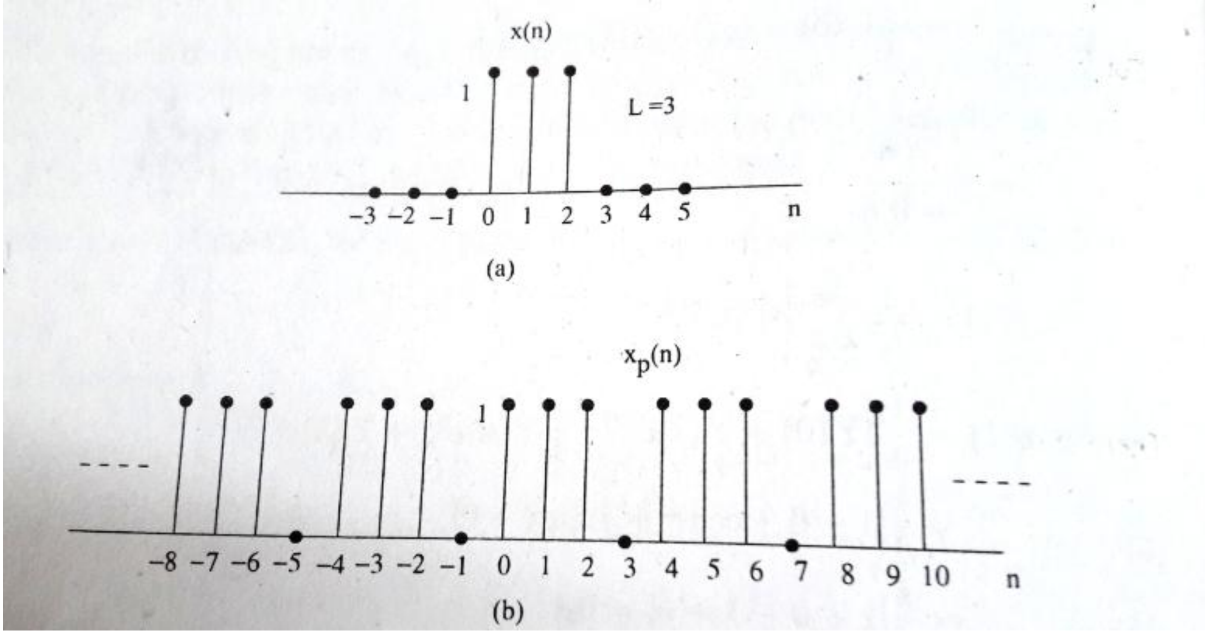


Fig a) Sequence given in problem b) Periodic extension of the sequence for N=4

For $N = 4$

$$X(k) = \sum_{n=0}^{3} x(n)e^{-j\pi nk/2} \quad k = 0, 1, 2, 3$$

For $k = 0$

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$
$$= 3$$

Therefore, $|X(0)| = 3, \angle X(0) = 0$

For $k = 1$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\pi n/2}$$
$$= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$$
$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi + 0$$
$$= 1 - j - 1 = -j$$

$$|X(1)| = 1, \quad \angle X(1) = \frac{-\pi}{2}$$

For $k = 2$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n}$$
$$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$
$$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi + 0$$
$$= 1 - 1 + 1 = 1$$

Therefore,

$$|X(2)| = 1, \quad \angle X(2) = 0$$

For $k = 3$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j3\pi n/2}$$
$$= x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2}$$
$$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi + 0$$
$$= 1 + j - 1 = j$$

Therefore $|X(3)| = 1, \quad \angle X(3) = \frac{\pi}{2}$

$$|X(k)| = \{3, 1, 1, 1\}$$
$$\angle X(k) = \left\{0, -\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$$

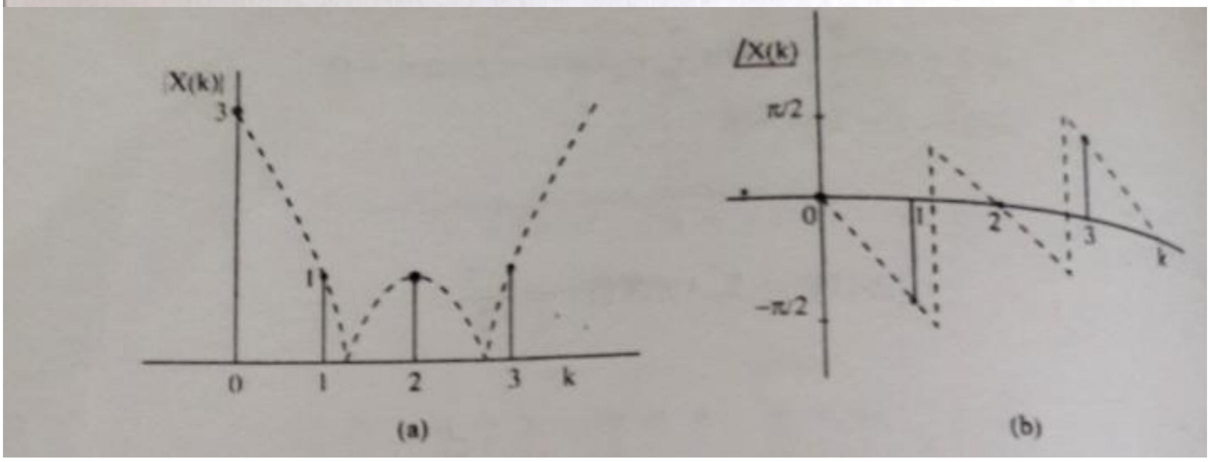


Fig: frequency response of x(n) for N=4

For $N = 8$

The periodic extension of $x(n)$ is shown in Fig. 3.7 can be obtained by adding five zeros ($\because$ $N - L$ zeros).

$$x(0) = x(1) = x(2) = 1 \quad \text{and} \quad x(n) = 0 \quad \text{for} \quad 3 \leq n \leq 7$$

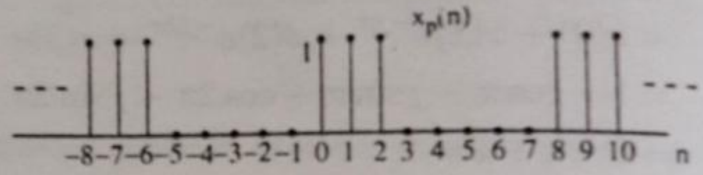


Fig. 3.7 Periodic extension of the sequence $x(n)$ for N = 8

For $N = 8$

$$X(k) = \sum_{n=0}^{7} x(n)e^{-j\pi nk/4} \quad k = 0, 1 \ldots 7$$

For $k = 0$

$$X(0) = \sum_{n=0}^{7} x(n)$$

$$X(0) = 1 + 1 + 1 + 0 + 0 + 0 + 0 + 0 = 3$$

Therefore, $|X(0)| = 3 \quad \angle X(0) = 0$

For $k = 1$

$$X(1) = \sum_{n=0}^{7} x(n)e^{-j\pi n/4}$$

$$= x(0) + x(1)e^{-j\pi/4} + x(2)e^{-j\pi/2}$$

$$= 1 + 0.707 - j0.707 + 0 - j$$

$$= 1.707 - j1.707$$

Therefore,

$$|X(1)| = 2.414, \quad \angle X(1) = \frac{-\pi}{4}$$

For $k = 2$

$$X(2) = \sum_{n=0}^{7} x(n)e^{-j\pi n/2}$$
$$= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi}$$
$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} \div \cos\pi - j\sin\pi$$
$$= 1 - j - 1 = -j$$

Therefore

$$|X(2)| = 1, \quad \angle X(2) = \frac{-\pi}{2}$$

For $k = 3$

$$X(3) = \sum_{n=0}^{7} x(n)e^{-j3\pi n/4}$$
$$= x(0) + x(1)e^{-j3\pi/4} + x(2)e^{-j3\pi/2}$$
$$= 1 + \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$
$$= 1 - 0.707 - j0.707 + j$$
$$= 0.293 + j0.293$$

Therefore, $|X(3)| = 0.414, \quad \angle X(3) = \frac{\pi}{4}.$

For $k = 4$

$$X(4) = \sum_{n=0}^{7} x(n)e^{-j\pi n}$$
$$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi}$$
$$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi$$
$$= 1 - 1 + 1 = 1$$

Therefore, $|X(4)| = 1, \quad \angle X(4) = 0$

For $k = 5$

$$X(5) = \sum_{n=0}^{7} x(n)e^{-j5\pi n/4}$$
$$= x(0) + x(1)e^{-j5\pi/4} + x(2)e^{-j5\pi/2}$$

$$= 1 + \cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4} + \cos\frac{5\pi}{2} - j\sin\frac{5\pi}{2}$$

$$= 1 - 0.707 + j0.707 - j$$

$$= 0.293 - j0.293$$

$$|X(5)| = 0.414, \quad \angle X(5) = -\frac{\pi}{4}$$

For $k = 6$

$$X(6) = \sum_{n=0}^{7} x(n)e^{-j3\pi n/2}$$

$$= x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi}$$

$$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi$$

$$= 1 + j - 1 = j$$

$$|X(6)| = 1, \quad \angle X(6) = -\frac{\pi}{2}$$

For $k = 7$

$$X(7) = \sum_{n=0}^{7} x(n)e^{-j7\pi n/4}$$

$$= 1 + e^{-j7\pi/4} + e^{-j7\pi/2}$$

$$= 1 + \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4} + \cos\frac{7\pi}{2} - j\sin\frac{7\pi}{2}$$

$$= 1 + 0.707 + j0.707 + j$$

$$= 1.707 + j1.707$$

$$|X(7)| = 2.414, \quad \angle X(7) = \frac{\pi}{4}$$

$$|X(k)| = \{3, 2.414, 1, 0.414, 1, 0.414, 1, 2.414\}$$

$$\angle X(k) = \left\{0, -\frac{\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, 0 \frac{-\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}\right\}$$

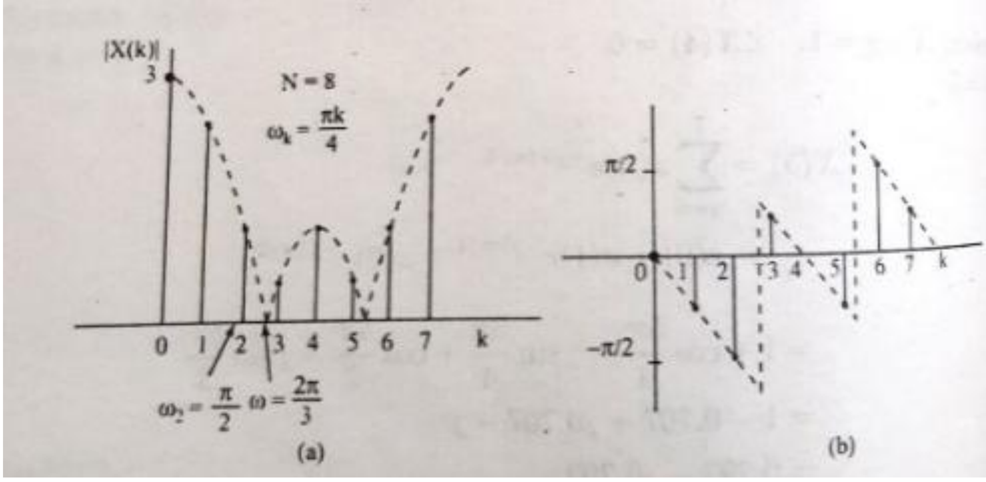


Fig: frequency response of x(n) for N=8

Convolution:

Two types

1.Linear Convolution
2.Circular Convolution

1.Linear Convolution

Formula:

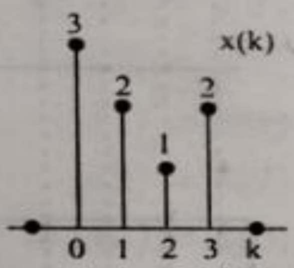
$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

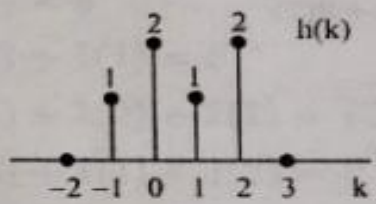
Example 1.11 Determine the convolution sum of two sequences
$x(n) = \{3, 2, 1, 2\}$; $h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$

Solution

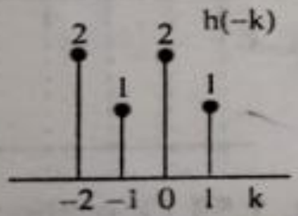
Step 1 The sequence $x(n)$ starts at $n = 0$ and $h(n)$ starts at $n_2 = -1$. Therefore the starting time for evaluating the output sequence $y(n)$ is $n = n_1 + n_2 = 0 + (-1) = -1$

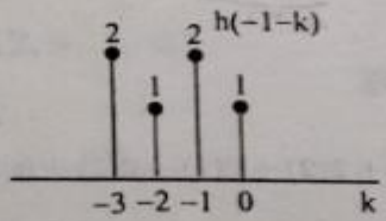
Step 2 Express both sequences in terms of the index k.





Step 3 Fold $h(k)$ about $k = 0$ to obtain $h(-k)$





As starting time to evaluate $y(n)$ is -1, shift $h(k)$ by one unit to left to obtain $h(-1-k)$

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k)$$

Multiply the two sequences $x(k)$ and $h(-1-k)$ element by element and sum the products

$$\Rightarrow y(-1) = 0(2) + 0(1) + 0(2) + 3(1) + 2(0) + 1(0) + 2(0)$$
$$= 3$$

Increment the index by 1, shift the sequence to right to obtain $h(-k)$ and multiply the two sequences $x(k)$ and $h(-k)$ element by element and sum the products

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$

$$= 0(2) + 0(1) + 3(2) + 2(1) + 1(0) + 2(0) = 8$$

Similarly

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k)$$

$$= 0(2) + 3(1) + 2(2) + 1(1) + 2(0)$$

$$= 8$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k)$$

$$= 3(0) + 2(1) + 1(2) + 2(1)$$

$$= 12$$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k)$$

$$= 3(0) + 2(2) + 1(1) + 2(2)$$

$$= 9$$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k)$$

$$= 3(0) + 2(0) + 1(2) + 2(1) + 0(2)$$

$$= 4$$

$$y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k)$$

$$= 3(0) + 2(0) + 1(0) + 2(2) + 0(1) + 0(2) + 0(1)$$

$$= 4$$

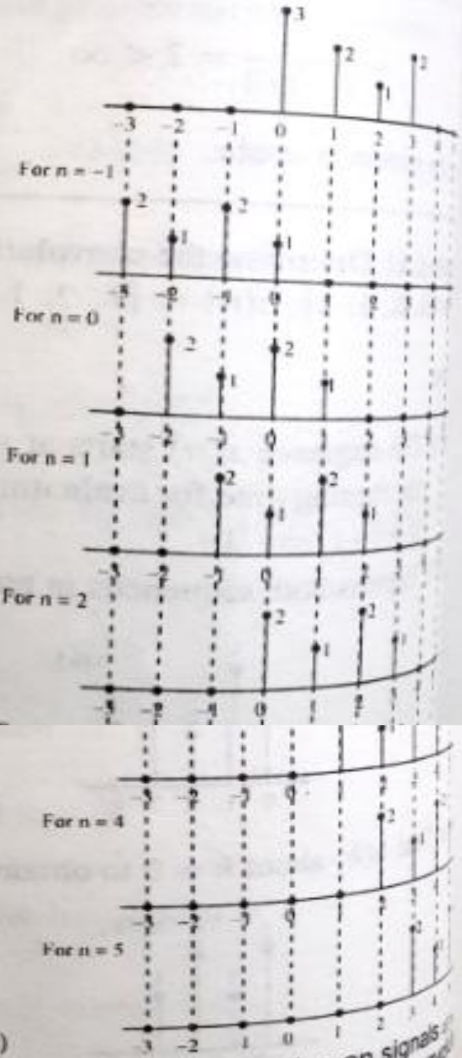
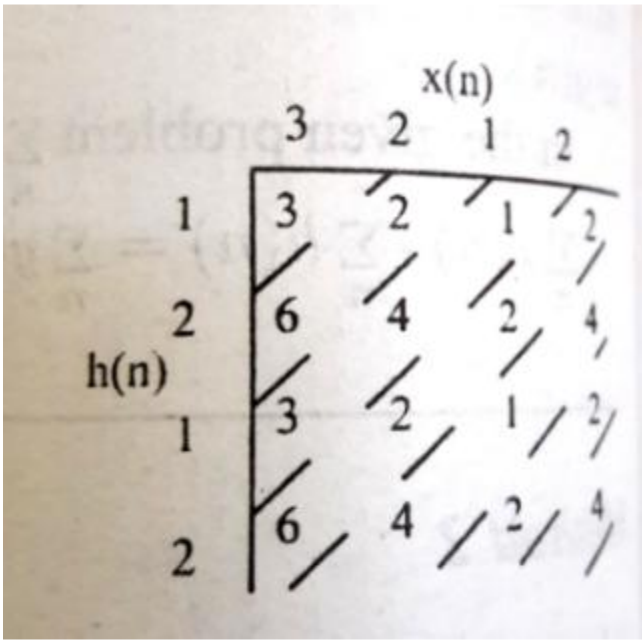


Fig. 1.58 Operation on signals ... $h(n)$ to compute conv...

y(n)= {3,8,8,12,9,4,4}
↑

method 2:



$$y(n) = \{3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4\}$$

2.Circular Convolution

The methods used to find the circular convolution of two sequences are

1) Concentric circle method 2) Matrix multiplication method

1) Concentric circle method

Given two sequences $x_1(n)$ and $x_2(n)$, the circular convolution of these two sequences $x_3(n) = x_1(n)$ Ⓝ $x_2(n)$ can be found by using the following steps

1. Graph N samples of $x_1(n)$ as equally spaced points around an outer circle in counterclockwise direction.
2. Start at the same point as $x_1(n)$ graph N samples of $x_2(n)$ as equally spaced points around an inner circle in clockwise direction.
3. Multiply corresponding samples on the two circles and sum the products to produce output.
4. Rotate the inner circle one sample at a time in counterclockwise direction and go to step 3 to obtain the next value of output.
5. Repeat step No.4 until the inner circle first sample lines up with the first sample of the exterior circle once again.

Problem 3:

Find the circular convolution of two finite duration sequences $x_1(n) = \{-1,-2,3,-1\}$ $x_2(n) = \{1,2,3\}$

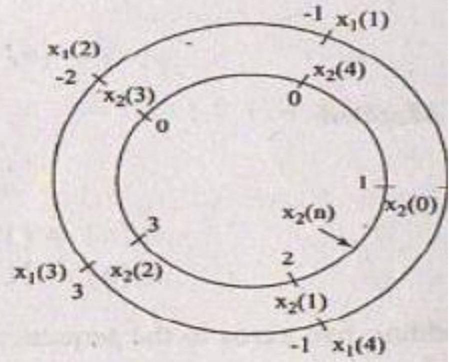
Solution To find circular convolution, both sequences must be of same length. So we append two zeros to the sequence $x_2(n)$ and use concentric circle method to find circular convolution.

We have

$x_1(n) = \{1, -1, -2, 3, -1\}$
$x_2(n) = \{1, 2, 3, 0, 0\}$

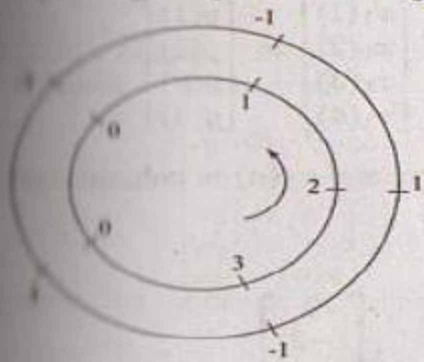
Graph all the points of $x_1(n)$ on the outer circle in the counterclockwise direction. Starting at same point as $x_1(n)$ graph all points of $x_2(n)$ on the inner circle in clockwise direction.

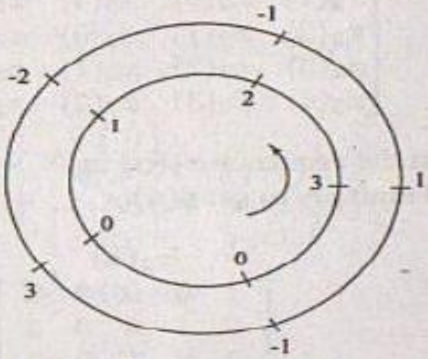
Multiply corresponding samples on the circle and add to obtain



$y(0) = 1(1) + 0(-1) + 0(-2) + 3(3) + 2(-1)$
$= 8$

Rotate the inner circle in counterclockwise direction by one sample, multiply corresponding samples to obtain $y(1)$.

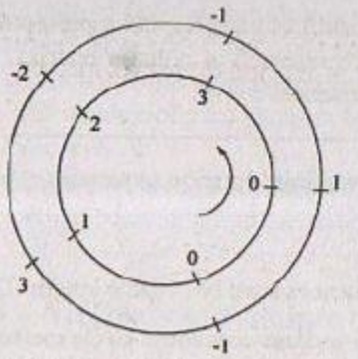


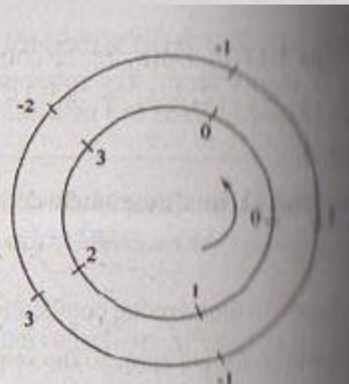


$y(1) = 1(2) + (-1)1 + (-2)0 + 3(0) + 3(-1);$
$= -2$

$y(2) = 3(1) + 2(-1) + 1(-2) + 0(3) + \ldots$
$= -1$

Obtain remaining samples by repeating above procedure until the inner circle first sample lines up with the first sample of the exterior circle.





$y(3) = (0)1 + 3(-1) + 2(-2) + 1(3) + (-1)(0);$
$= -4$

$y(4) = 0(1) + 0(-1) + 3(-2) + \ldots$
$= -1$

$y(n) = \{8, -2, -1, -4, -1\}$

Matrix Method

Given

$$x_1(n) = \{1, -1, -2, 3, -1\}$$
$$x_2(n) = \{1, 2, 3, \}$$

By adding two zeros to the sequence $x_2(n)$, we bring the length of the sequence $x_2(n)$ to 5.

Now

$$x_2(n) = \{1, 2, 3, 0, 0\}$$

The matrix form can be written by substituting $N = 5$ in Eq. (3.55).

$$\begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

Represent the sequence $x_2(n)$ in $N \times N$ matrix form and $x_1(n)$ in column matrix form and multiply to get $y(n)$.

Represent the sequence $x_2(n)$ in $N \times N$ matrix form and $x_1(n)$ in column matrix form and multiply to get $y(n)$.

$$\overset{x_2(n)}{\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix}} \overset{x_1(n)}{\begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix}} = \overset{y(n)}{\begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}}$$

$$y(n) = \{8, -2, -1, -4, -1\}$$

Problem 4:

Perform the circular convolution of the following sequences x(n)={1,1,2,1}, h (n)={1,2,3,4} using DFT and IDFT method.

We know $X_3(k) = X_1(k)X_2(k)$

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)e^{-j2\pi kn/N} \quad k = 0, 1, \ldots N-1$$

Given $x_1(n) = \{1, 1, 2, 1\}$ and $N = 4$

$$X_1(0) = \sum_{n=0}^{3} x_1(n) = 1 + 1 + 2 + 1 = 5$$

$$X_1(1) = \sum_{n=0}^{3} x_1(n)e^{-j\pi n/2} = 1 - j - 2 + j = -1$$

$$X_1(2) = \sum_{n=0}^{3} x_1(n)e^{-j\pi n} = 1 - 1 + 2 - 1 = 1$$

$$X_1(3) = \sum_{n=0}^{3} x_1(n)e^{-j3\pi n/2} = 1 + j - 2 - j = -1$$

$$X_1(k) = (5, -1, 1, -1)$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)e^{-j2\pi nk/N} \quad k = 0, 1, \ldots N-1$$

$$X_2(0) = \sum_{n=0}^{3} x_2(n) = 1+2+3+4 = 10$$

$$X_2(1) = \sum_{n=0}^{3} x_2(n)e^{-j\pi n/2} = 1 + 2(-j) + 3(-1) + 4(j) = -2 + j2$$

$$X_2(2) = \sum_{n=0}^{3} x_2(n)e^{-j\pi n} = 1 + 2(-1) + 3(1) + 4(-1) = -2$$

$$X_2(3) = \sum_{n=0}^{3} x_1(n)e^{-j3\pi n/2} = 1 + 2(j) + 3(-1) + 4(-j) = -2 - j2$$

$$X_2(k) = \{10, -2 + j2, -2, -2, -j2\}$$

$$X_3(k) = X_1(k)X_2(k) = \{50, 2 - j2, -2, 2 + j2\}$$

$$x_3(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)e^{j2\pi nk/N} \quad n = 0, 1, \ldots N-1$$

$$x_3(0) = \frac{1}{4}\sum_{k=0}^{3} X_3(k) = \frac{1}{4}(50 + 2 - j2 - 2 + 2 + j2) = 13$$

$$x_3(1) = \frac{1}{4}\left[\sum_{k=0}^{4} X_3(k)e^{j\pi k/2}\right]$$

$$= \frac{1}{4}[50 + (2 - j2)j + (-2)(-1) + (2 + j2)(-j)] = 14$$

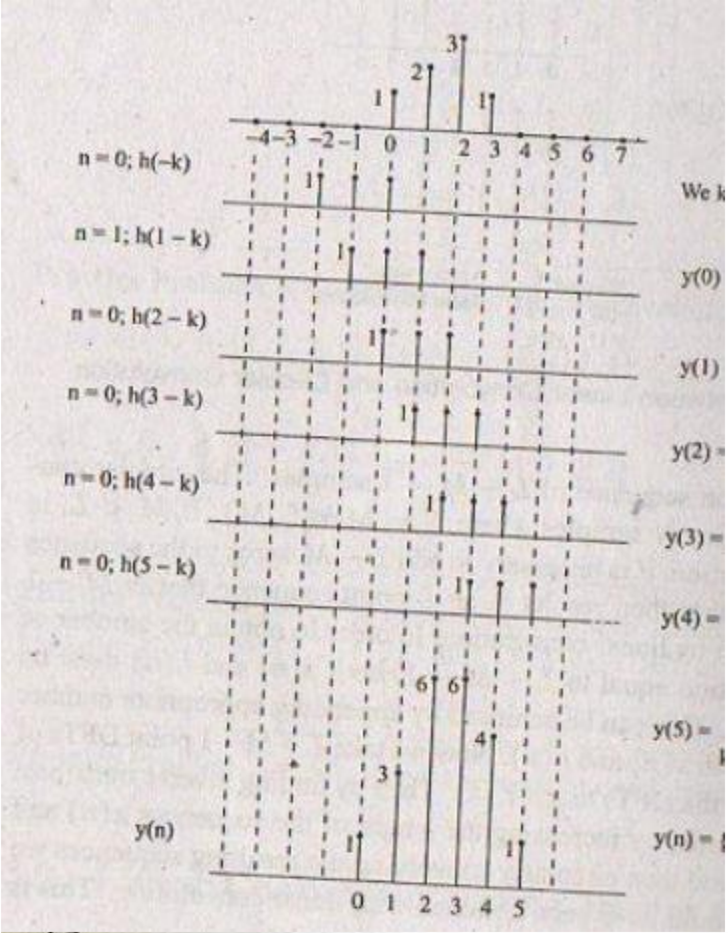
$$x_3(2) = \frac{1}{4}\left[\sum_{k=0}^{4} X_3(k)e^{j\pi k}\right]$$

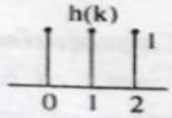
$$= \frac{1}{4}[50 + (2 - j2)(-j) + (-2)(-1) + (2 + j2)(j)] = 12$$

Ans: $x_3(n)$ = {13,14,11,12}

Problem 4:
Determine the output response y(n) if h(n)= {1,1,1}; x(n)= {1,2,3,1} by using i)Linear Convolution ii) Circular convolution iii) Circular convolution with zero padding

i)Linear Convolution





We know $y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k) = 1$$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k) = 1 + 2 = 3$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)\, h(2-k) = 1 + 2 + 3 = 6$$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)\, h(3-k) = 2 + 3 + 1 = 6$$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)\, h(4-k) = 3 + 1 = 4$$

$$y(5) = \sum_{k=-\infty}^{\infty} x(k)\, h(5-k) = 1$$

Solution

Given $x(n) = \{1,2,3,1\}, h(n) = \{1,1,1\}$

The number of samples in linear convolution is $L + M - 1 = 4 + 3 - 1 = 6.$

(ii) ***Circular Convolution***

$$x(n) = \{1,2,3,1\}; h(n) = \{1,1,1,0\}$$

Using matrix approach we can write $h(n)$ as $N \times N$ matrix form and $x(n)$ as column matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 6 \\ 6 \end{bmatrix}$$

$$y(n) = x(n) \circledN h(n) = \{5,4,6,6\}$$

(iii) ***Circular Convolution with Zero padding***

To get the result of linear convolution with circular convolution we have to add appropriate number of zeros to both sequences. Now

$$x(n) = \{1, 2, 3, 1, \overbrace{0, 0}^{(M-1) \text{ zeros appended}}\}$$

$$h(n) = \{1, 1, 1, 0, \underbrace{0, 0}_{(L-1) \text{ zeros appended}}\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

$$y(n) = \{1, 3, 6, 6, 4, 1\}$$

Fast Fourier Transform (FFT) algorithms:

Fast Fourier Transform (FFT):

Computing an N-point DFT using the direct formula

$$X(k)=\sum_{n=0}^{N-1} x(n)e^{-j2\pi(n/N)k}, \quad 0\le k\le N-1$$

requires N^2 complex multiplications and additions. This results in large computational time for large N values. Fast Fourier transform is an efficient way of computing an N-point DFT reducing this required computational time.

Several algorithms were developed to meet this based on several factors. The radix-2 algorithms were developed based on the factors that the N-point DFT is periodic with period N and N, for most cases, is an integer power of 2.

Radix-2 FFT algorithms:

3.4 Decimation-In-Time (DIT) FFT algorithm:

The algorithm in which the decimation is based on splitting the sequence x(n) into successively smaller sequences is called the decimation-in-time algorithm.

The N-point DFT of a sequence x(n) is given by

$$X(k)=\sum_{n=0}^{N-1} x(n)W_N^{nk}, \quad 0\le k\le N-1 \qquad (1)$$

where $W_N= e^{-j(2\pi/N)}$. X(k) is periodic with period N i.e., X(k+N)=X(k).

Splitting Equ(1) into two, one for even-indexed samples of x(n) and the other for odd-indexed samples of x(n), we have

$$X(k) = \sum_{n\ \text{even}} x(n)W_N^{nk} + \sum_{n\ \text{odd}} x(n)W_N^{nk} \qquad (2)$$

Substituting n=2n for n even and n=2n+1 for n odd, we have

$$X(k) = \sum_{n=0}^{N/2-1} x(2n)W_N^{2nk} + \sum_{n=0}^{N/2-1} x(2n+1)W_N^{(2n+1)k}$$

$$X(k) = \sum_{n=0}^{N/2-1} x(2n)W_N^{2nk} + W_N^k \sum_{n=0}^{N/2-1} x(2n+1)W_N^{2nk}$$

$$X(k) = \sum_{n=0}^{N/2-1} x(2n)W_{N/2}^{nk} + W_N^k \sum_{n=0}^{N/2-1} x(2n+1)W_{N/2}^{nk} \text{ -------------------- } (3)$$

Letting $x(2n)=x_{10}(n)$ and $x(2n+1)=x_{11}(n)$, we have

$$X(k) = \sum_{n=0}^{N/2-1} x_{10}(n)W_{N/2}^{nk} + W_N^k \sum_{n=0}^{N/2-1} x_{11}(n)W_{N/2}^{nk}$$

$$X(k) = X_{10}(k) + W_N^k X_{11}(k) \qquad (4)$$

where $X_{10}(k)$ and $X_{11}(k)$ are N/2-point DFTs given by

$$X_{10}(k) = \sum_{n=0}^{N/2-1} x_{10}(n)W_{N/2}^{nk}, \ 0 \le k \le (N/2)-1 \text{ -------------------- } (5)$$

$$x_{10}(n) = x(2n), \ 0 \le n \le (N/2)-1 \text{ -------------------- } (6)$$

$$X_{11}(k) = \sum_{n=0}^{N/2-1} x_{11}(n)W_{N/2}^{nk}, \ 0 \le k \le (N/2)-1 \text{ -------------------- } (7)$$

$$x_{11}(n) = x(2n+1), \ 0 \le n \le (N/2)-1 \text{ -------------------- } (8)$$

and hence $X_{10}(k)=X_{10}(k+N/2)$ & $X_{11}(k)=X_{11}(k+N/2)$.

Since $X_{10}(k)$ & $X_{11}(k)$ are periodic with period N/2, X(k) can be computed directly form Equ(3) only for $0 \le k \le (N/2)-1$ and X(k) for $N/2 \le k \le N-1$ are computed as follows.

$$X(k+N/2) = X_{10}(k+N/2) + W_N^{(k+N/2)} X_{11}(k+N/2)$$

Since $W_N^{(k+N/2)} = W_N^k . W_N^{(N/2)} = W_N^k . e^{-j\pi} = -W_N^k$, we have

$$X(k+N/2) = X_{10}(k) - W_N^k X_{11}(k) \text{ -------------------- } (9)$$

A similar procedure will decompose the two N/2-point DFTs, $X_{10}(k)$ & $X_{11}(k)$ into four N/4-point DFTs as follows.

$$X_{10}(k) = X_{20}(k) + W_N^{2k} X_{21}(k) \text{ -------------------- } (10)$$

$X_{10}(k+N/4) = X_{20}(k) – W_N^{2k}X_{21}(k)$ -- (11)

$X_{11}(k) = X_{22}(k) + W_N^{2k}X_{23}(k)$ --- (12)

$X_{11}(k+N/4) = X_{22}(k) – W_N^{2k}X_{23}(k)$ -- (13)

for $0 \le k \le (N/4)–1$ where $X_{20}(k)$, $X_{21}(k)$, $X_{22}(k)$ & $X_{23}(k)$ are N/4-point DFTs given by

$$X_{20}(k) = \sum_{n=0}^{N/4-1} x_{20}(n)W_{N/4}^{nk},\ 0 \le k \le (N/4)-1$$ -- (14)

$x_{20}(n) = x_{10}(2n),\ 0 \le n \le (N/4)–1$ -- (15)

$$X_{21}(k) = \sum_{n=0}^{N/4-1} x_{21}(n)W_{N/4}^{nk},\ 0 \le k \le (N/4)-1$$ -- (16)

$x_{21}(n) = x_{10}(2n+1),\ 0 \le n \le (N/4)–1$ --- (17)

$$X_{22}(k) = \sum_{n=0}^{N/4-1} x_{22}(n)W_{N/4}^{nk},\ 0 \le k \le (N/4)-1$$ -- (18)

$x_{22}(n) = x_{11}(2n),\ 0 \le n \le (N/4)–1$ -- (19)

$$X_{23}(k) = \sum_{n=0}^{N/4-1} x_{23}(n)W_{N/4}^{nk},\ 0 \le k \le (N/4)-1$$ -- (20)

$x_{23}(n) = x_{11}(2n+1),\ 0 \le n \le (N/4)–1$ --- (21)

If $N=2^r$, where r is a positive integer, then the algorithm terminates after r^{th} cycle of procedure i.e., after r^{th} decomposition. After r^{th} decomposition, we will have N 1-point DFTs, which are the sequence values themselves.

<u>Decomposition of 8-point DFT using DIT FFT algorithm: Butterfly Chart:</u>

A similar procedure for an 8-pt DFT

$$X(k) = \sum_{n=0}^{7} x(n)W_8^{nk},\quad 0 \le k \le 7$$

will terminate after 3^{rd} decomposition and will lead to the following sets of equations:

After 1st decomposition:

$X(k) = X_{10}(k) + W_8^k X_{11}(k)$, k=0,1,2,3
$X(k+4) = X_{10}(k) - W_8^k X_{11}(k)$, k=0,1,2,3

$x_{10}(n) = x(2n)$, n=0,1,2,3 i.e., $x_{10}(n)=\{x(0), x(2), x(4), x(6)\}$
$x_{11}(n) = x(2n+1)$, n=0,1,2,3 i.e., $x_{11}(n)=\{x(1), x(3), x(5), x(7)\}$

After 2nd decomposition:

$X_{10}(k) = X_{20}(k) + W_8^{2k} X_{21}(k)$, k=0,1
$X_{10}(k+2) = X_{20}(k) - W_8^{2k} X_{21}(k)$, k=0,1
$X_{11}(k) = X_{22}(k) + W_8^{2k} X_{23}(k)$, k=0,1
$X_{11}(k+2) = X_{22}(k) - W_8^{2k} X_{23}(k)$, k=0,1

$x_{20}(n) = x_{10}(2n) = x(4n)$, n=0,1 i.e., $x_{20}(n)=\{x(0), x(4)\}$
$x_{21}(n) = x_{10}(2n+1) = x(4n+2)$, n=0,1 i.e., $x_{21}(n)=\{x(2), x(6)\}$
$x_{22}(n) = x_{11}(2n) = x(4n+1)$, n=0,1 i.e., $x_{22}(n)=\{x(1), x(5)\}$
$x_{23}(n) = x_{11}(2n+1) = x(4n+3)$, n=0,1 i.e., $x_{23}(n)=\{x(3), x(7)\}$

After 3rd decomposition:

$X_{20}(k) = X_{30}(k) + W_8^{4k} X_{31}(k)$, k=0
$X_{20}(k+1) = X_{30}(k) - W_8^{4k} X_{31}(k)$, k=0
$X_{21}(k) = X_{32}(k) + W_8^{4k} X_{33}(k)$, k=0
$X_{21}(k+1) = X_{32}(k) - W_8^{4k} X_{33}(k)$, k=0
$X_{22}(k) = X_{34}(k) + W_8^{4k} X_{35}(k)$, k=0
$X_{22}(k+1) = X_{34}(k) - W_8^{4k} X_{35}(k)$, k=0
$X_{23}(k) = X_{36}(k) + W_8^{4k} X_{37}(k)$, k=0
$X_{23}(k+1) = X_{36}(k) - W_8^{4k} X_{37}(k)$, k=0

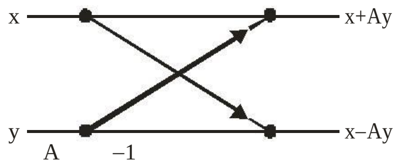
$x_{30}(n) = x_{20}(2n) = x(8n)$, n=0 i.e., $x_{30}(n)=\{x(0)\}$
$x_{31}(n) = x_{20}(2n+1) = x(8n+4)$, n=0 i.e., $x_{31}(n)=\{x(4)\}$
$x_{32}(n) = x_{21}(2n) = x(8n+2)$, n=0 i.e., $x_{32}(n)=\{x(2)\}$
$x_{33}(n) = x_{21}(2n+1) = x(8n+6)$, n=0 i.e., $x_{33}(n)=\{x(6)\}$
$x_{34}(n) = x_{22}(2n) = x(8n+1)$, n=0 i.e., $x_{34}(n)=\{x(1)\}$
$x_{35}(n) = x_{22}(2n+1) = x(8n+5)$, n=0 i.e., $x_{35}(n)=\{x(5)\}$
$x_{36}(n) = x_{23}(2n) = x(8n+3)$, n=0 i.e., $x_{36}(n)=\{x(3)\}$
$x_{37}(n) = x_{23}(2n+1) = x(8n+7)$, n=0 i.e., $x_{37}(n)=\{x(7)\}$

$X_{30}(k) = x_{30}(n) = x(8n)$, k=0 & n=0
$X_{31}(k) = x_{31}(n) = x(8n+4)$, k=0 & n=0
$X_{32}(k) = x_{32}(n) = x(8n+2)$, k=0 & n=0
$X_{33}(k) = x_{33}(n) = x(8n+6)$, k=0 & n=0
$X_{34}(k) = x_{34}(n) = x(8n+1)$, k=0 & n=0
$X_{35}(k) = x_{35}(n) = x(8n+5)$, k=0 & n=0
$X_{36}(k) = x_{36}(n) = x(8n+3)$, k=0 & n=0
$X_{37}(k) = x_{37}(n) = x(8n+7)$, k=0 & n=0

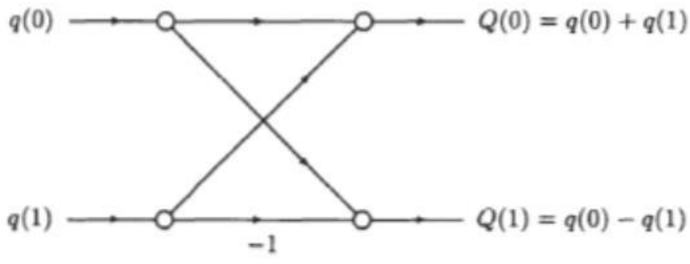
or simply

$X_{30}(0) = x(0)$
$X_{31}(0) = x(4)$
$X_{32}(0) = x(2)$
$X_{33}(0) = x(6)$
$X_{34}(0) = x(1)$
$X_{35}(0) = x(5)$
$X_{36}(0) = x(3)$
$X_{37}(0) = x(7)$

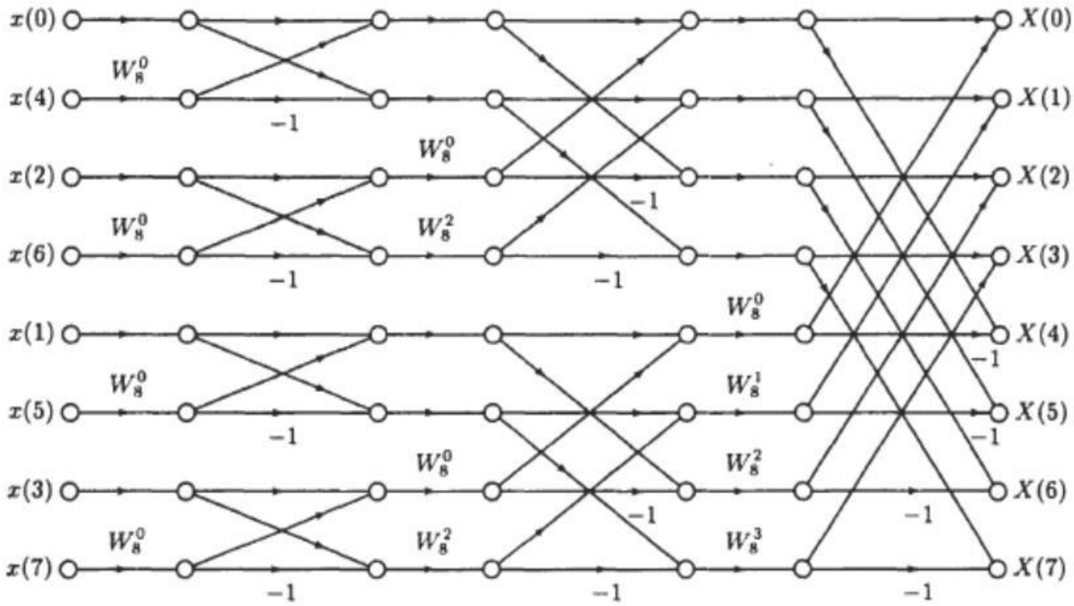
Butterfly Chart: The flow graph used to compute an N-point DFT using FFT algorithms pictorially is often called the butterfly chart. The basic butterfly chart for DIT FFT algorithm is shown below.



The butterfly chart for the DIT-FFT decomposition of an 8-point DFT has been shown below.



A two-point DFT.



A complete eight-point radix-2 decimation-in-time FFT.

3.5 Decimation-In-Frequency (DIF) FFT algorithm:

The algorithm in which the decimation is carried out with respect to the pseudo frequency index, k.

The N-point DFT of a sequence x(n) is given by

$$X(k)=\sum_{n=0}^{N-1} x(n)W_N^{nk},\ 0\le k\le N-1 \quad \text{------} \quad (1)$$

where $W_N= e^{-j(2\pi/N)}$. X(k) is periodic with period N i.e., X(k+N)=X(k).

Splitting Equ(1) into two about the midpoint of x(n), we have

$$X(k)=\sum_{n=0}^{N/2-1} x(n)W_N^{nk} + \sum_{n=N/2}^{N-1} x(n)W_N^{nk},\ 0\le k\le N-1 \quad \text{------} \quad (2)$$

Substituting n=n+N/2 in the second summation, we have

$$X(k)=\sum_{n=0}^{N/2-1} x(n)W_N^{nk} + \sum_{n=0}^{N/2-1} x(n+N/2)W_N^{(n+N/2)k},\ 0\le k\le N-1$$

$$X(k)=\sum_{n=0}^{N/2-1}\left[x(n) + W_N^{(N/2)k}\, x(n+N/2)\right] W_N^{nk},\ 0\le k\le N-1$$

Since $W_N^{(N/2)k}=e^{-j(2\pi/N)(N/2)k}=e^{-jk\pi}=(-1)^k$, we have

$$X(k)=\sum_{n=0}^{N/2-1}\left[x(n) + (-1)^k\, x(n+N/2)\right] W_N^{nk},\ 0\le k\le N-1 \quad \text{--} \ (3)$$

Splitting Equ(2) into two, one for even-indexed values of X(k) and the other for odd-indexed values of X(k), we have

$$X(\text{even } k)=X(2k)=\sum_{n=0}^{N/2-1}\left[x(n) + (-1)^{2k}\, x(n+N/2)\right] W_N^{n(2k)}$$

$$X(\text{even } k)=X(2k)=\sum_{n=0}^{N/2-1}\left[x(n) + x(n+N/2)\right] W_{N/2}^{nk}$$

$$X(\text{even } k)=X(2k)=X_{10}(k)=\sum_{n=0}^{N/2-1} x_{10}(n)\, W_{N/2}^{nk},\ 0\le k\le N/2-1 \quad (4)$$

Where

$$x_{10}(n)=\left[x(n) + x(n+N/2)\right],\ 0\le n\le N/2-1 \quad (5)$$

And

$$X(\text{odd } k)=X(2k+1)=\sum_{n=0}^{N/2-1}\left[x(n) + (-1)^{(2k+1)}\, x(n+N/2)\right] W_N^{n(2k+1)}$$

$$X(\text{odd } k)=X(2k+1)=\sum_{n=0}^{N/2-1}\left[x(n) - x(n+N/2)\right] W_N^{n}\, W_{N/2}^{nk}$$

$$X(\text{odd } k)=X(2k+1)=X_{11}(k)=\sum_{n=0}^{N/2-1} x_{11}(n)\, W_{N/2}^{nk},\ 0\le k\le N/2-1 \quad (6)$$

Where

$$x_{11}(n)=\left[x(n) - x(n+N/2)\right] W_N^{n},\ 0\le n\le N/2-1 \quad (7)$$

Following the procedure in a similar way for $x_{10}(n)$, $x_{11}(n)$, $X_{10}(k)$ & $X_{11}(k)$, we have

$$X_{10}(k)=\sum_{n=0}^{N/4-1}\left[x_{10}(n) + (-1)^{k}\, x_{10}(n+N/4)\right] W_{N/2}^{nk},\ 0\le k\le N/2-1 \quad (8)$$

$$X_{11}(k)=\sum_{n=0}^{N/4-1}\left[x_{11}(n) + (-1)^{k}\, x_{11}(n+N/4)\right] W_{N/2}^{nk},\ 0\le k\le N/2-1 \quad (9)$$

$$X_{10}(\text{even } k)=X_{10}(2k)=X_{20}(k)=\sum_{n=0}^{N/4-1} x_{20}(n)\, W_{N/4}^{nk},\ 0\le k\le N/4-1 \quad (10)$$

where

$$x_{20}(n)=\left[x_{10}(n) + x_{10}(n+N/4)\right], 0\le n\le N/4-1 \text{ ------------------------------------ } (11)$$

$$X_{10}(\text{odd } k)=X_{10}(2k+1)=X_{21}(k)=\sum_{n=0}^{N/4-1} x_{21}(n)\, W_{N/4}^{nk}, 0\le k\le N/4-1 \text{ ------------------------------------ } (12)$$

Where

$$x_{21}(n)=\left[x_{10}(n) - x_{10}(n+N/4)\right] W_N^{2n}, 0\le n\le N/4-1 \text{ ------------------------------------ } (13)$$

$$X_{11}(\text{even } k)=X_{11}(2k)=X_{22}(k)=\sum_{n=0}^{N/4-1} x_{22}(n)\, W_{N/4}^{nk}, 0\le k\le N/4-1 \text{ ------------------------------------ } (14)$$

Where

$$x_{22}(n)=\left[x_{11}(n) + x_{11}(n+N/4)\right], 0\le n\le N/4-1 \text{ ------------------------------------ } (15)$$

$$X_{11}(\text{odd } k)=X_{11}(2k+1)=X_{23}(k)=\sum_{n=0}^{N/4-1} x_{23}(n)\, W_{N/4}^{nk}, 0\le k\le N/4-1 \text{ ------------------------------------ } (16)$$

Where

$$x_{23}(n)=\left[x_{11}(n) - x_{11}(n+N/4)\right] W_N^{2n}, 0\le n\le N/4-1 \text{ ------------------------------------ } (17)$$

If $N=2^r$, where r is a positive integer, then the algorithm terminates after r^{th} cycle of procedure i.e., after r^{th} decomposition. After r^{th} decomposition, we will have N 1-point DFTs, which are the sequence values themselves.

<u>Decomposition of 8-point DFT using DIF FFT algorithm: Butterfly Chart:</u>

A similar procedure for an 8-pt DFT

$$X(k) = \sum_{n=0}^{7} x(n)W_8^{nk}, \quad 0\le k\le 7$$

will terminate after 3^{rd} decomposition and will lead to the following sets of equations: <u>After 1^{st} decomposition:</u>

$$X(\text{even } k)=X(2k)=X_{10}(k)=\sum_{n=0}^{3} x_{10}(n)\, W_4^{nk}, \quad k=0,1,2,3$$

$$X(\text{odd } k)=X(2k+1)=X_{11}(k)=\sum_{n=0}^{3} x_{11}(n)\, W_4^{nk}, \quad k=0,1,2,3$$

where

$x_{10}(n)=\left[x(n) + x(n+4)\right]$, n=0,1,2,3

i.e., $x_{10}(n)=\{x(0) + x(4), x(1) + x(5), x(2) + x(6), x(3) + x(7)\}$

$x_{11}(n)=\left[x(n) - x(n+4)\right] W_8^{n}$, n=0,1,2,3

i.e., $x_{11}(n)=\{[x(0) - x(4)]W_8^{0}, [x(1) - x(5)]W_8^{1}, [x(2) - x(6)]W_8^{2}, [x(3) - x(7)]W_8^{3}\}$ After 2nd decomposition:

$$X_{10}(\text{even } k)=X_{10}(2k)=X_{20}(k)=\sum_{n=0}^{1} x_{20}(n)\, W_2^{nk}, \quad k=0,1$$

$$X_{10}(\text{odd } k)=X_{10}(2k+1)=X_{21}(k)=\sum_{n=0}^{1} x_{21}(n)\, W_2^{nk}, \quad k=0,1$$

$$X_{11}(\text{even } k)=X_{11}(2k)=X_{22}(k)=\sum_{n=0}^{1} x_{22}(n)\, W_2^{nk}, \quad k=0,1$$

$$X_{11}(\text{odd } k)=X_{11}(2k+1)=X_{23}(k)=\sum_{n=0}^{1} x_{23}(n)\, W_2^{nk}, \quad k=0,1$$

where

$x_{20}(n)=\left[x_{10}(n) + x_{10}(n+2)\right]$, n=0,1

i.e., $x_{20}(n)=\{x_{10}(0) + x_{10}(2), x_{10}(1) + x_{10}(3)\}$

$x_{21}(n)=\left[x_{10}(n) - x_{10}(n+2)\right] W_8^{2n}$, n=0,1

i.e., $x_{21}(n)=\{[x_{10}(0) - x_{10}(2)]W_8^{0}, [x_{10}(1) - x_{10}(3)]W_8^{2}\}$

$x_{22}(n)=\left[x_{11}(n) + x_{11}(n+2)\right]$, n=0,1

i.e., $x_{22}(n)=\{x_{11}(0) + x_{11}(2), x_{11}(1) + x_{11}(3)\}$

$x_{23}(n)=\left[x_{11}(n) - x_{11}(n+2)\right] W_8^{2n}$, n=0,1

i.e., $x_{23}(n)=\{[x_{11}(0) - x_{11}(2)]W_8^{0}, [x_{11}(1) - x_{11}(3)]W_8^{2}\}$

<u>After 3rd decomposition:</u>

$X_{20}(\text{even } k)=X_{20}(2k)=X_{30}(k)=\sum x_{30}(n) W_1^{nk}$, k=0 n=0

$X_{20}(\text{odd } k)=X_{20}(2k+1)=X_{31}(k)=\sum x_{31}(n) W_1^{nk}$, k=0 n=0

$X_{21}(\text{even } k)=X_{21}(2k)=X_{32}(k)=\sum x_{32}(n) W_1^{nk}$, k=0 n=0

$X_{21}(\text{odd } k)=X_{21}(2k+1)=X_{33}(k)=\sum x_{33}(n) W_1^{nk}$, k=0 n=0

$X_{22}(\text{even } k)=X_{22}(2k)=X_{34}(k)=\sum x_{34}(n) W_1^{nk}$, k=0 n=0

$X_{22}(\text{odd } k)=X_{22}(2k+1)=X_{35}(k)=\sum x_{35}(n) W_1^{nk}$, k=0 n=0

$X_{23}(\text{even } k)=X_{23}(2k)=X_{36}(k)=\sum x_{36}(n) W_1^{nk}$, k=0 n=0

$X_{23}(\text{odd } k)=X_{23}(2k+1)=X_{37}(k)=\sum x_{37}(n) W_1^{nk}$, k=0 n=0
where

$x_{30}(n)=\left[x_{20}(n) + x_{20}(n+1)\right]$, n=0

i.e., $x_{30}(n)=\{x_{20}(0) + x_{20}(1)\}$

$x_{31}(n)=\left[x_{20}(n) - x_{20}(n+1)\right] W_8^{4n}$, n=0

i.e., $x_{31}(n)=\{[x_{20}(0) - x_{20}(1)]W_8^{0}\}$

$x_{32}(n)=\left[x_{21}(n) + x_{21}(n+1)\right]$, n=0

i.e., $x_{32}(n)=\{x_{21}(0) + x_{21}(1)\}$

$x_{33}(n)=\left[x_{21}(n) - x_{21}(n+1)\right] W_8^{4n}$, n=0

i.e., $x_{33}(n)=\{[x_{21}(0) - x_{21}(1)]W_8^{0}\}$

$x_{34}(n)=\left[x_{22}(n) + x_{22}(n+1)\right]$, n=0

i.e., $x_{34}(n)=\{[x_{22}(0) + x_{20}(1)]\}$

$x_{35}(n)=\left[x_{22}(n) - x_{22}(n+1)\right] W_8^{4n}$, n=0

i.e., $x_{35}(n)=\{[x_{22}(0) - x_{22}(1)]W_8^{0}\}$

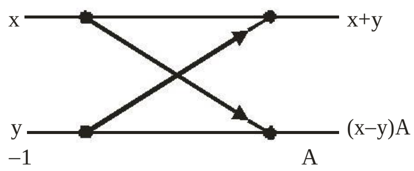
$x_{36}(n)=\left[x_{23}(n) + x_{23}(n+1)\right]$, n=0

i.e., $x_{36}(n)=\{[x_{23}(0) + x_{23}(1)]\}$

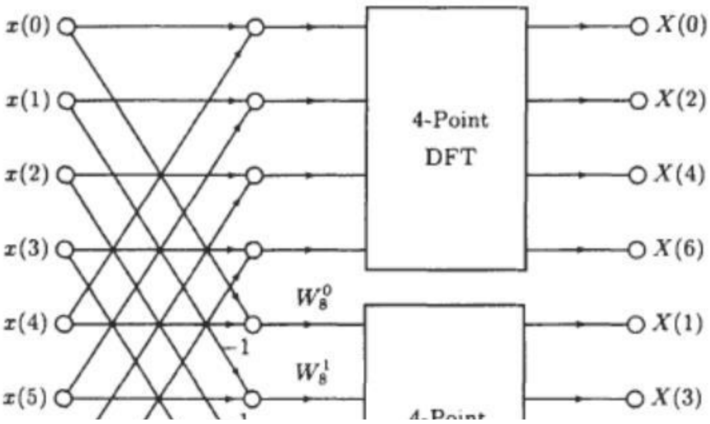
$x_{37}(n)=\left[x_{23}(n) - x_{23}(n+1)\right] W_8^{4n}$, n=0

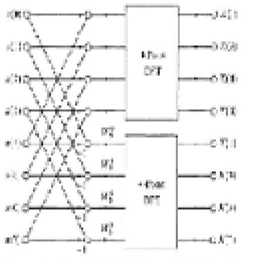
i.e., $x_{37}(n)=\{[x_{23}(0) - x_{23}(1)]W_8^{0}\}$

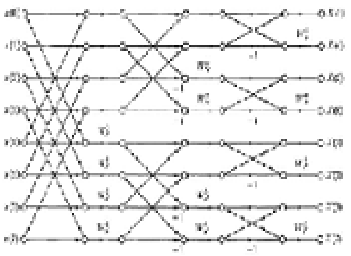
Butterfly Chart: The basic butterfly chart for DIF FFT algorithm is shown below.

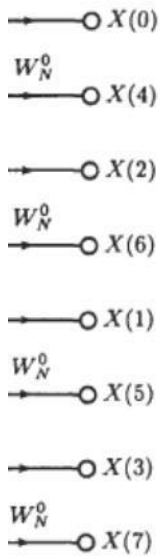


The butterfly chart for the DIF-FFT decomposition of an 8-point DFT has been shown below.


x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
4-Point
DFT
W_8^0
W_8^1
-1
X(0)
X(2)
X(4)
X(6)
X(1)
X(3)






X(0)
W_N^0
X(4)
X(2)
W_N^0
X(6)
X(1)
W_N^0
X(5)
X(3)
W_N^0
X(7)

Problem:

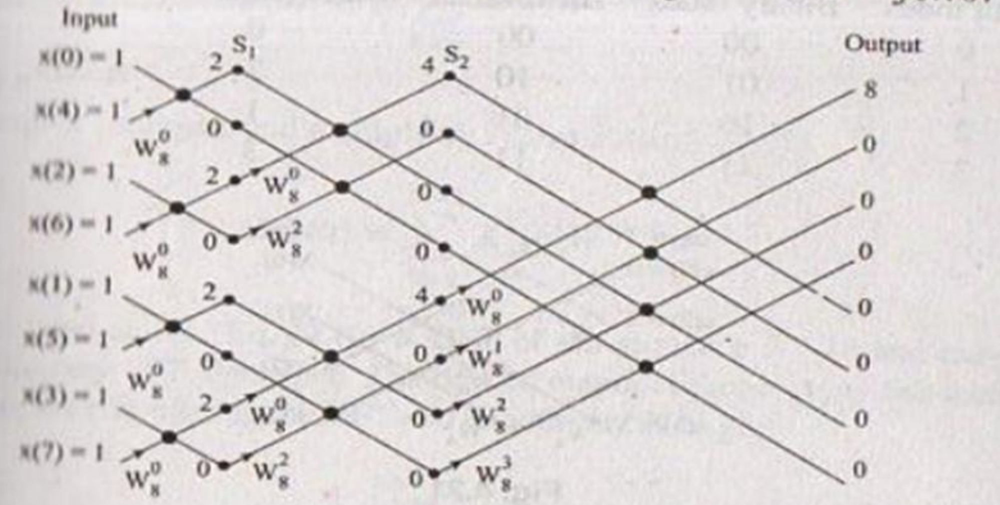
Compute the 8 point DFT of the sequence by using DIT and DIF algorithm

$$x(n) = \begin{cases} 1 & 0 \le n \le 7 \\ 0 & \text{otherwise} \end{cases}$$

DIT Algorithm:

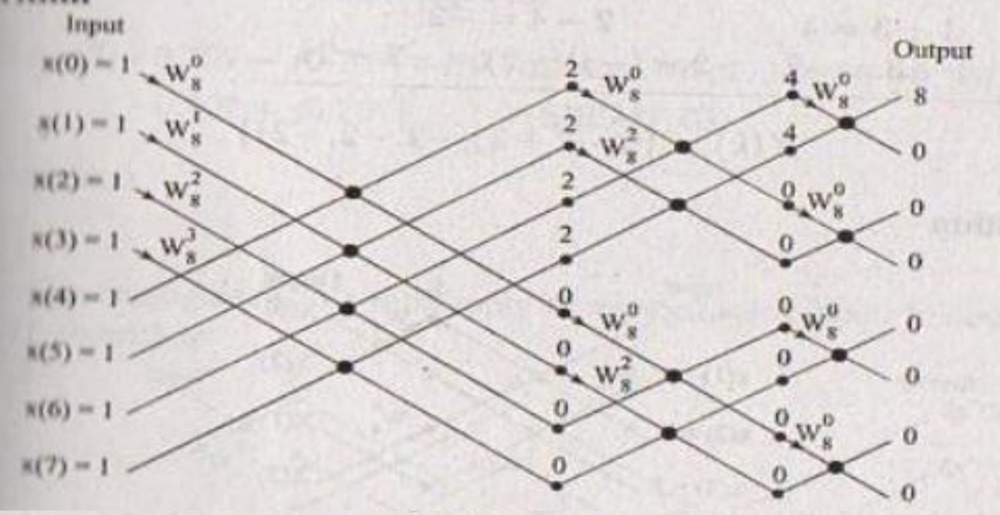
The twiddle factors associated with butterflies can be found as

$$W_8^0 = 1;\ W_8^1 = 0.707 - j0.707;\ W_8^2 = -j;\ W_8^3 = -0.707 - j0.707$$



$$X(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$$

DIF algorithm



$$X(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$$

Problem:

Find the IDFT of the sequences X(K)= {4,1-j2.414, 0, 1+j0.414, 0, 1-j0.414, 0,1-j2.414 } using DIF Algorithm

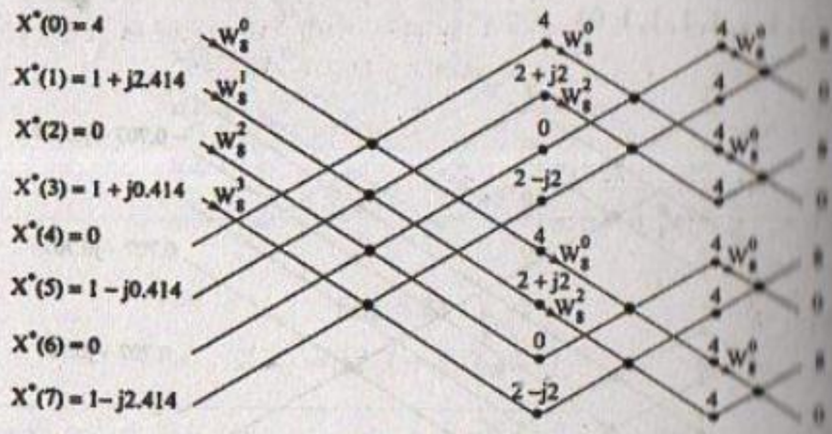


Fig. 4.30

The output $8x^*(n)$ is in bit reversal order.
Therefore

$$x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

Problem :

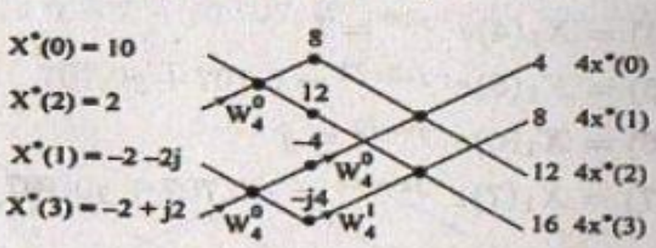
Find the IDFT of the sequence

$$X(k) = \{10, -2 + j2, -2, -2 - j2\}$$

using DIT algorithm.

Solution

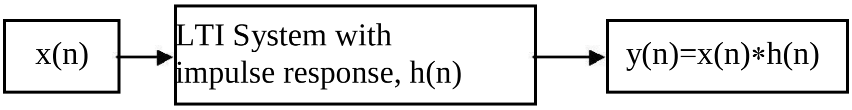
Then twiddle factors are $W_4^0 = 1; W_4^1 = -j$



The output $Nx^*(n)$ is normal order.
Therefore $x(n) = \{1, 2, 3, 4\}$.

3.6 Linear filtering through DFT (FFT):

Linear filtering refers to obtaining the output, y(n) of a linear, time-invariant (LTI) system with impulse response, h(n) to an input, x(n). This process is often termed as the convolution sum as shown in the following figure.



Let us first make the following assumptions:

(i) x(n) is a sequence of length P defined for $0 \le n \le P-1$ and

(ii) h(n) is a sequence of length M defined for $0 \le n \le M-1$.

The convolution of x(n) and h(n) called the linear convolution is computed through DFT (FFT) as follows:

Step(1): Choose $N \ge P+M-1$ (such that $N=2^r$ where r is a least positive integer). **Step(2):** Form the sequence $x^1(n)$ by padding N−P zeros to x(n).

$$x^1(n)=\begin{cases} x(n) & 0 \le n \le P-1 \\ 0 & P \le n \le N-1 \end{cases}$$

Step(3): Form the sequence $h^1(n)$ by padding N−M zeros to x(n).

$$h^1(n)=\begin{cases} h(n) & 0 \le n \le M-1 \\ 0 & M \le n \le N-1 \end{cases}$$

Step(4): Compute the N-point DFTs (FFTs), $X^1(k)$ and $H^1(k)$, of $x^1(n)$ and $h^1(n)$ i.e.,

$$X^1(k)=\sum_{n=0}^{N-1} x^1(n) W_N^{nk}, \quad k=0,1,2,\ldots,N-1$$

$$H^1(k)=\sum_{n=0}^{N-1} h^1(n) W_N^{nk}, \quad k=0,1,2,\ldots,N-1$$

where $W_N=\exp[-j(2\pi/N)]$.

Step(5): Compute the required output y(n) for $0 \le n \le P+M-2$ by computing IDFT of the product, $X^1(k)H^1(k)$ and retaining the first P+M−1 values of the result.

$$y(n)=\begin{cases} y^1(n) & 0 \le n \le P+M-2 \\ & \end{cases}$$

0 otherwise

Overlap-add method:

Let us first make the following assumptions:

(i) x(n) is a long sequence of length P>>M defined for $0 \le n \le P-1$ and

(ii) h(n) is a short sequence of length M defined for $0 \le n \le M-1$

Step(1): Choose a convenient, positive integer $L \ge 1$.

Step(2): Segment the long sequence x(n) into r* sequences, each of length L. Let the segmented sequences be $x_0(n)$, $x_1(n)$, $x_2(n)$,..., $x_{r-1}(n)$ where, in general

$$x_k(n) = \begin{cases} x(n+kL), & n=0,1,2,...,L-1 \\ 0, & \text{otherwise} \end{cases}$$

for k=0,1,2,...,r−1.

*r is the smallest positive integer chosen such that $rL \ge P$.

Step(3): Choose $N \ge L+M-1$ (such that $N=2^r$ where r is a least positive integer). **Step(4):** Compute the N-point circular convolution

$$y_k(n) = x_k(n) \circledN h(n), \quad 0 \le n \le N-1$$

using DFT (FFT).

Step(5): Form the sequences $y_k^1(n)$ by shifting the sequences $y_k(n)$ to the right by kL units for k=0,1,2,...,r−1 i.e.,

$$y_k^1(n) = \begin{cases} y_k(n-kL), & kL \le n \le (k+1)L+M-2 \\ 0, & \text{otherwise} \end{cases}$$

Here, the sequence $y_k^1(n)$ is nonzero for $kL \le n \le (k+1)L+M-2$, $y_{k-1}^1(n)$ is nonzero for $(k-1)L \le n \le kL+M-2$ and $y_{k+1}^1(n)$ is nonzero for $(k+1)L \le n \le (k+2)L+M-2$. This implies that the first (M−1) points of $y_k^1(n)$ overlap the last (M−1) points of $y_{k-1}^1(n)$ and the last (M−1) points of $y_k^1(n)$ overlap the first (M−1) points of $y_{k+1}^1(n)$ for all k as shown below:

Step(6): Compute the required output y(n) for $0 \le n \le P+M-2$ as follows:

$$y(n)=\sum_{k=0}^{r-1} y_k^1(n), \quad 0 \le n \le P+M-2$$

Overlap-save method:

Let us first make the following assumptions:

(i) x(n) is a long sequence of length P>>M defined for $0 \le n \le P-1$ and

(ii) h(n) is a short sequence of length M defined for $0 \le n \le M-1$

Step(1): Choose a convenient, positive integer $L \ge M$.

Step(2): Segment the long sequence x(n) into r sequences, each of length L. Let the segmented sequences be $x_0(n)$, $x_1(n)$, ..., $x_{r-1}(n)$, where, in general,

$$x_k(n)=\begin{cases} x[n+kL-(k+1)(M-1)], & n=0, 1, ..., L-1 \\ 0, & \text{otherwise} \end{cases}$$

for k=0, 1, ..., r–1.

$x_0(n)$ is chosen such that the first (M–1) points are zeros and the remaining (L–M+1) points are the first (L–M+1) points of x(n).

$x_1(n)$ is chosen such that the first (M–1) points are the last (M–1) points of $x_0(n)$ and the remaining (L–M+1) points are the second (L–M+1) points of x(n).

$x_2(n)$ is chosen such that the first (M–1) points are the last (M–1) points of $x_1(n)$ and the remaining (L–M+1) points are the third (L–M+1) points of x(n) and so on.

In general, $x_k(n)$ is chosen such that the first (M–1) points overlap the last (M–1) points of $x_{k-1}(n)$ and the last (M–1) points overlap the first (M–1) points of $x_{k+1}(n)$. **Step(3):** Compute the L-point circular convolution

$$y_k(n)=x_k(n) \textcircled{L} h(n), \quad 0 \le n \le L-1$$

using DFT (FFT).

Step(4): Form the sequences $y_k^1(n)$ by discarding the first (M–1) points of $y_k(n)$ for k=0, 1, ..., r–1.

Step(5): Compute the required output y(n) for $0 \le n \le P+M-2$ by appending the sequences $y_k^1(n)$ in order as follows:

$$y(n)=\{y_0^1(n), y_1^1(n), ..., y_{r-1}^1(n)\}, \quad 0 \le n \le P+M-2$$

3.7 Correlation through DFT (FFT):

The correlation of two finite length sequences, x(n) and y(n), each of length, N is the sequence, $r_{xx}(k)$ given by

$$r_{xy}(m)=\sum_{n=m}^{N-1} x(n)\; y(n-m) \quad \text{for } m\ge 0$$

$$r_{xy}(m)=\sum_{n=0}^{N-|m|-1} x(n)\; y(n-m) \quad \text{for } m<0$$

where the index, m is called the ***lag***. The correlation can be shown to be

$$r_{xy}(k)=x(k)*y(-k)$$

where y(–n) is the folded version of y(n). Hence the correlation can be computes using DFT (FFT) as follows.

The correlation of a sequence, x(n) to itself i.e., the correlation for the case when x(n)=y(n), is called the autocorrelation given by

$$r_{xx}(m)=\sum_{n=m}^{N-1} x(n)\, x(n-m) \quad \text{for } m\geq 0$$

$$r_{xx}(m)=\sum_{n=0}^{N-|m|-1} x(n)\, x(n-m) \quad \text{for } m<0$$

and

$$r_{xx}(k)=x(k)*x(-k)$$

Given $x(n)$ and $y(n)$, each of length, N

Step(1): Compute $(2N-1)$-point DFT (FFT), $X(k)$ of $x(n)$.
Step(2): Compute $(2N-1)$-point DFT (FFT), $Y_f(k)$ of $y_f(n)$ where $y_f(n)$ is the folded version of $y(n)$.
Step(3): Compute the product of $X(k)$ and $Y_f(k)$ and take the inverse DFT (FFT) of the result.