SMTX 1011 APPLIED NUMERICAL METHODS

COMMON TO ALL ENGINEERINGS EXCEPT BIO MED AND BIO INFO

III YEAR V SEMESTER (BATCH 2010 ONWARDS)

COURSE MATERIAL

COURSE OBJECTIVE: The ability to identify, reflect upon, evaluate and apply different types of information and knowledge to form independent judgments. Analytical, logical thinking and conclusions based on quantitative information will be the main objective of learning this subject.

UNIT II- INTERPOLATION NUMERICAL DIFFERENTIATION AND INTEGRATION

Interpolation – Newton's methods - Lagrange's Methods - Numerical differentiation and integration: Trapezoidal rule, Simpson's Rule-Finite difference Equation.

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INTERPOLATION

Interpolation with Equal Intervals :

Defn: Interpolation

Interpolation is the process of finding the intermediate values of a function [which is not explicitly known] from a set of its values at specific points given in a tabulated form.

Suppose that the following table represents a set of corresponding values of x and $y=f(x)$:

x	x_0	x_1	x_2	$\ldots x_n$
y	y_0	y_1	y_2	$\ldots y_n$

The process of computing y corresponding to x where $x_i < x < x_{i+1}$, $i=0,1,2\ldots n-1$ is interpolation

Defn: Extrapolation

If $x < x_0$ or $x > x_n$ then the process is called extrapolation.

Note :-

The term interpolation is used in both cases.

(2)

Defn :- Polynomial Interpolation

The process of representing $f(x)$ by a polynomial $p(x)$ called polynomial interpolation

Gregory Newton's Forward Interpolation Formula for Equal Intervals

If $y_0, y_1, y_2 \ldots y_n$ are the values of $y=f(x)$ corresponding to equidistant values of $x_0, x_1, x_2 \ldots x_n$ where $x_i - x_{i-1} = h$ for $i=1,2\ldots n$

then $$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots + \frac{u(u-1)\cdots(u-(n-1))}{n!}\Delta^n y_0$$

where $u = \frac{x-x_0}{h}$.

This result is known as Gregory-Newton forward interpolation (or) Newton's formula for equal intervals.

Notes :-

1. Since the formula derived involves the forward differences of y at y_0, it is called Newton's forward interpolation formula.

2. If only 2 values of y, namely y_0 and y_1 corresponding to $x=x_0$ and x_1 are given, the above formula takes the form

$$y = y_0 + \frac{(x-x_0)}{h}(y_1-y_0)$$

ie) $$y - y_0 = \left(\frac{y_1-y_0}{x_1-x_0}\right)(x-x_0)$$ which is called linear interpolation formula

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3. If 3 values of y, namely y_0, y_1 and y_2 corresponding to $x = x_0, x_1$ and x_2 are given then Newton's forward interpolation formula is called parabolic interpolation formula.

Gregory - Newton's Backward Interpolation Formula for Equal Intervals

If $y_0, y_1 \cdots y_n$ are the values of $y = f(x)$ corresponding to equidistant values of $x = x_0, x_1, \ldots x_n$ where $x_i - x_{i-1} = h$ for $i = 1, 2 \ldots n$

then
$$y = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots\cdots + \frac{u(u+1)\cdots(u+n-1)}{n!}\nabla^n y_n$$

where $u = \frac{x - x_n}{h}$.

Note:-

Since this formula involves the backward differences of y at y_n, it is called Newton's backward interpolation formula.

WORKED EXAMPLES

1. If $y(10) = 35.3$, $y(15) = 32.4$, $y(20) = 29.2$, $y(25) = 26.1$, $y(30) = 23.2$ and $y(35) = 20.5$, Find $y(12)$ using Newton's forward interpolation formula.

Solution:

The difference table is

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x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10 (x_0)	35.3 (y_0)					
		−2.9 (Δy_0)				
15	32.4		0.3 $(\Delta^2 y_0)$			
		−3.2		0.4 $(\Delta^3 y_0)$		
20	29.2		0.1		0.3 $(\Delta^4 y_0)$	
		−3.1		0.1		0.2 $(\Delta^5 y_0)$
25	26.1		0.2		−0.1	
		−2.9		0.0		
30	23.2		0.2			
		−2.7				
35	20.5					

Newton's forward interpolation formula is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y_0 \quad \text{——} (1)$$

where $u = \frac{x - x_0}{h} = \frac{12-10}{5} = 0.4$

Using the values of y_0 and the forward differences from the difference table in (1), we have

$$y(x=12) = y(u=0.4)$$

$$= 35.3 + \frac{(0.4)(-2.9)}{1!} + \frac{(0.4)(-0.6)(-0.3)}{2!} + \frac{(0.4)(-0.6)(-1.6)(0.4)}{3!} + \frac{(0.4)(-0.6)(-1.6)(-2.6)(-0.3)}{4!} + \frac{(0.4)(-0.6)(-1.6)(-2.6)(-3.6)}{5!}(0.2)$$

$$= 35.3 - 1.16 + 0.036 + 0.0256 + 0.01248 + 0.0059904$$

$$= 34.2200704$$

$$\simeq 34.22$$

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2. From the given table, compute the value of $\sin 38^\circ$.

x	0	10	20	30	40
$\sin x$	0	0.17365	0.34202	0.50000	0.64279

Solution :-

To determine the value of $y = \sin x$ near the lower end, we apply Newton's backward interpolation formula. The difference table is as given below.

x°	$y(x) = \sin x^\circ$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0°	0				
		0.17365			
10°	0.17365		-0.00528		
		0.16837		-0.00511	
20°	0.34202		-0.01039		0.00031 ($\nabla^4 y_0$)
		0.15798		-0.0048 ($\nabla^3 y_0$)	
30°	0.50000		-0.01519 ($\nabla^2 y_0$)		
		0.14279 (∇y_0)			
40° (x_0)	0.64279 (y_0)				

Newton's backward interpolation formula is

$$y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n \quad \text{where } u = \frac{x - x_n}{h}.$$

$$u = \frac{x - x_n}{h} = \frac{38-40}{10} = -0.2$$

$$y(38) = y(u = -0.2).$$

$$= 0.64279 + (-0.2)(0.14279) + \frac{(-0.2)(-0.2+1)}{2!}(-0.01519) + \frac{(-0.2)(-0.2+1)(-0.2+2)}{3!}(-0.0048) + \cdots$$

$$= 0.64279 - 0.028558 + 0.0012152 + 0.0002304 + \cdots$$

$\sin 38^\circ = 0.61568$

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Note:-

It is obvious that either of the two formulas may be used to interpolate (or) extrapolate y corresponding to any value of x, whatever be its position

3. The population of a town in the census is as given in the data. Estimate the population in the year 1996 using Newton's (i) forward interpolation and (ii) backward interpolation formula.

Year (x)	1961	1971	1981	1991	2001
Population (in 1000's)	46	66	81	93	101

Solution:- The difference table is

x	y	Δy (or) ∇y	$\Delta^2 y$ (or) $\nabla^2 y$	$\Delta^3 y$ (or) $\nabla^3 y$	$\Delta^4 y$ (or) $\nabla^4 y$
1961	46				
		20			
1971	66		−5		
		15		2	
1981	81		−3		−3
		12		−1	
1991	93		−4		
		8			
2001	101				

(i) Newton's forward interpolation formula is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \quad \text{——} (1)$$

where $u = \frac{x - x_0}{h} = \frac{1996 - 1961}{10} = 3.5$

Using the values of y_0 and the forward differences from the difference table in (1), we have

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$$y(x=1996) = y(u=3.5)$$

$$= 46 + \frac{3.5 \times 20}{1!} + \frac{(3.5)(2.5)(-5)}{2!} + \frac{(3.5)(2.5)(1.5)}{3!}(2)$$

$$+ \frac{(3.5)(2.5)(1.5)(0.5)}{4!}(-3)$$

$$= 46 + 70 - 21.875 + 4.375 - 0.8203125$$

$$= 97.6796875$$

(ii) Newton's backward interpolation formula is

$$y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots \quad \text{---} (2)$$

where $u = \frac{x - x_n}{h} = \frac{1996 - 2001}{10} = -0.5$

Using the values of y_n and the backward differences from the difference table, we have

$$y(x=1996) = y(u=-0.5)$$

$$= 101 - \frac{0.5 \times 8}{1!} + \frac{(-0.5)(0.5)}{2!}(-4) + \frac{(-0.5)(0.5)(1.5)}{3!}(-1)$$

$$+ \frac{(-0.5)(0.5)(1.5)(2.5)}{4!}(-3)$$

$$= 101 - 4.0 + 0.5 + 0.0625 + 0.1171875$$

$$= 97.6796875$$

4. Find $e^{-0.75}$ and $e^{-2.25}$ from the following data using both Newton's forward and backward formulas

x	1.00	1.25	1.50	1.74	2.00
$y = e^{-x}$	0.3679	0.2865	0.2231	0.1738	0.1353

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Solution :- The difference table is

x	y	Δy (or) ∇y	$\Delta^2 y$ (or) $\nabla^2 y$	$\Delta^3 y$ (or) $\nabla^3 y$	$\Delta^4 y$ (or) $\nabla^4 y$
1.00	0.3679				
		−0.0814			
1.25	0.2865		0.0180		
		−0.0634		−0.0039	
1.50	0.2231		0.0141		0.0006
		−0.0493		−0.0033	
1.75	0.1738		0.0108		
		−0.0385			
2.00	0.1353				

when $x = 0.75$

(i) Newtons forward formula is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots$$

where $u = \frac{x - x_0}{h} = \frac{0.75 - 1.00}{0.25} = -1$.

$$y(x=0.75) = 0.3679 + \frac{(-1)(-0.0814)}{1!} + \frac{(-1)(-2)}{2!}(0.0180)$$

$$+ \frac{(-1)(-2)(-3)}{3!}(-0.0039) + \frac{(-1)(-2)(-3)(-4)}{4!}(0.0006)$$

$$= 0.3679 + 0.0814 + 0.0180 + 0.0039 + 0.0006$$

$$= 0.4718$$

Newton's backward formula is

$$y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \cdots$$

where $u = \frac{x - x_n}{h} = \frac{0.75 - 2.00}{0.25} = -5$

$$y(x=0.75) = 0.1353 + \frac{(-5)}{1!}(-0.0385) + \frac{(-5)(-4)}{2!}(0.0108)$$

$$+ \frac{(-5)(-4)(-3)}{3!}(-0.0033) + \frac{(-5)(-4)(-3)(-2)}{4!}(-0.0006)$$

$$= 0.1353 + 0.1925 + 0.1080 + 0.0330 + 0.0030$$

$$= 0.4718$$

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(ii) when $x = 2.25$

Newton's forward formula is $y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \cdots$

where $u = \frac{x - x_0}{h} = \frac{2.25 - 1.00}{0.25} = 5$

$$y(x=2.25) = 0.3679 + \frac{5}{1!}(-0.0814) + \frac{(5)(4)}{2!}(0.0180)$$

$$+ \frac{(5)(4)(3)}{3!}(-0.0039) + \frac{(5)(4)(3)(2)}{4!}(0.0006)$$

$$= 0.3679 - 0.4070 + 0.1800 - 0.0390 + 0.0030$$

$$= 0.1049$$

Newton's backward formula is $y(x) = y_n + \frac{u}{1!}\nabla y_n + \cdots$

where $u = \frac{x - x_n}{h} = \frac{2.25 - 2.50}{0.25} = 1$.

$$y(x=2.25) = 0.1353 + \frac{(1)}{1!}(-0.0385) + \frac{(1)(2)}{2!}(0.0108) +$$

$$\frac{(1)(2)(3)}{3!}(-0.0033) + \frac{(1)(2)(3)(4)}{4!}(0.0006)$$

$$= 0.1353 - 0.0385 + 0.0108 - 0.0033 + 0.0006$$

$$= 0.1049$$

5. Find the interpolating polynomial for y from the following data using both Newton's forward and backward formulae

x	4	6	8	10
y	1	3	8	16

Solution:-

The difference table is

x	y	Δy (or) ∇y	$\Delta^2 y$ (or) $\nabla^2 y$	$\Delta^3 y$ (or) $\nabla^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

(10)

(i) Newton's forward formula is $y = y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \cdots$

where $u = \frac{x - x_0}{h} = \frac{x-4}{2}$

ie) $y = y(x) = 1 + \frac{\left(\frac{x-4}{2}\right)}{1!}(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2} - 1\right)}{2!}(3) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2} - 1\right)\left(\frac{x-4}{2} - 2\right)}{3!}(0)$

ie) $y = 1 + (x-4) + \frac{3}{8}(x-4)(x-6)$

ie) $y = \frac{1}{8}\left[3x^2 - 22x + 48\right]$ which is the required interpolating polynomial for y.

(ii) Newton's backward formula is

$y = y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \cdots$

where $u = \frac{x - x_n}{h} = \frac{x-10}{2}$

ie) $y = y(x) = 16 + \frac{\left(\frac{x-10}{2}\right)}{1!}(8) + \frac{\left(\frac{x-10}{2}\right)\left(\frac{x-10}{2} + 1\right)}{2!}(3)$

$= 16 + 4(x-10) + \frac{3}{8}(x-10)(x-8)$

$y = \frac{1}{8}\left[3x^2 - 22x + 48\right]$ which is the required interpolating polynomial for y.

(11)

Interpolation with Unequal Intervals

If the values of x are given at unequal intervals, the definition of differences for equal intervals is not applicable. In such situation, we make use of a new kind of differences called divided differences which take into consideration not only the changes in the values of $f(x)$ but also those in the values of x.

Divided Differences.

Let $f(x_0), f(x_1) \cdots f(x_n)$ be the values of the function $y=f(x)$ corresponding to $x = x_0, x_1, \cdots x_n$ which are not necessarily equally spaced. Where $x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}$ are not necessarily equal.

The first order divided difference of $f(x)$ for the arguments x_0 and x_1 is defined as

$$f(x_0, x_1) = \underset{x_1}{\Delta} f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\text{||}^{ly}\ f(x_1, x_2) = \underset{x_2}{\Delta} f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \cdots\cdots$$

In general $$f(x_{r-1}, x_r) = \underset{x_r}{\Delta} f(x_{r-1}) = \frac{f(x_r) - f(x_{r-1})}{x_r - x_{r-1}}$$

for $r = 1, 2 \ldots n$.

The second order divided difference of $f(x)$ for the three arguments x_0, x_1, x_2 is defined as

$$f(x_0, x_1, x_2) = \underset{x_1, x_2}{\Delta^2} f(x_0) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}.$$

(12)

The n^{th} order divided difference of $f(x)$ for the $(n+1)$ arguments $x_0, x_1 \ldots x_n$ is defined as

$$f(x_0, x_1, \ldots x_n) = \underset{x_1, x_2 \ldots x_n}{\Delta^n} f(x_0) = \frac{f(x_1, x_2 \ldots x_n) - f(x_0, x_1, \ldots x_{n-1})}{x_n - x_0}$$

Note :-

The value of any divided difference is got by dividing the difference between the two adjacent lower order divided differences lying on the immediately preceding column by the difference between the arguments corresponding to the terminal entries of the two diagonals emanating from the concerned divided difference.

Properties

1. The divided differences are symmetrical in all their arguments, viz., the value of any divided difference is independent of the order of the arguments.
2. The divided difference operator Δ is linear
3. The n^{th} order divided differences of a polynomial of degree n are constant, equal to the coefficient of x^n.
4. If the arguments $x_0, x_1 \ldots x_n$ are equally spaced such that $x_r - x_{r-1} = h$ $(r = 1, 2 \ldots n)$ then $\Delta^r f(x_0) = \frac{\Delta^r f(x_0)}{r!\, h^r}$, $r = 1, 2 \ldots n$.
5. If $f(x_0), f(x_1) \ldots f(x_n)$ are the values of $f(x)$ corresponding to the arguments $x_0, x_1, \ldots x_n$ that are not necessarily equally spaced, then $f(x) = f(x_0) + (x - x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\Delta^n f(x_0)$

1. Show that $\underset{x_1, x_2, x_3}{\Delta^3}\left(\frac{1}{x_0}\right) = -\frac{1}{x_0 x_1 x_2 x_3}$

Solution :-

$$\underset{x_1}{\Delta}\left(\frac{1}{x_0}\right) = \frac{\frac{1}{x_1} - \frac{1}{x_0}}{x_1 - x_0} = -\frac{1}{x_0 x_1}$$

$$\underset{x_1, x_2}{\Delta^2}\left(\frac{1}{x_0}\right) = \frac{\underset{x_2}{\Delta}\left(\frac{1}{x_1}\right) - \underset{x_1}{\Delta}\left(\frac{1}{x_0}\right)}{x_2 - x_0} = \frac{\frac{-1}{x_1 x_2} + \frac{1}{x_0 x_1}}{x_2 - x_0} = \frac{1}{x_0 x_1 x_2}$$

$$\underset{x_1, x_2, x_3}{\Delta^3}\left(\frac{1}{x_0}\right) = \frac{\underset{x_2, x_3}{\Delta^2}\left(\frac{1}{x_1}\right) - \underset{x_1, x_2}{\Delta^2}\left(\frac{1}{x_0}\right)}{x_3 - x_0} = \frac{\frac{1}{x_1 x_2 x_3} - \frac{1}{x_0 x_1 x_2}}{x_3 - x_0} = -\frac{1}{x_0 x_1 x_2 x_3}$$

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2. Using the following data find $f(x)$ as a polynomial in x and hence find $f(4)$.

x	0	1	2	5
$f(x)$	2	3	12	147

Solution

The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2			
		$\frac{3-2}{1-0}=1$		
			$\frac{9-1}{2-0}=4$	
1	3			$\frac{9-4}{5-0}=1$
		$\frac{12-3}{2-1}=9$		
			$\frac{45-9}{5-1}=9$	
2	12			
		$\frac{147-12}{5-2}=45$		
5	147			

By Newton's divided difference formula

$$f(x)=f(x_0)+(x-x_0)\Delta f(x_0)+(x-x_0)(x-x_1)\Delta^2 f(x_0)+(x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0)$$

$$f(x)=2+x\cdot(1)+x(x-1)(4)+x(x-1)(x-2)(1)$$

$$f(x)=x^3+x^2-x+2$$

$$f(4)=64+16-4+2=78$$

3. Use Newton's divided difference formula to find $f(x)$ from the following data:

x	0	2	3	4	6	7
$f(x)$	0	8	0	-72	0	1008

Solution:-

Since 6 values of $f(x)$ are given, we can assume $f(x)$ to be a polynomial of degree 5.

Since $f(0)=0$, $f(3)=0$, $f(6)=0$,

$x(x-3)(x-6)$ is a factor of $f(x)$.

Hence let $f(x)=x(x-3)(x-6)g(x)$

where $g(x)=\dfrac{f(x)}{x(x-3)(x-6)}$ is a quadratic expression

$$\therefore g(2)=\frac{f(2)}{2(-1)(-4)}=1,\quad g(4)=\frac{f(4)}{(4)(1)(-2)}=9,\quad g(7)=\frac{f(7)}{(7)(4)(1)}=36$$

Now let us find $g(x)$ by using Newton's divided difference formula

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x	$g(x)$	$\Delta g(x)$	$\Delta^2 g(x)$
2	1		
		4	
4	9		1
		9	
7	36		

By Newton's formula

$$g(x) = g(x_0) + (x-x_0)\Delta g(x_0) + (x-x_0)(x-x_1)\Delta^2 g(x_0)$$

$$= 1 + 4(x-2) + (x-2)(x-4)$$

$$g(x) = x^2 - 2x + 1 = (x-1)^2$$

$$\therefore f(x) = x(x-3)(x-6)(x-1)^2$$

LAGRANGE'S INTERPOLATION FORMULA FOR UNEQUAL INTERVALS

If $y_0, y_1 \ldots y_n$ are the values of a function $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots x_n$ which are not necessarily equally spaced then

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} y_1 + \ldots\ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} y_n$$

1. Determine by lagrange's method the percentage number of patients over 40 years, using the following data:

Age over (x) years :	30	35	45	55
% number (y) of patients :	148	96	68	34

Solution:- By lagrange's formula

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 \quad \text{—— (1)}$$

(15)

Put $x_0=30,\ x_1=35,\ x_2=45,\ x_3=55,\ y_0=148,\ y_1=96,\ y_2=68$ and $y_3=34$ in ①

$$y=\frac{(x-35)(x-45)(x-55)}{(-5)(-15)(-25)}\times 148+\frac{(x-30)(x-45)(x-55)}{(5)(-10)(-20)}\times 96+\frac{(x-30)(x-35)(x-55)}{(15)(10)(-10)}\times 68+\frac{(x-30)(x-35)(x-45)}{(25)(20)(10)}\times 34$$

$$[y]_{x=40}=-\frac{148}{5}+\frac{3}{4}\times 96+\frac{68}{2}-\frac{34}{20}$$

$$=74.7$$

2. Apply Lagrange's interpolation formula to find $f(x)$ if $f(1)=2,\ f(2)=4,\ f(3)=8,\ f(4)=16$ and $f(7)=128$. Hence find $f(5)$ and $f(6)$.

Solution Lagrange's interpolation formula is

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}f(x_0)+\cdots\cdots+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4) \quad \text{——①}$$

Putting $x_0=1,\ x_1=2,\ x_2=3,\ x_3=4,\ x_4=7$ and the given values of $f(x)$ in ①, we have

$$f(x)=\frac{(x-2)(x-3)(x-4)(x-7)}{(-1)(-2)(-3)(-6)}\times 2+\frac{(x-1)(x-3)(x-4)(x-7)}{(1)(-1)(-2)(-5)}\times 4+\frac{(x-1)(x-2)(x-4)(x-7)}{(2)(1)(-1)(-4)}\times 8+\frac{(x-1)(x-2)(x-3)(x-7)}{(3)(2)(1)(-3)}\times 16+\frac{(x-1)(x-2)(x-3)(x-4)}{(6)(5)(4)(3)}\times 128$$

$$f(x)=\frac{1}{90}\left[11x^4-80x^3+295x^2-310x+264\right] \quad \text{——②}$$

Put $x=5$ and $x=6$ in ②, we get $f(5)=32.93$ & $f(6)=66.67$

(16)

INVERSE LAGRANGE'S INTERPOLATION FORMULA

To find the values of y corresponding to some x. Here we treat y as a function of x. The process of finding x given y is called the inverse interpolation.

In such a case, we will take y as independent variable and x as dependent variable and use lagrange's interpolation formula.

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} \cdot x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} x_1 + \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})} x_n$$

This formula is called formula of inverse interpolation.

1. Find the age corresponding to the annuity value 13.6 given the table

Age(x)	30	35	40	45	50
Annuity Value (y)	15.9	14.9	14.1	13.3	12.5

Solution:

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)} x_0 + \cdots + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)} x_4$$

$$x = \frac{(13.6-14.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(15.9-14.9)(15.9-14.1)(15.9-13.3)(15.9-12.5)} \times 30 + \frac{(13.6-15.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(14.9-15.9)(14.9-14.1)(14.9-13.3)(14.9-12.5)} \times 35 + \frac{(13.6-15.9)(13.6-14.9)(13.6-13.3)(13.6-12.5)}{(14.1-15.9)(14.1-14.9)(14.1-13.3)(14.1-12.5)} \times 40 + \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-12.5)}{(13.3-15.9)(13.3-14.9)(13.3-14.1)(13.3-12.5)} \times 45 + \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.3)}{(12.5-15.9)(12.5-14.9)(12.5-14.1)(12.5-13.3)} \times 50$$

$x[y=13.6] = 43$

1. Define the first, second and third order divided differences.

2. Find $\Delta^3 y$ from the following data

x	1	2	4	7
y	22	30	82	106

3. State Newton's divided difference interpolation formula

4. State Lagrange's interpolation formula for unequal intervals

1. If $f(x) = \frac{1}{x^2}$, find the divided differences $f(x_0, x_1)$ $f(x_0, x_1, x_2)$ & $f(x_0, x_1, x_2, x_3)$

2. Find the value of y at $x = 20$ using Newton's divided difference formula given

x	12	18	22	24	32
y	146	836	1948	2796	9236

3. Use Newton's divided difference formula to find f(x) from the following data

x	0	1	4	5
$f(x)$	8	11	78	123

4. Use lagrange's interpolation formula to fit a polynomial to the following data:

x	0	1	3	4
y	-12	0	6	12

5. use lagranges inverse interpolation method to find the value of x corresponding to $y=100$ from the following data

x	3	5	7	9	11
y	6	24	58	108	174

1. Define interpolating polynomial
2. State Gregory - Newton forward interpolation formula
3. State Gregory - Newton backward interpolation formula.

1. Find the value of $f(1.02)$ from the following data correct to 5 places of decimals using Newton's
(i) forward interpolation formula &
(ii) backward interpolation formula

x	1.0	1.1	1.2	1.3	1.4	1.5
$f(x)$	0.84147	0.89121	0.93204	0.96356	0.98545	0.99749

2. Use both Newton's forward and backward interpolation formulas to find $\tan 17^\circ$ from the following data

x	0	4	8	12	16	20
$\tan x^\circ$	0	0.0699	0.1405	0.2126	0.2167	0.3640.

3. Find $y(5)$ using both Newton's forward and backward interpolation formulas if $y(10) = 35.3$, $y(15) = 32.4$, $y(20) = 29.2$, $y(25) = 26.1$, $y(30) = 23.2$ & $y(35) = 20.5$

NUMERICAL DIFFERENTIATION

Consider a set of values (x_i, y_i), $i = 0, 1, 2, \ldots, n$ of a function. The process of computing the derivative of the function y at a particular value of x from the given set of values is called Numerical differentiation. This may be done by first approximating the function by a suitable interpolation formula and then differentiating it as many times as desired.

Numerical Differentiation can be done for equal and unequal intervals.

DIFFERENTIATION FOR EQUAL INTERVALS

Let $x_0, x_1, x_2, \ldots, x_n$ be the values of x and $y_0, y_1, y_2, \ldots, y_n$ be the corresponding values of y, where the x values are equally spaced with a common interval of differencing h. Then

$x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $\ldots$, $x_n = x_0 + nh$

GREGORY-NEWTON'S FORWARD DIFFERENCE FORMULA FOR DERIVATIVES

At $x = x_0 + uh$

$$y'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$y''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_0 + \cdots\right]$$

$$y'''(x) = \frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \left(\frac{12u-18}{12}\right)\Delta^4 y_0 + \cdots\right]$$

$\ldots$ and so on

where $u = \frac{x - x_0}{h}$, x is the value at which the derivative needs to be found
x_0 is the first value of x
h is the common difference in x values

Particular case

At $x = x_0$

Then $u = \frac{x - x_0}{h} = 0$

Then the derivative formulae reduce to

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

GREGORY – NEWTON'S BACKWARD DIFFERENCE FORMULA FOR DERIVATIVES

At any $x = x_n + vh$

$$\frac{dy}{dx} = y'(x) = \frac{1}{h}\left[\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n + \left(\frac{4v^3+18v^2+22v+6}{24}\right)\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = y''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \left(\frac{6v^2+18v+11}{12}\right)\nabla^4 y_n + \cdots\right]$$

$$\frac{d^3y}{dx^3} = y'''(x) = \frac{1}{h^3}\left[\nabla^3 y_n + \left(\frac{12v+18}{12}\right)\nabla^4 y_n + \cdots\right]$$

where $v = \frac{x - x_n}{h}$, x is the value at which the derivative needs to be found, x_n is the last value of x, h is the common difference in the x values

Particular case

At $x = x_n$

Then $v = \frac{x - x_n}{h} = 0$

Then the derivative formulae reduce to

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

Note:

(1) If the derivative is required at a point nearer to the starting value in the table, Newton's forward difference formula for derivatives is used

(2) If the derivative at a point which is in the end of the table is required, then Newton's backward formula for derivatives is used.

PROBLEMS

1) Find the first two derivatives of $x^{1/3}$ at $x=50$ and $x=56$ given the table below.

x:	50	51	52	53	54	55	56
$y=x^{1/3}$:	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Solution

To find the derivatives at $x=50$, Newton's forward formula for derivatives is used and to find the derivatives at $x=56$, Newton's backward formula for derivatives is used.

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
50	3.6840			
		0.0244		
51	3.7084		−0.0003	
		0.0241		0
52	3.7325		−0.0003	
		0.0238		0
53	3.7563		−0.0003	
		0.0235		0
54	3.7798		−0.0003	
		0.0232		0
55	3.8030		−0.0003	
		0.0229		
56	3.8259			

At $x=50$

$$u=\frac{x-x_0}{h}=\frac{50-50}{1}\Rightarrow \boxed{u=0}$$

By Newton's forward formula for derivatives,

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \cdots\right]$$

$$= \frac{1}{1}\left[0.0244 - \frac{1}{2}(0.0003) + \frac{1}{3}(0)\right]$$

$$= 0.02455$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \cdots\right]$$

$$= \frac{1}{1}\left[-0.0003\right]$$

$$= -0.0003$$

At $x=56$

$$V = \frac{x-x_n}{h} = \frac{56-56}{1} \Rightarrow \boxed{V=0}$$

By Newton's backward formula for derivatives,

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$= \frac{1}{1}\left[0.0229 + \frac{1}{2}(-0.0003) + 0\right]$$

$$= 0.02275$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \cdots\right]$$

$$= \frac{1}{1}(-0.0003)$$

$$= -0.0003$$

2) The population of a certain town is shown in the following table

Year :	1931	1941	1951	1961	1971
Population : (In thousands)	40.6	60.8	79.9	103.6	132.7

Find the rate of growth of the population in the year 1961

Solution

Since we have to find rate of change of population, we have to find the first derivative. As 1961 lies in the end of the table, Newton's backward formula for derivatives is used.

Difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1931	40.6				
		20.2			
1941	60.8		-1.1		
		19.1		5.7	
1951	79.9		4.6		$\boxed{-4.9}$ $\nabla^4 y_n$
		23.7		$\boxed{0.8}$ $\nabla^3 y_n$	
1961	103.6		$\boxed{5.4}$ $\nabla^2 y_n$		
		$\boxed{29.1}$ ∇y_n			
1971	132.7				

Here $h=10 \quad x_n=1971$

$$v = \frac{x-x_n}{h} = \frac{1961-1971}{10} = -1$$

By Newton's backward formula for derivatives,

$$\left[\frac{dy}{dx}\right]_{x=x_n+vh} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n + \left(\frac{2v^3+9v^2+11v+3}{12}\right)\nabla^4 y_n + \cdots\right]$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1961} = \frac{1}{10}\left[29.1 + \left(\frac{-1}{2}\right)(5.4) + \left(\frac{-1}{6}\right)(0.8) + \left(\frac{-1}{12}\right)(-4.9)\right]$$

$$= 2.6775$$

3) A rod is rotating in a plane. The following table gives the angle θ (in radians) through which the rod has turned for various values of time t (seconds). Calculate the angular velocity and angular acceleration of the rod at $t=0.6$ seconds.

t:	0	0.2	0.4	0.6	0.8	1.0
θ:	0	0.12	0.49	1.12	2.02	3.20

Solution

Since $x=0.6$ is towards the end, backward difference formula for derivatives is used.

Difference table

t	θ	$\nabla\theta$	$\nabla^2\theta$	$\nabla^3\theta$	$\nabla^4\theta$
0	0				
		0.12			
0.2	0.12		0.25		
		0.37		0.01	
0.4	0.49		0.26		0
		0.63		0.01	
0.6	1.12		0.27		0
		0.90		0.01	
0.8	2.02		0.28		
		1.18			
1.0	3.20				

Here $h = 0.2$, $x_n = 1.0$

$$\Rightarrow V = \frac{x - x_n}{h} = \frac{0.6 - 1.0}{0.2} = -2$$

To find angular velocity

By Newton's backward difference formula for derivatives,

$$\left[\frac{dy}{dx}\right]_{x=x_n+vh} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n + \left(\frac{4v^3+18v^2+22v+6}{24}\right)\nabla^4 y_n + \cdots\right]$$

$$\Rightarrow \left[\frac{d\theta}{dt}\right]_{t=0.6} = \frac{1}{0.2}\left[1.18 - \frac{3}{2}(0.28) + \frac{1}{3}(0.01)\right]$$

$$= 3.81665 \text{ radians/sec.}$$

To find angular acceleration

By Newton's backward difference formula for derivatives,

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n+vh} = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \cdots\right]$$

$$\Rightarrow \left[\frac{d^2\theta}{dt^2}\right]_{t=0.6} = \frac{1}{0.04}\left[0.28 - 0.01\right]$$

$$= 6.75 \text{ radians/sec}^2$$

4) From the values in the table given below, find the value of $\sec 31^\circ$

θ (in degrees)	31	32	33	34
$\tan\theta$	0.6008	0.6249	0.6494	0.6745

Solution

Since $\frac{d}{d\theta}(\tan\theta) = \sec^2\theta$, we first find the first order derivative. As $\theta = 31$ lies in the beginning of the table, Newton's forward interpolation formula for derivatives is used.

Difference table

$\theta(x)$	$y = \tan\theta$	Δy	$\Delta^2 y$	$\Delta^3 y$
31	0.6008			
		0.0241		
32	0.6249		0.0004	
		0.0245		0.0002
33	0.6494		0.0006	
		0.0251		
34	0.6745			

Here $h = 1^\circ$, $x_0 = 31^\circ$

$$u = \frac{x - x_0}{h} = \frac{31^\circ - 31^\circ}{1^\circ} = 0$$

By Newton's forward formula for derivatives,

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \cdots\right]$$

$$\Rightarrow \left[\frac{d}{d\theta}(\tan\theta)\right]_{\theta=31^\circ} = \frac{1}{1^\circ}\left[0.0241 - \frac{1}{2}(0.0004) + \frac{1}{3}(0.0002)\right]$$

$$= \frac{1}{0.01745}(0.0240) \quad (\text{since } 1^\circ = 0.01745 \text{ radians})$$

$$= 1.3754$$

$$\Rightarrow \sec^2 31^\circ = 1.3754$$

Thus $\sec 31^\circ = \sqrt{1.3754} = 1.1728$

NUMERICAL DIFFERENTIATION FOR UNEQUAL INTERVALS

Divided Difference

When the x values are not equally spaced, then the interval of differencing is not constant. In such cases, we express y as a polynomial in x using Newton's divided difference formula or Lagrange's interpolation formula and then differentiating it w.r.t x, the derivatives at any x in the given range can be found.

PROBLEMS

1) Using Lagrange's formula, find y' and y'' at $x=2$ for the following data

x:	0	1	3	6
y:	18	10	−18	90

Solution

Since the x values are unequally spaced, first Lagrange's interpolation formula is used to find y as a polynomial in x.

Given $x_0=0,\ x_1=1,\ x_2=3,\ x_3=6$

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\times y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\times y_1+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\times y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\times y_3$$

$$=\frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}\times 18+\frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10)+$$
$$\frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(18)+\frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}(90)$$

$$=-(x-1)(x-3)(x-6)+x(x-3)(x-6)+x(x-1)(x-6)+x(x-1)(x-3)$$

$$f(x)=2x^3-10x^2+18$$

Hence $f'(x) = 6x^2 - 20x$

$\Rightarrow f'(2) = -16$

Also $f''(x) = 12x - 20$

$\Rightarrow f''(2) = 4$

$\therefore$ y' and y'' at $x=2$ are -16 and 4 respectively

2) Given the following data, find $y'(6)$ and the maximum value of y.

x:	0	2	3	4	7	9
y:	4	26	58	112	466	922

Solution

Since the x values are not equally spaced, Newton's divided difference formula is used to find y as a function of x.

Divided difference table

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	4	11	7	1	0
2	26	32	11	1	0
3	58	54	16	1	
4	112	118	22		
7	466	228			
9	922				

By Newton's divided difference formula,

$$y=f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + \dots$$

$$= 4 + (x-0)11 + (x-0)(x-2)(7) + (x-0)(x-2)(x-3)(1)$$

$$= x^3 + 2x^2 + 3x + 4$$

$$\Rightarrow f'(x) = 3x^2 + 4x + 3$$

$$\Rightarrow f'(6) = 3(6)^2 + 4(6) + 3 = 135$$

$y(x)$ is maximum if $y'(x) = 0$

$\Rightarrow 3x^2 + 4x + 3 = 0$

But the roots of this equation are imaginary.
Hence there is no extremum value in the range.

NUMERICAL INTEGRATION

Let $x_0, x_1, x_2, \ldots, x_n$ be the values of x and $y_0, y_1, y_2, \ldots, y_n$ be the corresponding values of y, where the x values are equally spaced with a common interval of differencing h. Then $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \ldots x_n = x_0 + nh$

DEFINITION

The process of computing $\int_a^b y\,dx$, where $y = f(x)$ is given by a set of tabulated values $[x_i, y_i]$, $i = 0, 1, 2, \ldots, n$ and $a = x_0$, $b = x_n$ is called numerical integration.

Geometrically, $\int_a^b y\,dx$ represents the area under the curve $y = f(x)$ between the ordinates $x = a$ and $x = b$.

(1) TRAPEZOIDAL RULE

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$= \frac{h}{2}\left[(\text{sum of the first and last ordinates}) + 2(\text{sum of the remaining ordinates})\right]$$

where n = number of intervals

$h = \frac{x_n - x_0}{n}$ is the common difference in x values

Note:

(1) This rule is the simplest one but is the least accurate

(2) The error in the Trapezoidal rule is of order h^2

(2) SIMPSON'S ONE-THIRD RULE

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\cdots) + 4(y_1+y_3+y_5+\cdots)\right]$$

$$= \frac{h}{3}\left[(\text{sum of the first and last ordinates}) + 2(\text{sum of ordinates with even suffixes}) + 4(\text{sum of ordinates with odd suffixes})\right]$$

Note:

(1) This rule is the most accurate of the three rules

(2) Simpson's one-third rule can be applied only when n, the number of intervals is even

(3) The error in Simpson's one-third rule is of order h^4

(3) SIMPSON'S THREE-EIGHTH RULE

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots) + 2(y_3+y_6+y_9+\cdots)\right]$$

$= \frac{3h}{8}$[(sum of the first and last ordinates) + 2(sum of ordinates with suffixes as multiples of 3) + 3(sum of remaining ordinates)]

Note:

Simpson's three eigth rule can be applied only when n, the number of intervals is a multiple of 3.

REMARK:

In all the three rules, the accuracy of the result increases as the value of h decreases and the value of n increases

PROBLEMS

1) From the following table, find the area bounded by the curve and the x axis from $x = 7.47$ to $x = 7.52$

x:	7.47	7.48	7.49	7.50	7.51	7.52
$y = f(x)$:	1.93	1.95	1.98	2.01	2.03	2.06

Solution

Here $n=5$. Since Simpson's rules cannot be used, Trapezoidal rule is used.

$$\text{Area} = \int_{x=7.47}^{x=7.52} y\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$\text{Area} = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right], \text{ where } h = 0.0$$

$$= \frac{0.01}{2}\left[(1.93 + 2.06) + 2(1.95 + 1.98 + 2.01 + 2.03)\right]$$

$$= 0.09965$$

2) Evaluate $\int_0^4 e^x dx$ taking $h = 1$

<u>Solution</u>

Given $h = 1$, $x_0 = 0$, $x_n = 4$

Since the x values start from 0 and go till 4 with a common difference of 1, the x values are 0, 1, 2, 3, 4

x:	0	1	2	3	4
$y = e^x$:	1	2.7183	7.3891	20.0855	54.5982
	y_0	y_1	y_2	y_3	y_4

Since $n = 4$, it is even. Hence Simpson's one third rule can be applied

By Simpson's one-third rule,

$$\int_0^4 e^x dx = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + \cdots) + 4(y_1 + y_3 + y_5 + \cdots)\right]$$

$$= \frac{h}{3}\left[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)\right]$$

$$= \frac{1}{3}\left[(1 + 54.5982) + 2(7.3891) + 4(2.7183 + 20.0855)\right]$$

$$= 53.8639$$

3) Evaluate $\int_0^1 \frac{dx}{1+x}$ correct to three decimal places.

Hence evaluate $\log_e 2$.

<u>Solution</u>

To use Simpson's one third rule, n the number of intervals has to be even.

Let $n = 6$. Then $h = \frac{x_n - x_0}{n} = \frac{1-0}{6} = \frac{1}{6}$.

$\therefore$ The x values vary from 0 to 1 with a common difference of $\frac{1}{6}$.

x:	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y=\frac{1}{1+x}$:	1	0.8571	0.75	0.6667	0.6	0.5455	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By simpson's one third rule,

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}\left[(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\right]$$

$$= \frac{\frac{1}{6}}{3}\left[(1+0.5)+2(0.75+0.6)+4(0.8571+0.6667+0.5455)\right]$$

$\int_0^1 \frac{dx}{1+x} = 0.6932 \simeq 0.693$ correct to 3 decimal places

To evaluate $\log_e 2$

By actual integration, we get

$$\left[\log_e(1+x)\right]_0^1 = 0.6932$$

$$\Rightarrow \log_e 2 - \log_e 1 = 0.6932$$

$$\Rightarrow \boxed{\log_e 2 = 0.6932}$$

4) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule, with $h=0.2$

Hence determine the value of π.

Solution

Here $h=0.2$, $x_0=0$, $x_n=1$

$\therefore$ x values vary from 0 to 1 with a common difference of 0.2

x:	0	0.2	0.4	0.6	0.8	1.0
$y=\frac{1}{1+x^2}$:	1	0.9615	0.8621	0.7353	0.6098	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

By Trapezoidal rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.2}{2}\left[1.5 + 2(3.1687)\right]$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.78374$$

To find the value of π

we know that, by actual integration,

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

$$\Rightarrow 0.78374 = \frac{\pi}{4}$$

$$\Rightarrow \boxed{\pi = 3.13496}$$

5) By dividing the range into ten equal parts, evaluate $\int_0^{\pi} \sin x\,dx$ by Trapezoidal and simpson's rule. Verify your answer with integration.

Solution

Given $n = 10$, $x_0 = 0$, $x_n = \pi$

$$\therefore h = \frac{x_n - x_0}{n} = \frac{\pi - 0}{10} = \frac{\pi}{10} \Rightarrow \boxed{h = \frac{\pi}{10}}$$

$\therefore x$ values vary from 0 to π with a common difference of $\frac{\pi}{10}$

x:	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$
$y = \sin x$:	0.0 (y_0)	0.3090 (y_1)	0.5878 (y_2)	0.8090 (y_3)	0.9511 (y_4)	1.0 (y_5)

x:	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π
$y = \sin x$:	0.9511 (y_6)	0.8090 (y_7)	0.5878 (y_8)	0.3090 (y_9)	0 (y_{10})

Trapezoidal Rule

$$\int_0^{\pi} \sin x\,dx = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + \cdots + y_9)\right]$$

$$= \frac{\pi}{20}\left[(0+0) + 2(0.3090 + 0.5878 + 0.8090 + 0.9511 + 1.0 + 0.9511 + 0.8090 + 0.5878 + 0.3090)\right]$$

$$= 1.9843$$

Simpson's one third rule

Since $n=10$ is even, Simpson's one-third rule can be applied.

By Simpson's one third rule,

$$\int_0^{\pi} \sin x\,dx = \frac{h}{3}\left[(y_0+y_{10}) + 2(y_2+y_4+y_6+y_8) + 4(y_1+y_3+y_5+y_7+y_9)\right]$$

$$= \frac{\pi}{30}\left[(0+0) + 2(0.5878+0.9511+0.9511+0.5878) + 4(0.3090+0.8090+1+0.8090+0.3090)\right]$$

$$= 2.0001$$

Actual Integration

By actual integration,

$$\int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} = 1+1 = 2$$

Hence Simpson's one third rule is more accurate than Trapezoidal rule.

6) The velocity v of a particle at distance s from a point on its' path is given by the following table:

S (feet) :	0	10	20	30	40	50	60
V (feet/sec) :	47	58	64	65	61	52	38

Estimate the time taken to travel 60 feet using Simpson's one third rule. Compare the result with Simpson's three eigth rule.

Solution

Velocity = rate of change of displacement

$$\Rightarrow V = \frac{ds}{dt} \quad \Rightarrow dt = \frac{1}{V}ds$$

$\therefore$ Time taken to travel 60 feet is given by

$$t = \int_0^{60} \frac{1}{V}\,ds \quad \left(\text{Here } x = s,\ y = \frac{1}{V}\right)$$

The new table is

$S(x)$:	0	10	20	30	40	50	60
$\frac{1}{V}(y)$:	0.0213	0.0172	0.0156	0.0154	0.0164	0.0192	0.0263
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's $\frac{1}{3}^{rd}$ Rule

$$t=\int_0^{60}\frac{1}{v}\,ds=\frac{h}{3}\left[(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\right]$$

$$=\frac{10}{3}\left[(0.0213+0.0263)+2(0.0156+0.0164)+4(0.0172+0.0154+0.0192)\right]$$

Time $=1.0635$ seconds

Simpson's $\frac{3}{8}^{th}$ Rule

$$t=\int_0^{60}\frac{1}{v}\,ds=\frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$=\frac{30}{8}\left[(0.0213+0.0263)+3(0.0172+0.0156+0.0164+0.0192)+2(0.0154)\right]$$

$=1.0644$ seconds

7) The table below gives the velocity v of a moving particle at time t seconds. Find the distance covered by the particle in 12 seconds and also the acceleration at t=2 seconds.

t :	0	2	4	6	8	10	12
v :	4	6	16	34	60	94	136

Solution

To find distance

We know $v=\frac{ds}{dt}$

$ds=v\,dt$

$\Rightarrow s=\int_0^{12}v\,dt$

$n=6$ is a multiple of 3. Hence Simpson's $\frac{3}{8}^{th}$ rule can be applied [As n is also even, $\frac{1}{3}^{rd}$ rule can also be applied]

$\therefore$ Distance covered in 12 seconds $= \int_0^{12} v\,dt$

$$= \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{3\times 2}{8}\left[(4+136)+3(6+16+60+94)+2(34)\right]$$

$$= 552 \text{ metres}$$

To find acceleration

We know $a = \left(\frac{dv}{dt}\right)_{t=2}$ = acceleration

Hence we use Newton's forward difference formula for derivatives. The forward difference table is

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$
0	4	2	8	0
2	6	10	8	0
4	16	18	8	0
6	34	26	8	0
8	60	34	8	
10	94	42		
12	136			

$u = \frac{x-x_0}{h} = \frac{2-0}{2} = 1$

$\Rightarrow \boxed{u=1}$

$\therefore$ Acceleration $= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \cdots\right]$

$$= \frac{1}{2}\left[2+\frac{1}{2}(8)+0\right]$$

$$= 3 \text{ metre/sec}^2$$

1

Set I

DIFFERENCE EQUATIONS

An equation which expresses a relation between the independent variable x, dependent variable $f(x)$ and the differences of various orders of $f(x)$ or successive values of the function $f(x)$ is called a Difference Equation.

Difference equations can be written in various forms : (i) in terms of Δ, (ii) in terms of E and (iii) in terms of successive values of $f(x)$.

eg; 1) $\Delta^3 y_x - 4\Delta y_x + 7y_x = x^2 + \cos x + 7$

2) $(E^2 - 4E + 6)y_x = x^2$

3) $y_{x+3} - 5y_{x+2} + 3y_{x+1} - 2y_x = 10x$.

Order and degree of a difference equation

The "order" of a difference equation written in the form free from Δ's, is the difference between the highest and lowest subscripts of y. Thus,

1) Order of $y_{x+3} - 5y_{x+2} + y_{x+1} = 0$ is $(x+3) - (x+1) = 2$

2) Order of $y_{x+3} - 5y_{x+2} + 7y_{x+1} + y_x = 10x$ is $(x+3) - (x) = 3$.

The "degree" of a difference equation written in the form free from Δ's, is the highest power of y's.

eg; 1) $y_{x+1}\, y^5_{x+2} - y_{x+1}\, y_x + y^2_{x+3} = \cos x$ is of degree 5 and of order 3

2) $(E^2 - 5E + 16)y_x = e^x$ is of order 2 and degree 1

3) $\Delta^2 u_x - 5\Delta u_x + 7u_x = 0 \Rightarrow (E^2 - 7E + 13)u_x = 0$ is of order 2 & deg 1

Note: The equation $\Delta^2 y_x + 2\Delta y_x + y_x = x^2$ which on simplification gives $E^2 y_x = x^2$ or $y_{x+2} = x^2$ which involves the value of the dependent variable and hence not at all a difference equation. Thus we cannot determine the order...

Linear Difference Equations with constant coefficients.

An equation of the form,

$$a_0 y_{x+n} + a_1 y_{x+n-1} + a_2 y_{x+n-2} + \cdots + a_{n-1} y_{x+1} + a_n y_x = f(x) \quad \text{—} (1)$$

ie, $$(a_0 E^n + a_1 E^{n-1} + a_2 E^{n-2} + \cdots + a_{n-1} E + a_n) y_x = f(x) \quad \text{—} (2)$$

where $a_0, a_1, a_2, \ldots a_n$ are constants and $f(x)$, a known function of 'x' is called a Linear Difference Equation in y_x with constant coefficients.

In a linear difference equation, the successive values of 'y' viz, $y_x, y_{x+1}, y_{x+2}, \ldots$ occur in the equation only in first degree and are not multiplied together.

(1) or (2) can also be rewritten as, $\phi(E) y_x = f(x)$ — (3) where $\phi(E)$ is a polynomial expression in E.

If RHS of (3) is zero, then $\phi(E) y_x = 0$ is called the Homogeneous equation corresponding to (3). If RHS of (3) is non-zero then the equation is called Non-homogeneous Linear equation

Solution of (3) consists of two parts (ie)
(i) Complementary Function (C.F) and (ii) Particular Integral (P.I)

Thus the general solution of (3) is $y_x = C.F + P.I$

If the RHS of (3) is zero, then the general solution of (3) is $y_x = C.F$ as there is no particular solution.

③.

To find the complementary function

Replacing E by 'a' in $\phi(E)$, we get $\phi(a)=0$ which is called the Auxiliary Equation of ③. Let the roots of $\phi(a)=0$ be $a_1, a_2, \dots a_n$.

Nature of the roots	C.F
1) Roots are real and distinct (ie) $a_1 \neq a_2 \neq \dots \neq a_n$	$c_1 a_1^x + c_2 a_2^x + \dots + c_n a_n^x$
2) Two real roots are equal (ie) $a_1 = a_2 = a \neq a_3 \neq \dots \neq a_n$	$(c_1 + c_2 x) a^x + c_3 a_3^x + \dots c_n a_n^x$.
3) Three real roots are equal (ie) $a_1 = a_2 = a_3 = a \neq a_4 \neq \dots \neq a_n$	$(c_1 + c_2 x + c_3 x^2) a^x + c_4 a_4^x + \dots$
4) Roots are complex (ie) $\alpha \pm i\beta$	$r^x[c_1 \cos\theta x + c_2 \sin\theta x] + c_3 a_3^x + \dots + c_n a_n^x$. where $r = \sqrt{\alpha^2+\beta^2}$, $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$
5) complex roots repeated twice (ie) $\alpha \pm i\beta, \alpha \pm i\beta$	$\left[(c_1 + c_2 x)\cos x\theta + (c_3 + c_4 x)\sin x\theta\right] r^x + c_5 a_5^x + \dots + c_n a_n^x$

Problems (Homogeneous)

1. Solve: $y_{x+2} - 8y_{x+1} + 15y_x = 0$.

The given equation can be rewritten as,
$$(E^2 - 8E + 15) y_x = 0.$$

A.E is $a^2 - 8a + 15 = 0 \Rightarrow a = 3, 5$.

$\therefore$ The complete solution is $y_x = A.3^x + B.5^x$

2. $(E^2+6E+9)\,y_n = 0$.

A.E is $a^2+6a+9=0 \Rightarrow a=-3,-3$.

$C.F = (A+Bn)(-3)^n$.

The complete solution is $y_n = (A+Bn)(-3)^n$.

3. Solve: $y_{n+2} + 2y_{n+1} + 4y_n = 0$.

(ie) $(E^2+2E+4)\,y_n = 0$.

A.E is, $a^2+2a+4=0 \Rightarrow a = \dfrac{-2\pm\sqrt{4-16}}{2}, \dfrac{-2\pm\sqrt{-12}}{2}$

$\Rightarrow \dfrac{-2\pm 2i\sqrt{3}}{2} \Rightarrow -1\pm i\sqrt{3}$

Thus $\alpha=-1,\ \beta=\sqrt{3}$.

Now $r = \sqrt{\alpha^2+\beta^2} = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{4} = 2$.

$\tan\theta = \dfrac{\sqrt{3}}{-1} = -\sqrt{3} \Rightarrow \theta = 120^\circ = \dfrac{2\pi}{3}$.

$\{\because \tan(180-\theta) = -\tan\theta;\ \tan\theta = -\sqrt{3} \Rightarrow \tan\theta = -\tan 60^\circ$
$\Rightarrow \tan\theta = \tan(180-60^\circ) = \tan 120^\circ\}$.

$C.F = 2^n\left\{A\cos\dfrac{2n\pi}{3} + B\sin\dfrac{2n\pi}{3}\right\}$.

$\therefore$ The complete solution is, $y_n = 2^n\left[A\cos\dfrac{2n\pi}{3} + B\sin\dfrac{2n\pi}{3}\right]$.

(2)

4. Solve: $y_{x+3} - 2y_{x+2} - y_{x+1} + 2y_x = 0$.

The equation can be rewritten as $(E^3 - 2E^2 - E + 2)y_x = 0$.

A.E is $a^3 - 2a^2 - a + 2 = 0 \Rightarrow a = 1, -1, 2$.

$\therefore$ The complete solution is $y_x = A\cdot(1)^x + B(-1)^x + C(2)^x$.

5. Solve the difference equation $y_{n+3} - 3y_{n+1} + 2y_n = 0$ given $y_1 = 0$, $y_2 = 8$ and $y_3 = -8$

$(E^3 - 3E + 2)y_n = 0$; A.E is $a^3 - 3a + 2 = 0 \Rightarrow a = 1, -2, 1$

$\therefore y_n = (A + Bn)(1)^n + C(-2)^n$ (ie) $y_n = A + Bn + C(-2)^n$ — (*)

Given $y_1 = 0 \Rightarrow A + B - 2C = 0$. — (1)

$y_2 = 8 \Rightarrow A + 2B + 4C = 8$ — (2)

$y_3 = -8 \Rightarrow A + 3B - 8C = -8$ — (3).

$$\begin{array}{l} 2A + 2B - \cancel{4C} = 0 \\ A + 2B + \cancel{4C} = 8 \\ \hline 3A + 4B = 8 \quad \text{— (4)} \end{array} \qquad \begin{array}{l} 2A + 4B + \cancel{8C} = 16 \\ A + 3B - \cancel{8C} = -8 \\ \hline 3A + 7B = 8 \quad \text{— (5)} \end{array}$$

(4) − (5) $\Rightarrow -3B = 0 \Rightarrow \boxed{B = 0}$

$\therefore \boxed{A = \frac{8}{3}}$, sub. A & B in (1), $\frac{8}{3} = 2C$ or $\boxed{C = \frac{4}{3}}$

$\therefore$ The solution is, $y_n = \frac{8}{3} + \frac{4}{3}(-2)^n$.

Hw 1) Solve the difference equation $y_{n+3} - 3y_{n+1} + 2y_n = 0$ given $y_1 = 0$, $y_2 = 8$, $y_3 = -2$ [Ans $y_n = 2n + (-2)^n$]

6. The integers 0, 1, 1, 2, 3, 5, 8, 13, 21, ... are said to form a Fibonacci sequence. Form the Fibonacci difference equation and solve it.

Soln. Let y_n be the n^{th} number in the sequence 0, 1, 1, 2, 3, ...
In this, each number (beyond the second) is equal to the sum of its 2 previous numbers. Hence $y_n = y_{n-1} + y_{n-2}$ for $n > 2$

(ie) $y_{n+2} = y_{n+1} + y_n$ for $n > 0$ (or) $y_{n+2} - y_{n+1} - y_n = 0$, $n > 0$.

where $y_1 = 0$, $y_2 = 1$

The difference equation can be rewritten as $(E^2 - E - 1)y_n = 0$.

A.E is $(a^2 - a - 1) = 0 \Rightarrow a = \dfrac{1 \pm \sqrt{5}}{2}$

∴ The solution is $y_n = A\left(\dfrac{1+\sqrt{5}}{2}\right)^n + B\left(\dfrac{1-\sqrt{5}}{2}\right)^n$ — ①, $n > 0$.

Since $y_1 = 0$; $A\left(\dfrac{1+\sqrt{5}}{2}\right) + B\left(\dfrac{1-\sqrt{5}}{2}\right) = 0 \Rightarrow A(1+\sqrt{5}) + B(1-\sqrt{5}) = 0$ — ②.

$y_2 = 1 \Rightarrow A\left(\dfrac{1+\sqrt{5}}{2}\right)^2 + B\left(\dfrac{1-\sqrt{5}}{2}\right)^2 = 1 \Rightarrow A(1+\sqrt{5})^2 + B(1-\sqrt{5})^2 = 4$ — ③.

$\cancel{A+\sqrt{5}}$) $A(1-5) + B(1-\sqrt{5})^2 = 0$

$A(1+\sqrt{5})^2 + \cancel{B(1-\sqrt{5})^2} = 4$
$(-) \quad (-) \quad (-)$

$+4A + A(1+\sqrt{5})^2 = +4$

$\Rightarrow A\{4 + 1 + 5 + 2\sqrt{5}\} = 4$

$\Rightarrow A = \dfrac{4}{10 + 2\sqrt{5}}$ (or) $A = \dfrac{2}{5+\sqrt{5}} \times \dfrac{(5-\sqrt{5})}{5-\sqrt{5}}$

$\Rightarrow A = \dfrac{2(5-\sqrt{5})}{25-5} \Rightarrow \boxed{A = \dfrac{5-\sqrt{5}}{10}}$

and $B = \dfrac{-A(1+\sqrt{5})}{(1-\sqrt{5})10} = \dfrac{-(5-\sqrt{5})(1+\sqrt{5})}{(1-\sqrt{5})10} = \dfrac{(\sqrt{5}-5)(1+\sqrt{5})}{(1-\sqrt{5})10}$

7.

$$B = \frac{\sqrt{5}\,\cancel{(1-\sqrt{5})}\,(1+\sqrt{5})}{\cancel{(1-\sqrt{5})}\,10} \Rightarrow \boxed{B = \frac{5+\sqrt{5}}{10}}$$

Hence the n^{th} term of the sequence is,

$$y_n = \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

HW

1) $\Delta^3 u_n - 5\Delta u_n + 4u_n = 0$ [Ans: $y_n = A.2^n + B.\left(\frac{1+\sqrt{17}}{2}\right)^n + C\left(\frac{1-\sqrt{17}}{2}\right)^n$].

2). $y_{x+3} - 3y_{x+2} - 10y_{x+1} + 24y_x = 0$ [Ans: $c_1(2)^x + c_2(-3)^x + c_3.(4)^x = y_x$]

3) $u_{n+4} - 8u_{n+3} + 18u_{n+2} - 27u_n = 0$ [Ans: $c_1(-1)^n + (c_2 + c_3 n + c_4 n^2)(3)^n$]

4) $y_{x+1} - 2y_x \cos\alpha + y_{x-1} = 0$. {Ans: $r = 1, \theta = \alpha$; $y_{x-1} = [c_1 \cos\alpha(x-1) + c_2 \sin\alpha(x-1)]\,(1)^{x-1}$

(ie) $y_x = c_1 \cos\alpha x + c_2 \sin\alpha x$}

Set II . (1)

To find the Particular integral of $\phi(E)y_x = f(x)$:

Type I : $f(x) = a^x$ where a is a constant [Constant power variable]

$$P.I = \frac{1}{\phi(E)} a^x = \frac{1}{\phi(a)} a^x \text{ provided } \phi(a) \neq 0.$$

If $\phi(a)=0$, then, $\frac{1}{\phi(E)} a^x = \frac{1}{(E-a)\psi(a)} a^x = \frac{1}{\psi(a)}\left(\frac{a^x}{E-a}\right)$

$$= \frac{1}{\psi(a)} x \cdot a^{x-1} \text{ provided } \psi(a) \neq 0.$$

(ie) $\frac{1}{\phi(E)} a^x = \frac{1}{\psi(a)} x a^{x-1}$ where $\frac{a^x}{E-a} = x a^{x-1}$

Similarly, $\frac{a^x}{(E-a)^2} = \frac{x(x-1)}{2!} a^{x-2}$, $\frac{a^x}{(E-a)^3} = \frac{x(x-1)(x-2)}{3!} a^{x-3}$

and so on.

Problems.

1. Solve the equation $y_{x+2} - 5y_{x+1} + 6y_x = 6^x$.

$(E^2-5E+6)y_x = 6^x \Rightarrow$ A.E is $a^2-5a+6=0 \Rightarrow a=3,2$.

C.F $= A(2)^x + B.(3)^x$

$$P.I = \left(\frac{1}{E^2-5E+6}\right) 6^x = \frac{1}{(E-3)(E-2)} 6^x = \frac{1}{3 \cdot 4} 6^x = \frac{6^x}{12}$$

The complete solution is $y_x = A(2)^x + B.(3)^x + \frac{6^x}{12}$.

2. $y_{k+2} - 6y_{k+1} + 8y_k = 4^k$.

$(E^2-6E+8)y_k = 4^k \Rightarrow$ A.E is $a^2-6a+8=0 \Rightarrow a=4,2$

$C.F = A \cdot 2^k + B \cdot 4^k$

$P.I = \frac{1}{(E-4)(E-2)} 4^k = \frac{1}{(4-2)} \left(\frac{1}{E-4} 4^k \right) = \frac{1}{2} \cdot k \cdot 4^{k-1}$

The complete solution is $y_k = A(2)^k + B \cdot (4)^k + \frac{k \cdot 4^{k-1}}{2}$.

3. $y_{n+2} - 3y_{n+1} + 2y_n = 5^n + 2^n$.

$(E^2 - 3E + 2) y_n = 5^n + 2^n$; $C.F = A \cdot (1)^n + B \cdot (2)^n = A + B(2)^n$

$P.I = \frac{1}{(E-1)(E-2)} 5^n + \frac{1}{(E-1)(E-2)} 2^n = \frac{1}{4 \cdot 3} \cdot 5^n + \frac{1}{(2-1)} \left(\frac{1}{E-2} 2^n \right)$

$= \frac{1}{12} 5^n + n \cdot 2^{n-1}$

Complete solution is $y_n = A + B(2)^n + \frac{5^n}{12} + n \cdot 2^{n-1}$.

4. $y_{x+2} - 8y_{x+1} + 16 y_x = 4^x$.

$(E^2 - 8E + 16) y_x = 4^x \Rightarrow C.F = (A + Bx)(4)^x$.

$P.I = \frac{1}{(E-4)^2} 4^x = \frac{x(x-1)}{2!} 4^{x-2}$

complete solution is $y_x = (A + Bx)(4)^x + \frac{x(x-1)}{2!} 4^{x-2}$.

5. $u_{x+2} - 5u_{x+1} + 6u_x = 36$

$(E^2 - 5E + 6) u_x = 36 = 36 \cdot (1)^x \Rightarrow C.F = A(2)^x + B(3)^x$.

$P.I = \frac{1}{(E-2)(E-3)} 36 (1)^x = 36 \cdot \frac{1}{(-1)(-2)} = 18$.

$u_x = A(2)^x + B \cdot (3)^x + 18$.

3

6. $u_{n+2} - 2u_{n+1} + 6u_n = 4$

$(E^2 - 2E + 6)u_n = 4(1)^n \Rightarrow$ A.E is $a^2 - 2a + 6 = 0 \Rightarrow a = 1 \pm i\sqrt{5}$.

C.F $= 6^{n/2}[A\cos n\theta + B\sin n\theta]$ where $\tan\theta = \sqrt{5}$, $r = \sqrt{1+5} = \sqrt{6}$

$P.I = \left(\frac{1}{E^2 - 2E + 6}\right) 4(1)^n = 4 \cdot \frac{1}{1 - 2 + 6} = \frac{4}{5}$.

Complete solution is, $y_n = 6^{n/2}[A\cos n\theta + B\sin n\theta] + \frac{4}{5}$.

7. $(E^3 - 5E^2 + 3E + 9)y_x = 3^x$.

A.E is $a^3 - 5a^2 + 3a + 9 = 0 \Rightarrow a = -1, 3, 3$.

C.F $= A(-1)^x + (B + Cx)(3)^x$.

$P.I = \frac{1}{(E-3)^2(E+1)}(3)^x = \frac{1}{3+1} \cdot \left[\frac{1}{(E-3)^2}(3)^x\right] = \frac{1}{4} \cdot \frac{x(x-1)3^{x-2}}{2!}$

$= \frac{x(x-1)3^{x-2}}{8}$

Complete solution is $y_x = A(-1)^x + (B + Cx)(3)^x + \frac{x(x-1)3^{x-2}}{8}$

Hw 1) $y_{x+2} - 6y_{x+1} + 9y_x = 8(3)^x \rightarrow (A + Bx)(3)^x + 4x(x-1)3^{x-2}$

2) $y_{n+2} + 10y_{n+1} + 20y_n = 2^n + 10 \rightarrow A(-5+\sqrt{5})^n + B(-5-\sqrt{5})^n + \frac{2^n}{44} + \frac{10}{31}$

3) $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0 \rightarrow \left(\frac{-1}{25} + \frac{1}{15}n\right)(-3)^n + \frac{1}{25}2^n$

4) $u_{n+3} - 6u_{n+2} + 11u_{n+1} - 6u_n = 3^n \rightarrow A + B(2)^n + C(3)^n + \frac{n3^{n-1}}{2}$

5) $6y_{n+2} + 5y_{n+1} - 6y_n = 2^n \rightarrow A\left(\frac{2}{3}\right)^n + B\left(\frac{-3}{2}\right)^n + \frac{2^n}{28}$

6) $y_{n+2} - \sqrt{2}y_{n+1} + y_n = 2^{3n/2} \rightarrow A\cos\frac{n\pi}{4} + B\sin\frac{n\pi}{4} + \frac{2^{3n/2}}{5}$

Type II: $f(x) = \sin px$ or $\cos px$

$$P.I = \frac{1}{\phi(E)} \sin px \,[\text{or } \cos px] = \frac{1}{\phi(E)} \left[I.P \,(\text{or } R.P) \text{ of } e^{ipx} \right]$$

$(\because e^{ipx} = \cos px + i \sin px)$

$$= I.P \,(\text{or } R.P) \text{ of } \frac{1}{\phi(E)} \left(e^{ip}\right)^x \text{ and proceed as in}$$

type I as e^{ip} is a constant.

Problems

1. Solve $y_{n+2} - 16 y_n = \cos \frac{1}{2} n$.

$(E^2 - 16) y_n = \cos \frac{n}{2} \Rightarrow$ A-E is $a^2 - 16 = 0 \Rightarrow a = \pm 4$.

$C.F = A(4)^n + B(-4)^n$.

$$P.I = \frac{1}{E^2 - 16} \cos\left(\frac{n}{2}\right) = R.P. \frac{1}{(E^2 - 16)} e^{\frac{in}{2}} = R.P. \left[\frac{1}{E^2 - 16} \left(e^{i/2}\right)^n \right]$$

$$= R.P. \left(\frac{1}{e^{i} - 16} \right) e^{in/2} = R.P \text{ of } \frac{(e^{-i} - 16)(e^{in/2})}{(e^{i} - 16)(e^{-i} - 16)}$$

$$= R.P \left[\frac{e^{i(n/2 - 1)} - 16 e^{in/2}}{1 - 16e^{i} - 16e^{-i} + 256} \right] = R.P \left[\frac{e^{i(\frac{n}{2} - 1)} - 16 e^{\frac{in}{2}}}{257 - 16(2\cos 1)} \right]$$

$$P.I = \frac{\cos\left(\frac{n}{2} - 1\right) - 16 \cos\left(\frac{n}{2}\right)}{257 - 32 \cos 1}$$

The complete solution is, $y_n = A(4)^n + B(-4)^n + \dfrac{\cos\left(\frac{n}{2} - 1\right) - 16 \cos\left(\frac{n}{2}\right)}{257 - 32 \cos 1}$

(5)

2. Solve $\Delta u_x + \Delta^2 u_x = \sin x$

This can be written as $(E-1)u_x + (E-1)^2 u_x = \sin x$

$\Rightarrow (E-1+E^2-2E+1)u_x = \sin x \Rightarrow (E^2-E)u_x = \sin x$

A.E is $a^2 - a = 0 \Rightarrow a = 0, 1$

$C.F = A(1)^x + B(0)^x \Rightarrow C.F = A$.

$$P.I = \frac{1}{E^2-E}\sin x = \frac{1}{E^2-E} \text{ I.P of } e^{ix} = \text{I.P of } \left[\frac{1}{E^2-E}(e^i)^x\right]$$

$$= \text{I.P}\left\{\frac{1 \cdot e^{ix}}{(e^{2i}-e^i)}\right\} = \text{I.P of}\left[\frac{(e^{-2i}-e^{-i})\, e^{ix}}{(e^{2i}-e^{i})(e^{-2i}-e^{-i})}\right]$$

$$= \text{I.P of}\left[\frac{e^{i(x-2)} - e^{i(x-1)}}{1-e^{i}-e^{-i}+1}\right] = \text{I.P of}\left(\frac{e^{i(x-2)}-e^{i(x-1)}}{2-2\cos 1}\right)$$

$$P.I = \frac{\sin(x-2)-\sin(x-1)}{2(1-\cos 1)}$$

Complete solution is $u_x = A + \dfrac{\sin(x-2)-\sin(x-1)}{2(1-\cos 1)}$

HW

1. $y_{n+2} - 2\cos\alpha\, y_{n+1} + y_n = \cos \alpha n \rightarrow A\cos n\alpha + B\sin n\alpha + \dfrac{n\sin(n-1)\alpha}{2\sin\alpha}$

2. $(E^2+1)y_x = \sin x \rightarrow A\cos\left(\frac{\pi x}{2}\right) + B\sin\left(\frac{\pi x}{2}\right) + \dfrac{\sin x + \sin(x-2)}{2(1+\cos 2)}$

Type III: $f(x) = x^p$ [Variable power constant]

$$P.I = \frac{1}{\phi(E)} x^p = \frac{1}{\phi(1+\Delta)} x^p = [\phi(1+\Delta)]^{-1} x^p$$

Expand $[\phi(1+\Delta)]^{-1}$ by using Binomial expansion and then operate each term on x^p.

Problems:

1. Solve $y_{x+2} - 4y_x = 9x^2$.

This equation becomes $(E^2-4) y_x = 9x^2$.

A.E is $a^2-4=0 \Rightarrow a = \pm 2$

$C.F = A(-2)^x + B(2)^x$.

$$P.I = \frac{1}{E^2-4}(9x^2) = 9\left[\frac{1}{(1+\Delta)^2-4}\right](x^2) = 9\left[\frac{1}{\Delta^2+2\Delta-3}(x^2)\right]$$

$$= 9\left[\frac{1}{-3\left(1+\frac{\Delta^2+2\Delta}{-3}\right)}(x^2)\right] = -3\left[1-\left(\frac{\Delta^2+2\Delta}{3}\right)\right]^{-1}(x^2)$$

$$= (-3)\left\{1+\left(\frac{\Delta^2+2\Delta}{3}\right)+\left(\frac{\Delta^2+2\Delta}{3}\right)^2+\cdots\right\}(x^2)$$

$$= (-3)\left\{1+\frac{2\Delta}{3}+\frac{\Delta^2}{3}+\frac{4\Delta^2}{9}\right\}(x^2) \quad \text{(Neglecting } \Delta^3 \text{ \& higher powers of } \Delta)$$

$$= (-3)\left\{1+\frac{2\Delta}{3}+\frac{7\Delta^2}{9}\right\}(x^2) = (-3)\left\{x^2+\frac{2}{3}\Delta(x^2)+\frac{7}{9}\Delta^2(x^2)\right\}$$

$[\Delta(x^2) = (x+1)^2 - x^2 = 2x+1;\quad \Delta^2(x^2) = 1.2!\,1^2$ using $\Delta^n f(x) = a_0 n!\, h^n]$

$$P.I = (-3)\left\{x^2+\frac{2}{3}(2x+1)+\frac{7}{9}(2)\right\} = (-3)\left\{x^2+\frac{4x}{3}+\frac{2}{3}+\frac{14}{9}\right\}$$

$$= (-3)\left\{x^2+\frac{4x}{3}+\frac{20}{9}\right\}$$

Complete solution is $y_x = A(-2)^x + B(2)^x - 3\left(x^2+\frac{4x}{3}+\frac{20}{9}\right)$

(7).

2. Solve $y_{n+2} - 4y_n = n^2+n-1$

$(E^2-4)\,y_n = 0 \Rightarrow$ A.E is $a^2-4=0 \Rightarrow a = \pm 2$

$C.F = A(-2)^n + B(2)^n$

$$P.I = \frac{1}{E^2-4}(n^2+n-1) = \frac{1}{(1+\Delta)^2-4}(n^2+n-1) = \frac{1}{\Delta^2+2\Delta-3}(n^2+n-1)$$

$$= \frac{-1}{3}\left[1-\left(\frac{\Delta^2+2\Delta}{3}\right)\right]^{-1}(n^2+n-1) = \frac{-1}{3}\left\{1+\frac{2\Delta}{3}+\frac{7\Delta^2}{9}\right\}(n^2+n-1)$$

$\Delta^2(n^2+n-1) = 1.2!(1)^2 = 2$; $\Delta(n^2+n-1) = \left[(n+1)^2+(n+1)-1\right] - \left[n^2+n-1\right]$

$= \cancel{n^2}+2n+1+\cancel{n}+\cancel{1}-\cancel{1}-\cancel{n^2}-\cancel{n}+1 = 2n+2$

$$P.I = \frac{-1}{3}\left\{(n^2+n-1)+\frac{2}{3}(2n+2)+\frac{7}{9}(2)\right\} = \frac{-1}{3}\left\{n^2+n-1+\frac{4n}{3}+\frac{4}{3}+\frac{14}{9}\right\}$$

$$= \frac{-1}{3}\left\{n^2+\frac{7n}{3}+\frac{17}{9}\right\}$$

Complete solution is $y_n = A(-2)^n + B(2)^n - \frac{1}{3}\left\{n^2+\frac{7n}{3}+\frac{17}{9}\right\}$

3. $(\Delta^2+\Delta+1)\,y_x = x^2$.

$\left[(E-1)^2+E-1+1\right]y_x = x^2 \Rightarrow (E^2-E+1)\,y_x = x^2$

A.E is $a^2-a+1=0 \Rightarrow a = \frac{1\pm i\sqrt{3}}{2}$

$\alpha = \frac{1}{2},\ \beta = \frac{\sqrt{3}}{2} \Rightarrow r = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1,\quad \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

$C.F = (1)^x\left\{A\cos\frac{\pi x}{3} + B\sin\frac{\pi x}{3}\right\}$.

$$P.I = \left(\frac{1}{E^2-E+1}\right)x^2 = \frac{1}{1+\Delta+\Delta^2}(x^2) = \left[1+(\Delta+\Delta^2)\right]^{-1}(x^2)$$

$\left[1-(\Delta+\Delta^2)+(\Delta+\Delta^2)^2-\cdots\right]x^2 = \left[1-\Delta-\cancel{\Delta^2}+\Delta^2\right]x^2$ (neglecting Δ^3 & higher powers of x)

$\Rightarrow \quad x^2-\Delta(x^2) = x^2-(2x+1) = x^2-2x-1$

Complete solution is $y_x = A\cos\left(\frac{\pi x}{3}\right)+B\sin\left(\frac{\pi x}{3}\right)+x^2-2x-1$

<u>Hw</u>

1. Solve $y_{x+2}-5y_{x+1}+6y_x = x^2+x+1 \longrightarrow A(2)^x+B(3)^x+\frac{1}{2}\left(x^2+4x+\frac{15}{2}\right)$

2. Solve $u_{n+2}-2u_{n+1}+u_n = 3n+4 \longrightarrow An+B+\frac{n(n-1)(n+2)}{2}$

3. Solve $y_{x+2}-4y_x = x^2-1 \rightarrow y_x = A(2)^x+B(-2)^x-\frac{1}{27}\left(9x^2+12x+11\right)$

Set III ①

Type IV: $f(x) = a^x F(x)$ where $F(x)$ is some function of 'x'

$$P.I = \frac{1}{\phi(E)} a^x F(x) = a^x \frac{1}{\phi(aE)} F(x)$$ and then

proceed as in Type II & Type III.

Problems:

~~1. Solve $u_{n+2} - 7u_{n+1}$~~

1. Solve $y_{n+2} + y_{n+1} - 56y_n = 2^n(n^2-3)$

$$(E^2+E-56)\, y_n = 2^n(n^2-3) \Rightarrow \text{A.E is } a^2+a-56=0 \Rightarrow a = -8, 7$$

$$C.F = A\cdot(-8)^n + B\cdot(7)^n.$$

$$P.I = \frac{1}{E^2+E-56} 2^n(n^2-3) = 2^n\left(\frac{1}{4E^2+2E-56}\right)(n^2-3)$$

$$= 2^n\left[\frac{1}{4(1+\Delta)^2+2(1+\Delta)-56}\right](n^2-3) = 2^n\left[\frac{1}{4\Delta^2+10\Delta-50}\right](n^2-3)$$

$$= \frac{-2^n}{50}\left\{1-\left(\frac{4\Delta^2+10\Delta}{50}\right)\right\}^{-1}(n^2-3) = \frac{-2^n}{50}\left\{1-\left(\frac{2\Delta^2}{25}+\frac{\Delta}{5}\right)\right\}^{-1}(n^2-3)$$

$$= \frac{-2^n}{50}\left\{1+\frac{2\Delta^2}{25}+\frac{\Delta}{5}+\frac{\Delta^2}{25}\right\}(n^2-3) \quad (\text{Neglecting } \Delta^3 \text{ \& higher powers of } \Delta)$$

$$= \frac{-2^n}{50}\left\{1+\frac{\Delta}{5}+\frac{3\Delta^2}{25}\right\}(n^2-3) = \frac{-2^n}{50}\left\{(n^2-3)+\frac{1}{5}\Delta(n^2-3)+\frac{3}{25}\Delta^2(n^2-3)\right\}$$

$$= \frac{-2^n}{50}\left\{(n^2-3)+\frac{1}{5}(2n+1)+\frac{3}{25}(2)\right\}$$

$$= \frac{-2^n}{50}\left\{n^2+\frac{2n}{5}-3+\frac{1}{5}+\frac{6}{25}\right\} = \frac{-2^n}{50}\left\{n^2+\frac{2n}{5}-\frac{64}{25}\right\}$$

Complete solution is $y_n = A(-8)^n + B(7)^n - \frac{2^n}{50}\left\{n^2+\frac{2n}{5}-\frac{64}{25}\right\}$

2. Solve $u_{x+2} - 4u_{x+1} + 4u_x = 2^x \sin x$

$(E^2 - 4E + 4)u_x = 2^x \sin x$; A.E is $a^2 - 4a + 4 = 0$

$\Rightarrow a = 2, 2$

$C.F = (A + Bx)(2)^x$.

$$P.I = \left(\frac{1}{E^2 - 4E + 4}\right)(2^x \sin x) = 2^x \cdot \left(\frac{1}{4E^2 - 8E + 4}\right) \sin x$$

$$= \frac{2^x}{4}\left(\frac{1}{E^2 - 2E + 1}\right)\sin x = \frac{2^x}{4}\left[\frac{1}{E^2 - 2E + 1} \text{ I.P of } e^{ix}\right]$$

$$= \frac{2^x}{4} \cdot \text{I.P of } \left(\frac{1}{E^2 - 2E + 1}\right)(e^i)^x = \frac{2^x}{4}\left(\frac{e^{ix}}{e^{2i} - 2e^i + 1}\right)$$

$$= \frac{2^x}{4} \text{I.P}\left\{\frac{(e^{ix})(e^{-2i} - 2e^{-i} + 1)}{(e^{2i} - 2e^i + 1)(e^{-2i} - 2e^{-i} + 1)}\right\} = \frac{2^x}{4} \text{ I.P of } \left\{\frac{e^{i(x-2)} - 2e^{i(x-1)} + e^{ix}}{DR}\right\}$$

$$DR = 1 - 2e^i + e^{2i} - 2e^{-i} + 4 - 2e^i + e^{-2i} - 2e^{-i} + 1$$

$$= 6 - 4e^i - 4e^{-i} + 2\cos 2 = 6 - 4(2\cos 1) + 2\cos 2$$

$$DR = 6 - 8\cos 1 + 2\cos 2$$

$$P.I = \frac{2^x}{8}\left[\frac{\sin(x-2) - 2\sin(x-1) + \sin x}{\cos 2 - 4\cos 1 + 3}\right]$$

Complete solution is, $u_x = (A + Bx)(2)^x + \dfrac{2^x}{8}\left\{\dfrac{\sin(x-2) - 2\sin(x-1) + \sin x}{\cos 2 - 4\cos 1 + 3}\right\}$

(3)

HW

1. $u_{n+2} - 7u_{n+1} - 8u_n = 2^n n^2 \rightarrow A(-1)^n + B(8)^n - \frac{2^n}{18}\left(n^2 - \frac{2n}{3} + \frac{1}{3}\right)$

2. $u_{n+1} - 3u_n = n^2 2^n \rightarrow A\cdot(3)^n - 2^n\left(n^2 + 4n + 10\right)$

3. $u_{x+2} - 4u_{x+1} + 4u_x = 2^x x^2 \rightarrow (A+Bx)(2)^x + \frac{2^x x(x-1)^2(x-2)}{48}$

[NOTE: The above problem can be done only by integration using Factorial polynomial concept which is not included in the current syllabus].

FORMATION OF DIFFERENCE EQUATION.

1. Form the difference equation by eliminating the arbitrary constants from $y = a\cdot 2^x + b\cdot 3^x$.

$\boxed{\text{Method I}}$

Let $y_x = a\cdot 2^x + b\cdot 3^x$ —— (1)

Then, $y_{x+1} = a\cdot 2^{x+1} + b\cdot 3^{x+1}$ —— (2)

$y_{x+2} = a\cdot 2^{x+2} + b\cdot 3^{x+2}$ —— (3).

$(1)\times 2 \Rightarrow \quad 2y_x = a\cdot 2^{x+1} + 2b\cdot 3^x$

$(2) \Rightarrow \quad y_{x+1} = a\cdot 2^{x+1} + b\cdot 3^{x+1}$

$\quad (-) \quad (-) \quad (-)$

$2y_x - y_{x+1} = b\cdot 3^x(2-3) = -b\cdot 3^x$

$\Rightarrow 2y_x - y_{x+1} = -b\cdot 3^x$ —— (4).

$\therefore 2y_{x+1} - y_{x+2} = -6\cdot 3^{x+1}$ — (5)

$(5)/(4) \Rightarrow \dfrac{2y_{x+1} - y_{x+2}}{2y_x - y_{x+1}} = \dfrac{-6\cdot 3^{x+1}}{-6\cdot 3^x} = 3.$

$\Rightarrow 2y_{x+1} - y_{x+2} = 6y_x - 3y_{x+1}$

$\Rightarrow \boxed{y_{x+2} - 5y_{x+1} + 6y_x = 0}$

Method II

Eliminating $a\cdot 2^x$, $b\cdot 3^x$ from (1), (2) & (3) using determinants, we have

$$\begin{vmatrix} y_x & 1 & 1 \\ y_{x+1} & 2 & 3 \\ y_{x+2} & 4 & 9 \end{vmatrix} = 0.$$ Expanding along the 1st column, $y_x(18-12) - y_{x+1}(9-4) + y_{x+2}(3-2) = 0$

$\Rightarrow \boxed{y_{x+2} - 5y_{x+2} + 6y_x = 0}$

[Note: Normally determinant is expanded using 1st row. Here we used 1st column to avoid involvement of y_x, y_{x+1}, y_{x+2} more than once]

2. Eliminate the arbitrary constants in $y_x = (A+Bx)2^x$ and form the difference equation.

$y_x = (A+Bx)2^x = A\cdot 2^x + B\cdot x\, 2^x$ — (1)

$y_{x+1} = A\cdot 2^{x+1} + B(x+1)2^{x+1}$ — (2)

$y_{x+2} = A\cdot 2^{x+2} + B(x+2)2^{x+2}$ — (3)

(5)

Eliminating $A\cdot 2^x$, $B\cdot 2^x$ from (1), (2) & (3) using determinants, we have,

$$\begin{vmatrix} y_x & 1 & x \\ y_{x+1} & 2 & 2(x+1) \\ y_{x+2} & 4 & 4(x+2) \end{vmatrix} = 0.$$ Expanding along 1st column we have, $y_x\left[8(x+2)-8(x+1)\right] - y_{x+1}\left[4(x+2)-4x\right]$

$$+ y_{x+2}\left[2(x+1)-2x\right] = 0.$$

$$\Rightarrow \quad 8y_x - 8y_{x+1} + 2y_{x+2} = 0 \Rightarrow \boxed{y_{x+2} - 4y_{x+1} + 4y_x = 0}$$

3. Find the difference equation satisfied by $y = ax^2 - bx$.

Let $y_x = ax^2 - bx$ — (1). Then

$$y_{x+1} = a(x+1)^2 - b(x+1) \quad \text{— (2)}$$

$$y_{x+2} = a(x+2)^2 - b(x+2) \quad \text{— (3)}.$$

Eliminating $a, -b$ from (1), (2) & (3) using determinants, we have,

$$\begin{vmatrix} y_x & x^2 & x \\ y_{x+1} & (x+1)^2 & x+1 \\ y_{x+2} & (x+2)^2 & x+2 \end{vmatrix} = 0.$$ Expanding along 1st column, we have,

$$y_x\left\{(x+1)^2(x+2) - (x+1)(x+2)^2\right\} - y_{x+1}\left\{x^2(x+2) - x(x+2)^2\right\} + y_{x+2}\left\{x^2(x+1) - x(x+1)^2\right\} = 0$$

$$\Rightarrow y_x\left\{(x+1)(x+2)\left[\cancel{x}+1-\cancel{x}-2\right]\right\} - y_{x+1}\left\{x(x+2)\left[\cancel{x}-\cancel{x}-2\right]\right\}$$

$$+ y_{x+2}\left\{x(x+1)\left[\cancel{x}-\cancel{x}-1\right]\right\} = 0.$$

$$-(x^2+3x+2)y_x + 2x(x+2)y_{x+1} - x(x+1)y_{x+2} = 0.$$

$$\Rightarrow \boxed{x(x+1)y_{x+2} - 2x(x+2)y_{x+1} + (x^2+3x+2)y_x = 0}.$$

Hw

Form the difference equation by eliminating the arbitrary constants from the following.

1) $y = A \cdot 4^x + B \cdot 5^x \longrightarrow y_{x+2} - 9y_{x+1} + 20y_x = 0$.

2) $y_n = A\cos n\alpha + B\sin n\alpha \rightarrow y_{n+2} - 2y_{n+1}\cos\alpha + y_n = 0$

3) $y = ax^2 + bx - 3 \longrightarrow (x^2+x)y_{x+2} - 2(x^2+2x)y_{x+1} + (x^2+3x+2)y_x + 6 = 0$

4) $y = A \cdot 2^n + B \rightarrow y_{n+2} - 3y_{n+1} + 2y_n = 0$.