UNIT - II

Response Analysis of Control Systems

Time response of a control system means how a system behaves with time when a specified i/p signal is applied. It consist of

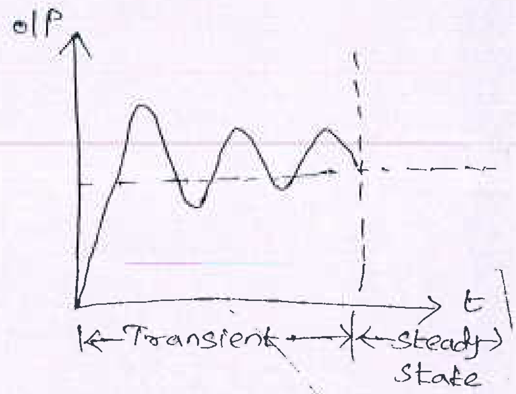
(i) Transient response

Transient response means which goes from initial state to the final state.

(ii) Steady state response

The Steady state response is the time response as the time approached infinity.

i.e) The response of the system at $t \to \infty$ is called stead state response.



Standard Test signals

(i) step signal

(ii) Ramp signal

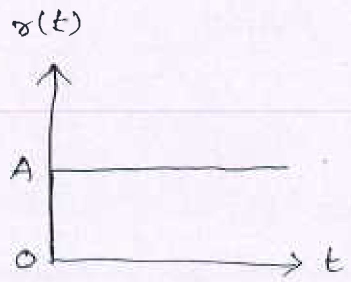
(iii) Parabolic signal

(iv) Impulse signal

(v) Sinusoidal signal

step signal

The step signal is a signal whose value changes from zero to A at $t=0$ and remains constant at A for $t>0$

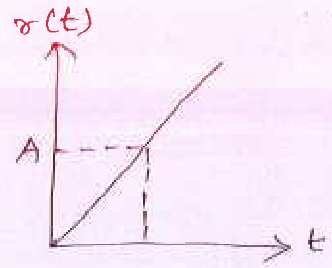


$$r(t) = A\,u(t)$$

$u(t)=1 \;;\; t \ge 0 \qquad u(t)=0 \;;\; t<0$

Ramp signal

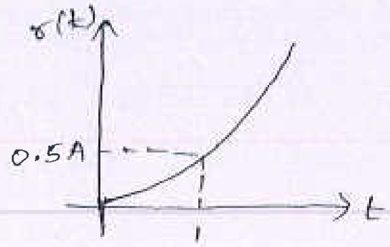
Ramp signal is a signal whose value increases linearly with time from an initial value of zero at $t=0$



$$r(t) = At \quad ; \quad t \geq 0$$
$$= 0 \quad ; \quad t < 0$$

Parabolic signal

The instantaneous value varies as square of time from an initial value of zero at $t=0$.

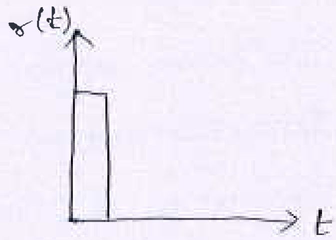


$$r(t) = \frac{At^2}{2} \quad ; \quad t \geq 0$$
$$= 0 \quad ; \quad t < 0$$

Impulse signal

A signal which is available for very short duration is called impulse signal.

$$\delta(t) = 0 \quad \text{for } t \neq 0$$



Name of the signal	Time domain eqn of signal $r(t)$	Laplace Transform of the signal $R(s)$
Step	A	A/s
unit step	1	$1/s$
Ramp	At	A/s^2
unit ramp	t	$1/s^2$
Parabolic	$At^2/2$	A/s^3
unit Parabolic	$t^2/2$	$1/s^3$
Impulse	$\delta(t)$	1

Laplace transform

$$e^{-at} = \frac{1}{s+a} \qquad \cos \omega t = \frac{s}{s^2+\omega^2}$$

$$e^{at} = \frac{1}{s-a} \qquad \sinh \omega t = \frac{\omega}{s^2-\omega^2}$$

$$te^{-at} = \frac{1}{(s+a)^2} \qquad \cosh \omega t = \frac{s}{s^2-\omega^2}$$

$$\sin \omega t = \frac{\omega}{s^2+\omega^2}$$

Order of a system

The order of the system is given by the order of the differential equation governing the system.

$$T(s) = K\frac{P(s)}{Q(s)}$$

where K – Constant

$P(s)$ – Numerator Polynomial

$Q(s)$ – Denominator Polynomial

The order of the system is given by the maximum Power of s in the denominator Polynomial, $Q(s)$

Type of a system

The type number is given by no of Poles of loop transfer function at the origin.

1. For the system with following T.F, determine type & order of the system.

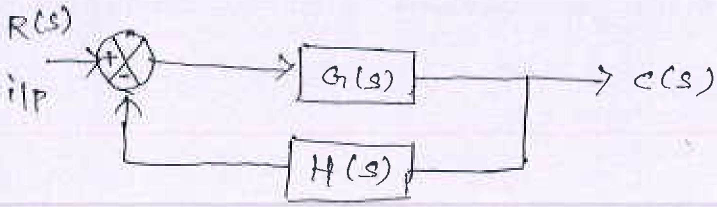
(i) $G(s)H(s) = \dfrac{K}{s(s+1)(s^2+6s+8)}$ (ii) $G(s)H(s) = \dfrac{20(s+2)}{s^2(s+3)(s+0.5)}$

(iii) $G(s)H(s) = \dfrac{s+4}{(s-2)(s+0.25)}$ (iv) $G(s)H(s) = \dfrac{10}{s^3(s^2+2s+1)}$

(i) Type – 1 , order – 4
(ii) Type – 2 , order – 4
(iii) Type – 0 , order – 2
(iv) Type – 3 , order – 5

Time response

The time response is the o/p of the closed loop system as a function of time. It is denoted by c(t). It is given by inverse laplace of the product of input and transfer function of the system.



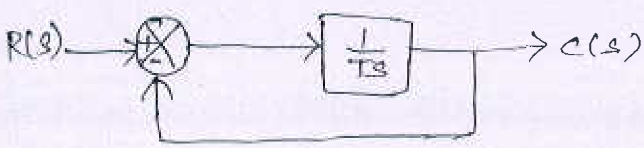
$$T.F \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$C(s) = R(s)\frac{G(s)}{1+G(s)H(s)}$$

Response in time domain $c(t) = L^{-1}[C(s)]$

$$= L^{-1}\left[R(s)\frac{G(s)}{1+G(s)H(s)}\right]$$

Response of first order system for unit step input



$$T.F \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{1/sT}{1+\left(\frac{1}{sT}\times 1\right)}$$

$$= \frac{\frac{1}{sT}}{\frac{sT+1}{sT} \times 1} = \frac{1}{sT+1}$$

$$\frac{C(s)}{R(s)} = \frac{1}{sT+1} \qquad C(s) = R(s) \times \frac{1}{Ts+1}$$

If the i/p is unit step then $r(t) = 1$ and $R(s) = \frac{1}{s}$

The response in s domain, $C(s) = R(s) \frac{1}{1+Ts}$

$$C(s) = \frac{1}{s} \times \frac{1}{sT+1} = \frac{1}{s(sT+1)}$$

Take Partial Fraction

$$= \frac{K_1}{s} + \frac{K_2}{sT+1}$$

$$K_1 = \left(\frac{1}{sT+1}\right)_{s=0} = 1$$

$$K_2 = \left(\frac{1}{s}\right)_{s=-1/T} = -T$$

$sT+1 = 0$
$sT = -1$
$s = -1/T$

$$\therefore C(s) = \frac{1}{s} - \frac{T}{sT+1}$$

$$= \frac{1}{s} - \frac{T}{T(s+1/T)}$$

$$= \frac{1}{s} - \frac{1}{s+1/T}$$

The response in time domain is given by

$$c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s} - \frac{1}{s+1/T}\right]$$

$$= 1 - e^{-t/T}$$

when $t = 0$, $c(t) = 1 - e^{0} = 0$

$t = 1T$, $c(t) = 1 - e^{-1} = 0.632$

$t = 2T$, $c(t) = 1 - e^{-2} = 0.865$

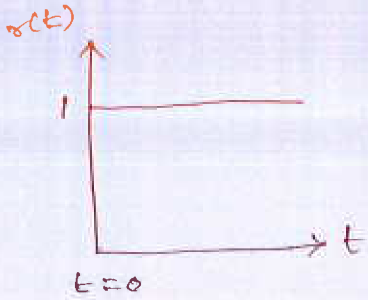
$t = 3T$, $c(t) = 1 - e^{-3} = 0.95$

$t = 4T$, $c(t) = 1 - e^{-4} = 0.9817$

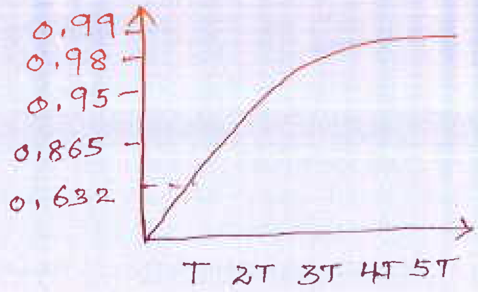
$t = 5T$, $c(t) = 1 - e^{-5} = 0.993$

$t = \infty$, $c(t) = 1 - e^{-\infty} = 1$

The i/p & o/p of the System



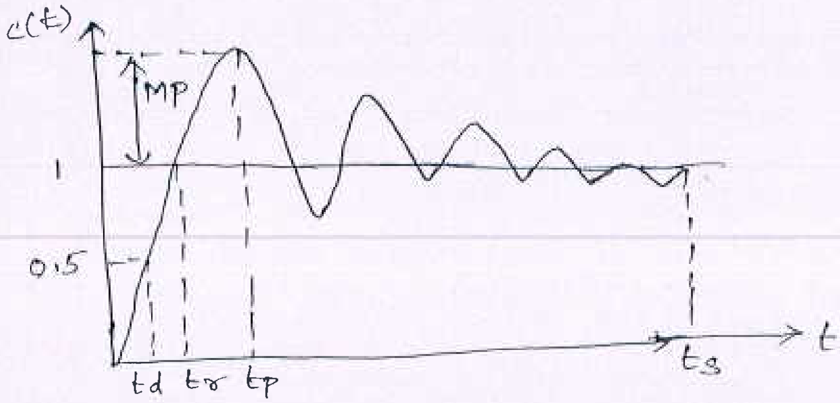
unit step i/p



Response of first order System to unit step i/p

Time domain specifications

Damped oscillatory response of second order system



(i) Delay time (td)

It is the time taken for response to reach 50% of the final value, for the very first time.

$$t_d = \frac{1 + 0.7\varepsilon}{\omega_n}$$

(ii) Rise time (tr)

It is the time taken for response to raise from 0 to 100% for the very first time.

$$t_r = \frac{\pi - \theta}{\omega_d}$$

where $\theta = \tan^{-1}\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}$

damped frequency oscillation $\omega_d = \omega_n\sqrt{1-\varepsilon^2}$

(iii) Peak time (tp)

It is the time taken for the response to reach the peak value for the very first time.

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\omega_d} \qquad \omega_d = \omega_n\sqrt{1-\xi^2}$$

Peak overshoot (Mp)

It is defined as the ratio of the max peak value measured from final value to final value.

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

$$= e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$$

Settling time

It is defined as the time taken by the response to reach and stay with in a specified error (specified as % of final value)

$$t_s = \frac{4}{\xi\omega_n} = 4T \text{ (for 2\% error settling time)}$$

$$t_s = 3T \text{ (for 5\% error)} \quad \left(\because T = \frac{1}{\xi\omega_n}\right)$$

Damping ratio (ξ)

The damping ratio is defined as the ratio of the actual damping to the critical damping.

Second order system

The standard form of closed loop T.F of second order system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where ω_n - undamped natural frequency rad/sec

ξ - damping ratio

Depending on the value of ξ, the system can be classified into

Case (i) : Undamped system $\xi = 0$

Case (ii) : Under damped system $0 < \xi < 1$

case (iii) : Critically damped system $\xi = 1$

case (iv) : overdamped system $\xi > 1$

The characteristic eqn of the second order system is

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

It is a quadratic eqn and the roots are

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$A^2 + \boxed{B}S + C$

$$= \frac{-2\xi\omega_n \pm \sqrt{4\omega_n^2(\xi^2 - 1)}}{2}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$

1. Response of undamped second order system for unit step input

The standard form of closed loop T.f of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For undamped system $\xi = 0$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

when the i/p is unit step, $r(t) = 1$ and $R(s) = \frac{1}{s}$

$\therefore$ the response is s-domain

$$C(s) = R(s)\frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{1}{s}\frac{\omega_n^2}{s^2 + \omega_n^2}$$

By Partial fraction

$$C(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

A is obtained by multiplying C(s) by s & put s = 0

$$A = C(s)\,s = \frac{\omega_n^2 \cancel{s}}{\cancel{s}(s^2 + \omega_n^2)} = \left.\frac{\omega_n^2}{s^2 + \omega_n^2}\right|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$\boxed{A = 1}$

Multiply ~~to~~ $c(s)$ by $s^2+\omega_n^2$ for B & Put $s^2=-\omega_n^2$ or $s=j\omega_n$

$$B = c(s)\, s^2+\omega_n^2 = \frac{\omega_n^2}{s(\cancel{s^2+\omega_n^2})} \times \cancel{(s^2+\omega_n^2)} = \frac{\omega_n^2}{s}\Big|_{s=j\omega_n}$$

$$= \frac{\omega_n^{\cancel{2}}}{j\cancel{\omega_n}} = \frac{\omega_n}{j} = -j\omega_n = -s \qquad \boxed{B=-s}$$

$s^2 = -\omega_n^2$

$s^2 = j^2\omega_n^2$

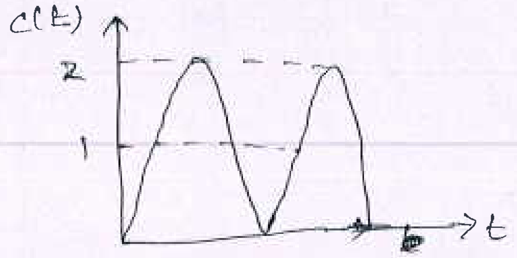
$s = j\omega_n$

$$\therefore\ C(s) = \frac{A}{s} + \frac{B}{s^2+\omega_n^2} = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$$

The time domain response $c(t) = L^{-1}[c(s)]$

$$= L^{-1}\left[\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right] = 1-\cos\omega_n t$$

Response of undamped second order system for unit step i/p



The response of undamped second order system for unit step i/p is completely oscillatory.

2. Response of second order system for critically damped case and when i/p is unit step

The standard form of closed loop transfer function of second order system is

$$\frac{c(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

For critical damping $\xi=1$

$$\therefore\ \frac{c(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

when i/p is unit step $r(t)=1$ & $R(s)=1/s$

$\therefore$ The response is s domain

$$c(s) = R(s)\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s}\,\frac{\omega_n^2}{(s+\omega_n)^2}$$

$$= \frac{\omega_n^2}{s(s+\omega_n)^2}$$

By Partial fraction

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$A = s\, C(s)\Big|_{s=0} = \frac{\omega_n^2}{\cancel{s}(s+\omega_n)^2} \times \cancel{s} = \frac{\omega_n^2}{(s+\omega_n)^2}\Big|_{s=0}$$

$$\boxed{A=1} \qquad = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+\omega_n)^2 C(s)\Big|_{s=-\omega_n} = \frac{\omega_n^2}{s(s+\omega_n)^2}(s+\omega_n)^2\Big|_{s=-\omega_n}$$

$$= \frac{\omega_n^2}{s}\Big|_{s=-\omega_n} = \frac{\omega_n^2}{-\omega_n} = -\omega_n$$

$$\boxed{B = -\omega_n}$$

$$C = \frac{d}{ds}\left[(s+\omega_n)^2 C(s)\right]\Big|_{s=-\omega_n}$$

$$= \frac{d}{ds}\left[\frac{\omega_n^2}{\cancel{(s+\omega_n)^2}\, s}\cancel{(s+\omega_n)^2}\right]\Big|_{s=-\omega_n}$$

$$= \frac{d}{ds}\left(\frac{\omega_n^2}{s}\right)\Big|_{s=-\omega_n}$$

$$= \frac{-\omega_n^2}{s^2}\Big|_{s=-\omega_n}$$

$$= \frac{-\omega_n^2}{(-\omega_n)^2} = -1$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

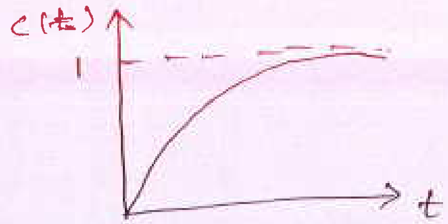
The response in time domain

$$c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right]$$

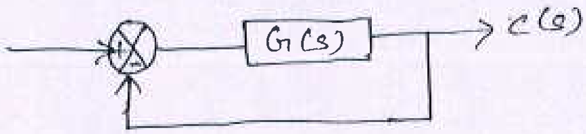
$$c(t) = 1 - \omega_n t e^{-\omega_n t} - e^{-\omega_n t}$$

$$c(t) = 1 - e^{-\omega_n t}\left[1 + \omega_n t\right]$$

The response of critically damped second order system has no oscillations.



1. Obtain the response of unity feedback system whose open loop T.F is $G(s) = \frac{4}{s(s+5)}$ and when the i/p is unit step.



The closed loop T.F

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s).1}$$

$$\frac{C(s)}{R(s)} = \frac{4/s(s+5)}{1+\frac{4}{s(s+5)}} = \frac{4/\cancel{s(s+5)}}{\frac{s(s+5)+4}{\cancel{s(s+5)}}} = \frac{4}{s(s+5)+4}$$

$$= \frac{4}{s^2+5s+4} = \frac{4}{(s+4)(s+1)}$$

The response in s-domain

$$c(s) = R(s)\frac{4}{(s+4)(s+1)}$$

Since the i/p is unit step, $r(t) = 1$ and $R(s) = \frac{1}{s}$

$$\therefore c(s) = \frac{4}{s(s+4)(s+1)}$$

By Partial Fraction

$$C(s) = \frac{4}{s(s+4)(s+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = c(s)\, s\Big|_{s=0} = \frac{4}{(s+4)(s+1)}\Big|_{s=0} = \frac{4}{4} = 1 \qquad \boxed{A=1}$$

$$B = C(s)(s+1)\Big|_{s=-1} = \frac{4}{s(s+4)}\Big|_{s=-1} = \frac{4}{-1(-1+4)} = -4/3$$

$$\boxed{B = -4/3}$$

$$C = C(s)(s+4)\Big|_{s=-4} = \frac{4}{s(s+1)} = \frac{4}{-4(-4+1)} = \frac{4}{12} = \frac{1}{3}$$

$$\boxed{C = 1/3}$$

$$C(s) = \frac{1}{s} - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s+4}$$

The response in time domain

$$c(t) = L^{-1}[C(s)]$$

$$= L^{-1}\left[\frac{1}{s} - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s+4}\right]$$

$$= 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t}$$

$$c(t) = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$$

3. Response of second order system for overdamped case and when i/p is unit step.

The standard form of closed loop transfer fn of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For overdamped system $\xi > 1$

$$s_a, s_b = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$= -\left[\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}\right]$$

Let $s_1 = -s_a$ & $s_2 = -s_b$

$$\therefore \quad s_1 = \xi\omega_n - \omega_n\sqrt{\xi^2-1}$$

$$s_2 = \xi\omega_n + \omega_n\sqrt{\xi^2-1}$$

The closed loop T.F can be written in terms of s_1 & s_2

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+s_1)(s+s_2)}$$

For unit step i/p $r(t) = 1$ & $R(s) = 1/s$

$$\therefore C(s) = R(s)\,\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

By Partial Fraction

$$C(s) = \frac{\omega_n^2}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{(s+s_1)} + \frac{C}{(s+s_2)}$$

$$A = C(s)\,s\Big|_{s=0} = \frac{\omega_n^2\,\cancel{s}}{\cancel{s}(s+s_1)(s+s_2)}\Big|_{s=0} = \frac{\omega_n^2}{s_1 s_2}$$

$$= \frac{\omega_n^2}{\left[\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right]\left[\xi\omega_n + \omega_n\sqrt{\xi^2-1}\right]}$$

$$= \frac{\omega_n^2}{\xi^2\omega_n^2 - \omega_n^2(\xi^2-1)} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+s_1)\,C(s)\Big|_{s=-s_1} = \frac{\omega_n^2}{s(s+s_2)}\Big|_{s=-s_1} = \frac{\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{s_1\left[-\xi\omega_n + \omega_n\sqrt{\xi^2-1} + \xi\omega_n + \omega_n\sqrt{\xi^2-1}\right]}$$

$$= \frac{-\omega_n^2}{s_1\left(2\omega_n\sqrt{\xi^2-1}\right)} = \frac{-\omega_n}{2\sqrt{\xi^2-1}}\,\frac{1}{s_1}$$

$$C = C(s)(s+s_2)\Big|_{s=-s_2} = \frac{\omega_n^2}{s(s+s_1)}\Big|_{s=-s_2}$$

$$= \frac{\omega_n^2}{-s_2(-s_2+s_1)} = \frac{\omega_n^2}{-s_2\left(-\xi\omega_n - \omega_n\sqrt{\xi^2-1} + \xi\omega_n - \omega_n\sqrt{\xi^2-1}\right)}$$

$$= \frac{\omega_n^2}{s_2\left(2\omega_n\sqrt{\xi^2-1}\right)} = \frac{\omega_n}{2\sqrt{\xi^2-1}}\,\frac{1}{s_2}$$

$$\therefore\ C(s) = \frac{1}{s} - \frac{\omega_n}{2\sqrt{\varepsilon^2-1}}\frac{1}{s_1}\frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\varepsilon^2-1}}\frac{1}{s_2}\frac{1}{(s+s_2)}$$

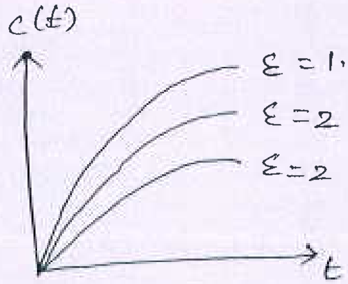
On taking inverse L.T

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\varepsilon^2-1}}\frac{1}{s_1}e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\varepsilon^2-1}}\frac{1}{s_2}e^{-s_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\varepsilon^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right)$$

where $s_1 = \varepsilon\omega_n - \omega_n\sqrt{\varepsilon^2-1}$

$s_2 = \varepsilon\omega_n + \omega_n\sqrt{\varepsilon^2-1}$



4. Response of second order system for underdamped case and when the i/p is unit step

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\varepsilon\omega_n s + \omega_n^2}$$

For underdamped system, $0 < \varepsilon < 1$

The roots of the denominators are

$$s = -\varepsilon\omega_n \pm \omega_n\sqrt{\varepsilon^2-1}$$

Since $\varepsilon < 1$, $\varepsilon^2 < 1$

$$\therefore\ s = -\varepsilon\omega_n \pm \omega_n\sqrt{(-1)(1-\varepsilon^2)}$$

$$= -\varepsilon\omega_n \pm j\omega_n\sqrt{1-\varepsilon^2}$$

The response is s domain

$$C(s) = R(s)\,\frac{\omega_n^2}{s^2+2\varepsilon\omega_n s+\omega_n^2}$$

$$\therefore\ C(s) = \frac{\omega_n^2}{s(s^2+2\varepsilon\omega_n s+\omega_n^2)}$$

By Partial Fraction

$$C(s) = \frac{A}{s} + \frac{Bs+C}{s^2+2\varepsilon\omega_n s+\omega_n^2}$$

$A = s\,C(s)\big|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$ $\boxed{A=1}$

On cross multiplication and equating the coefficients of like power of s we get,

$$\frac{\omega_n^2}{s(s^2+2\varepsilon\omega_n s+\omega_n^2)} = \frac{1}{s} + \frac{Bs+C}{s^2+2\varepsilon\omega_n s+\omega_n^2}$$

$$\omega_n^2 = s^2+2\varepsilon\omega_n s+\omega_n^2 + Bs^2 + Cs$$

Equating coefficients of s^2 we get

$$s^2 + Bs^2 = 0$$

$$1+B = 0$$

$$\boxed{B=-1}$$

Equating coefficient of s we get

$$2\varepsilon\omega_n s + Cs = 0$$

$$2\varepsilon\omega_n + C = 0$$

$$\boxed{C = -2\varepsilon\omega_n}$$

$$c(s) = \frac{1}{s} - \frac{s+2\varepsilon\omega_n}{s^2+2\varepsilon\omega_n s+\omega_n^2}$$

Adding & subtracting $\varepsilon^2\omega_n^2$ to the denominator of the second term in the above equation

$$= \frac{1}{s} - \frac{s+2\varepsilon\omega_n}{s^2+2\varepsilon\omega_n s+\omega_n^2+\varepsilon^2\omega_n^2-\varepsilon^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\varepsilon\omega_n}{(s^2+2\varepsilon\omega_n s+\varepsilon^2\omega_n^2)+(\omega_n^2-\varepsilon^2\omega_n^2)}$$

$$= \frac{1}{s} - \frac{s+2\varepsilon\omega_n}{(s+\varepsilon\omega_n)^2+\omega_n^2(1-\varepsilon^2)} = \frac{1}{s} - \left(\frac{s+2\varepsilon\omega_n}{den} + \frac{\varepsilon\omega_n}{den}\right)$$

$\because \omega_d = \omega_n\sqrt{1-\varepsilon}$

$\omega_d^2 = \omega_n^2(1-\varepsilon^2)$

$$= \frac{1}{s} - \frac{(s+\varepsilon\omega_n)}{(s+\varepsilon\omega_n)^2+\omega_d^2} - \frac{\varepsilon\omega_n}{(s+\varepsilon\omega_n)^2+\omega_d^2}$$

Multiply & divide by ω_d

$$= \frac{1}{s} - \frac{s+\varepsilon\omega_n}{(s+\varepsilon\omega_n)^2+\omega_d^2} - \frac{\varepsilon\omega_n}{\omega_d}\,\frac{\omega_d}{(s+\varepsilon\omega_n)^2+\omega_d^2}$$

on taking inverse L.T

$$c(t) = 1 - e^{-\xi \omega_n t} \cos \omega_d t - \frac{\xi \omega_n}{\omega_d} e^{-\xi \omega_n t} \sin \omega_d t$$

$$= 1 - e^{-\xi \omega_n t} \left(\cos \omega_d t + \frac{\xi \omega_n}{\omega_n \sqrt{1-\xi^2}} \sin \omega_d t \right]$$

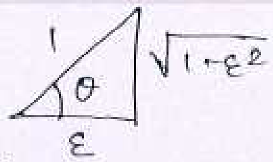
$$= 1 - e^{-\xi \omega_n t} \left(\cos \omega_d t + \frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_d t \right)$$

$$= 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \left(\sqrt{1-\xi^2} \cos \omega_d t + \xi \sin \omega_d t \right)$$

$$= 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \left(\sin \omega_d t \cdot \xi + \cos \omega_d t \sqrt{1-\xi^2} \right)$$

On constructing right angle triangle with ξ & $\sqrt{1-\xi^2}$ we get

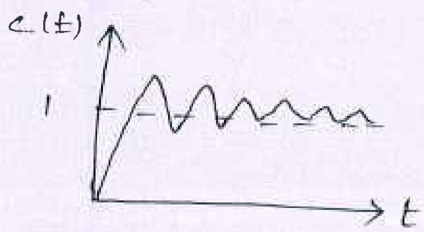
$$\sin \theta = \sqrt{1-\xi^2}, \quad \cos \theta = \xi \quad \& \quad \tan \theta = \frac{\sqrt{1-\xi^2}}{\xi}$$



$$= 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \left(\sin \omega_d t \cos \theta + \cos \omega_d t \sin \theta \right)$$

$$= 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

The response of under-damped second order system oscillates before settling to a final value. The oscillations depends on the value of damping ratio.



2. The unity feed back system is characterized by an open loop T.F $G(s) = \frac{K}{s(s+10)}$. Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine settling time, peak overshoot and time to peak overshoot for a unit step i/p.

The closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K/\cancel{s(s+10)}}{\frac{s(s+10)+K}{\cancel{s(s+10)}}} = \frac{K}{s(s+10)+K}$$

$$= \frac{K}{s^2+10s+K}$$

The value of K can be evaluated by comparing the system T.F with standard form of second order T.F

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{K}{s^2+10s+K}$$

On comparing we get

$$\omega_n^2 = K \qquad 2\xi\omega_n = 10 \qquad \omega_n^2 = K$$

$$\omega_n = \sqrt{K} \qquad 2\times 0.5\,\omega_n = 10 \qquad \omega_n^2 = 100$$

$$2\times 0.5\times\sqrt{K} = 10 \qquad \omega_n = 10 \text{ rad/sec}$$

$$\sqrt{K} = 10$$

$$K = 100$$

% Peak overshoot $(\%M_p) = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100$$

$$= 0.163 \times 100 = 16.3\%$$

Peak time $(t_p) = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{10\sqrt{1-(0.5)^2}} = 0.363$ sec

The value of gain $(K) = 100$

% Peak overshoot $(\%M_p) = 16.3\%$

Peak time $(t_p) = 0.363$ sec

3. The response of a feedback system to a step i/p is
$c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$

(a) obtain the expression for the closed loop transfer function

(b) Determine the undamped natural frequency and damping ratio of the system

$R(s) = \frac{1}{s} \qquad c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$

Taking L.T for $c(t)$

$$C(s) = \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}$$

$$= \frac{1}{s} + \frac{0.2(s+10) - 1.2(s+60)}{(s+60)(s+10)}$$

$$= \frac{1}{s} + \frac{0.2s + 2 - 1.2s - 72}{(s+60)(s+10)}$$

$$= \frac{1}{s} + \frac{(-s-70)}{s^2+70s+600}$$

$$= \frac{\cancel{s^2} + \cancel{70s} + 600 - \cancel{s^2} - \cancel{70s}}{s(s^2+70s+600)}$$

$$= \frac{600}{s(s^2+70s+600)}$$

Closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{600}{s^2+70s+600}$$

This is a second order system. For this type standard form of $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$$\omega_n^2 = 600$$

$$\omega_n = \sqrt{600} \qquad \boxed{\omega_n = 24.5 \text{ rad/sec}}$$

$$2\xi\omega_n = 70$$

$$\therefore \xi = \frac{70}{2\omega_n} = \frac{70}{2\times 24.5} = 1.43$$

Note as $\xi > 1$, it is overdamped system.

4. A unity feed back control system has an open loop t/f function, $G(s) = \frac{10}{s(s+2)}$. Find the rise time, Percentage overshoot, Peak time and settling time for a step i/p of 12 units.

T.F ; $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$G(s) = \frac{10}{s(s+2)}$

$\therefore \frac{C(s)}{R(s)} = \frac{\frac{10}{s(s+2)}}{1+\frac{10}{s(s+2)}} = \frac{\frac{10}{s(s+2)}}{\frac{s(s+2)+10}{s(s+2)}} = \frac{10}{s^2+2s+10}$

$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{10}{s^2+2s+10}$

on comparing

$\omega_n^2 = 10$

$\omega_n = \sqrt{10}$

$\omega_n = 3.162$ rad/sec

$2\xi\omega_n = 2$

$\xi = \frac{2}{2\omega_n} = \frac{1}{\omega_n} = \frac{1}{3.162} = 0.316$

(i) Rise time $t_r = \frac{\pi-\theta}{\omega_d}$

$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\frac{\sqrt{1-(0.316)^2}}{0.316} = 1.249$ rad

$\omega_d = \omega_n\sqrt{1-\xi^2} = 3.162\sqrt{1-(0.316)^2} = 3$ rad/sec

$t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-1.249}{3} = 0.63$ sec

(ii) Percentage overshoot (% MP) $= e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100$

$= e^{\frac{-0.316\pi}{\sqrt{1-(0.316)^2}}} \times 100$

$= 35.12\ \%$

(iii) Peak overshoot $= \frac{35.12}{100} \times 12 = 4.2144$ units

(iv) Peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{3} = 1.047$ sec

(v) Time constant $T = \frac{1}{\xi\omega_n} = \frac{1}{0.316 \times 3.162} = 1$ sec

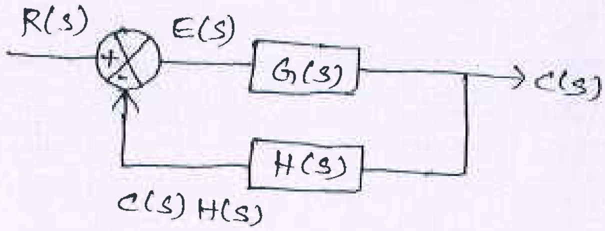
For 5% error settling time $t_s = 3T = 3$ sec

For 2% error settling time $t_s = 4T - ..$

Steady state error (ess)

The steady state error is the value of error signal, e(t), when t tends to infinity. The steady state error is a measure of system accuracy. These errors arise from the nature of the i/p's, type of system and from non linearity of system components.

Consider a closed loop system



$R(s) \rightarrow$ i/p signal

$E(s) \rightarrow$ Error signal

$C(s)H(s) \rightarrow$ Feedback signal

$C(s) \rightarrow$ o/p signal (or) response

$$E(s) = R(s) - C(s)H(s) \qquad \text{where } C(s) = E(s)G(s)$$

$$= R(s) - [E(s)G(s)]H(s)$$

$$E(s) + E(s)G(s)H(s) = R(s)$$

$$E(s)[1 + G(s)H(s)] = R(s)$$

$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

Let $e(t)$ = error signal in time domain

$$\therefore e(t) = L^{-1}[E(s)] = L^{-1}\left[\frac{R(s)}{1+G(s)H(s)}\right]$$

$$\therefore e_{ss} = \underset{t\to\infty}{Lt}\, e(t)$$

Using Final value theorem

$$e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, sE(s) = \underset{s\to 0}{Lt}\, \frac{sR(s)}{1+G(s)H(s)}$$

Static error constants

There are three types of error constants.

(i) Positional error constant. $K_p = Lt\ G(s)H(s)$

(ii) Velocity error constant, $K_V = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s)$

(iii) Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\ s^2 G(s)\,H(s)$

$$e_{ssp} = \frac{A_1}{1+K_p}$$

$$e_{ssv} = \frac{A_2}{K_v}$$

$$e_{ssa} = \frac{A_3}{K_a}$$

The K_p, K_v & K_a are in general called static error constants

General equation for magnitude

The i/p eqn $r(t) = A_1 + A_2 t + A_3 \frac{t^2}{2}$

$A_1 \downarrow$ Step i/p; $A_2 t \downarrow$ Ramp i/p; $A_3 \frac{t^2}{2} \downarrow$ Parabolic i/p

1. A unity Feedback system has $G(s) = \dfrac{40(s+2)}{s(s+1)(s+4)}$

Find the (a) Type of the system

(b) All error constants

(c) Calculate ess for ramp i/p with magnitude

(a) Type of the system = 1

(b) error constants

$$K_p = \underset{s\to 0}{Lt}\ G(s)H(s) = \underset{s\to 0}{Lt}\ \frac{40(s+2)}{s(s+1)(s+4)} = \frac{40\times 2}{0} = \infty$$

$$K_v = \underset{s\to 0}{Lt}\ s\,G(s)H(s) = \underset{s\to 0}{Lt}\ \frac{\cancel{s}\cdot 40(s+2)}{\cancel{s}(s+1)(s+4)} = \frac{40\times 2}{4} = 20$$

$$K_a = \underset{s\to 0}{Lt}\ s^2 G(s)H(s) = \underset{s\to 0}{Lt}\ \frac{s^2\, 40(s+2)}{s(s+1)(s+4)} = 0$$

(c) To find ess for ramp i/p with magnitude

$$e_{ss} = e_{ss1} + e_{ss2} + e_{ss3}$$

$$= \frac{A_1}{1+K_p} + \frac{A_2}{K_v} + \frac{A_3}{K_a}$$

$$e_{ss} = \frac{4}{20} = 0.2$$

$e_{ss} = 0.2 \Rightarrow$ Final steady state error.

2. Determine the static error constants and the steady state error for i/p $r(t) = 2t^2 + 5t + 10$. For a open loop T.F of feed back system.

$$G(s)H(s) = \frac{100}{s^2(s+4)(s+12)}$$

Error constants

$$K_p = \lim_{s\to 0} G(s)H(s) = \frac{100}{s^2(s+4)(s+12)} = \frac{100}{0} = \infty$$

$$K_v = \lim_{s\to 0} s\,G(s)H(s) = \frac{\cancel{s} \times 100}{s^{\cancel{2}}(s+4)(s+12)} = \infty$$

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \frac{\cancel{s^2} \times 100}{\cancel{s^2}(s+4)(s+12)} = \frac{100}{4\times 12} = \frac{100}{48} = 2.08$$

$$r(t) = A_1 + A_2 t + A_3 \frac{t^2}{2}$$

General Form $= A_3 \frac{t^2}{2} + A_2 t + A_1$ $\qquad r(t) = 2t^2 + 5t + 10$

$$= 4\frac{t^2}{2} + 5t + 10$$

$A_1 = 10$

$A_2 = 5$

$A_3 = 4$

e_{ss} for $K_p = \frac{A_1}{1+K_p} = \infty$

e_{ss} for $K_v = \frac{A_2}{K_v} = \infty$

e_{ss} for $K_a = \frac{A_3}{K_a} = \frac{4}{2.08} = 1.923$

Alternate method for

Generalised error series or Dynamic error coefficients

The error signal is s-domain

$$E(s) = \frac{R(s)}{1+G(s)H(s)} \qquad \therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

The above eqn can be expressed as a power series of s

$$F(s) = \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} = C_0 + C_1 s + \frac{C_2}{2!}s^2 + \frac{C_3}{3!}s^3 + \ldots$$

$$\therefore E(s) = C_0 R(s) + C_1 s R(s) + \frac{C_2}{2!} s^2 R(s) + \frac{C_3}{3!} s^3 R(s) + \ldots$$

On taking inverse laplace transform

$$e(t) = c_0 r(t) + c_1 \dot{r}(t) + \frac{c_2}{2!}\ddot{r}(t) + \frac{c_3}{3!}\dddot{r}(t) + \text{---}$$

where c_0, c_1, c_2 --- are generalized error coefficient

$$\therefore e_{ss} = \underset{t\to\infty}{Lt}\ e(t)$$

$$\dot{r}(t) = \frac{d}{dt} r(t)$$

$$\ddot{r}(t) = \frac{d^2}{dt^2} r(t)$$

$$\dddot{r}(t) = \frac{d^3}{dt^3} r(t)$$

$$c_0 = \underset{s\to 0}{Lt}\ F(s)$$

$$c_1 = \underset{s\to 0}{Lt}\ \frac{d}{ds} F(s)$$

$$c_2 = \underset{s\to 0}{Lt}\ \frac{d^2}{ds^2} F(s)$$

(1) For a unity feed back control system the open loop T.F

$$G(s) = \frac{10(s+2)}{s^2(s+1)}.$$ Find

(a) the position, velocity and acceleration error constants

(b) the steady state error when the i/p is

$$R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$$

(a) To find static error constants

$$H(s) = 1$$

Position error constant $K_p = \underset{s\to 0}{Lt}\ G(s)H(s)$

$$= \underset{s\to 0}{Lt}\ \frac{10(s+2)}{s^2(s+1)} = \infty$$

Velocity error constant $K_v = \underset{s\to 0}{Lt}\ s\,G(s)H(s)$

$$= \underset{s\to 0}{Lt}\ \frac{s\times 10(s+2)}{s^2(s+1)} = \infty$$

Acceleration error constant $K_a = \underset{s\to 0}{Lt}\ s^2 G(s)H(s)$

$$= \underset{s\to 0}{Lt}\ \frac{\cancel{s^2}\times 10(s+2)}{\cancel{s^2}(s+1)} = \frac{20}{1} = 20$$

(b) To find steady state error

(I) method

$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

$$R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3} \qquad G(s) = \frac{10(s+2)}{s^2(s+1)} \qquad H(s) = 1$$

$$\therefore E(s) = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{1 + \frac{10(s+2)}{s^2(s+1)}} = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{\frac{s^2(s+1) + 10(s+2)}{s^2(s+1)}}$$

$$= \frac{3}{s}\left(\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right) - \frac{2}{s^2}\left(\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right) + \frac{1}{3s^3}\left(\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right)$$

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, sE(s)$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\, s\left\{\frac{3}{s}\left(\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right) - \frac{2}{s^2}\left(\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right) + \frac{1}{3s^3}\left(\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right)\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)} - \frac{2s(s+1)}{s^2(s+1)+10(s+2)} + \frac{(s+1)}{3s^2(s+1)+30(s+2)}\right\}$$

$$= 0 - 0 + \frac{1}{60}$$

$$\boxed{e_{ss} = \frac{1}{60}}$$

To find steady state error

Method - II

Steady state error for non-standard input is obtained using generalized error series, given below.

The error signal $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!}$

Given that $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

Input signal in time domain, $r(t) = L^{-1}[R(s)]$

$$= L^{-1}\left[\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}\right]$$

$$= 3 - 2t + \frac{1}{3}\frac{t^2}{2!} = 3 - 2t + \frac{t^2}{6}$$

$$\dot{r}(t) = \frac{d}{dt}\{r(t)\} = -2 + \frac{1}{6}2t = -2 + \frac{t}{3}$$

$$\ddot{r}(t) = \frac{d^2}{dt^2} r(t) = \frac{d}{dt}(\dot{r}(t)) = \frac{1}{3}$$

$$\dddot{r}(t) = \frac{d^3}{dt^3} r(t) = \frac{d}{dt}(\ddot{r}(t)) = 0$$

The derivatives of $r(t)$ is zero after second derivative. Hence we have to evaluate only three constants C_0, C_1 and C_2.

The generalized error constants are given by

$$C_0 = \underset{s\to 0}{Lt}\, F(s) \qquad C_1 = \underset{s\to 0}{Lt}\, \frac{d}{ds} F(s) \qquad C_2 = \underset{s\to 0}{Lt}\, \frac{d^2}{ds^2} F(s)$$

$$F(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+G(s)} = \frac{1}{1+\frac{10(s+2)}{s^2(s+1)}} = \frac{s^2(s+1)}{s^2(s+1)+10(s+2)} = \frac{s^3+s^2}{s^3+s^2+10s+20}$$

$$C_0 = \underset{s\to 0}{Lt}\, F(s) = \underset{s\to 0}{Lt}\left[\frac{s^3+s^2}{s^3+s^2+10s+20}\right] = 0$$

$$C_1 = \underset{s\to 0}{Lt}\, \frac{d}{ds} F(s) = \underset{s\to 0}{Lt}\, \frac{d}{ds}\left[\frac{s^3+s^2}{s^3+s^2+10s+20}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(s^3+s^2+10s+20)(3s^2+2s)-(s^3+s^2)(3s^2+2s+10)}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{3s^5+2s^4+3s^4+2s^3+30s^3+20s^2+60s^2+40s-3s^5-2s^4-10s^3-3s^4-2s^3-10s^2}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{20s^3+70s^2+40s}{(s^3+s^2+10s+20)^2}\right] = 0$$

$$C_2 = \underset{s\to 0}{Lt}\, \frac{d^2}{ds^2} F(s) = \underset{s\to 0}{Lt}\, \frac{d}{ds}\left[\frac{d}{ds}F(s)\right] = \underset{s\to 0}{Lt}\, \frac{d}{ds}\left[\frac{20s^3+70s^2+40s}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(s^3+s^2+10s+20)^2(60s^2+140s+40)-(20s^3+70s^2+40s)\,2\times(s^3+s^2+10s+20)(3s^2+2s+10)}{(s^3+s^2+10s+20)^4}\right]$$

$$= \frac{20^2\times 40}{20^4} = \frac{1}{10}$$

Error signal $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!}$

$$= \left(3-2t+\frac{t^2}{6}\right)\times 0 + \left(-2+\frac{t}{3}\right)\times 0 + \frac{1}{3}\times\frac{1}{10}\times\frac{1}{2!}$$

$$= \frac{1}{60}$$

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{t\to 0}{Lt}\ \frac{1}{60} = \frac{1}{60}$

Determine the static error coefficients for

$$G(s) = \frac{1}{s(s+1)(s+10)} \quad ; \quad H(s)=1$$

$$K_p = \underset{s\to 0}{Lt}\ G(s)H(s) = \underset{s\to 0}{Lt}\ \frac{1}{s(s+1)(s+10)} = \infty$$

$$K_v = \underset{s\to 0}{Lt}\ sG(s)H(s) = \underset{s\to 0}{Lt}\ \frac{s}{s(s+1)(s+10)} = 0.1$$

$$K_a = \underset{s\to 0}{Lt}\ s^2G(s)H(s) = \underset{s\to 0}{Lt}\ \frac{s^2}{s(s+1)(s+10)} = 0$$

A feed back control system is described as $G(s) = \frac{50}{(s+2)(s+5)s}$ and $H(s) = \frac{1}{s}$. For a unit step determine steady state error constants and errors

$$K_p = \underset{s\to 0}{Lt}\ G(s)H(s) = \underset{s\to 0}{Lt}\ \frac{50}{s^2(s+2)(s+5)} = \infty$$

$$K_v = \lim_{s \to 0} s\,G(s)\,H(s) = \infty$$

$$K_a = \lim_{s \to 0} s^2 G(s)\,H(s) = 5$$

steady state error, $e_{ss} = \lim_{s \to 0} \dfrac{s\,R(s)}{1+G(s)H(s)}$

$$= \lim_{s \to 0} \frac{s \times \frac{1}{s}}{1 + \frac{50}{s^2(s+2)(s+5)}}$$

$$= 0$$

A certain feedback control system is described by following T.F

$$G(s) = \frac{K}{s^2(s+20)(s+30)} \; ; \; H(s) = 1$$

Determine the steady state error coefficients and also determine the value of K, to limit the steady to 10 units due to i/p $r(t) = 1 + 10t + 20t^2$

$$K_p = \lim_{s \to 0} G(s)\,H(s) = \lim_{s \to 0} \frac{K}{s^2(s+20)(s+30)} = \infty$$

$$K_v = \lim_{s \to 0} s\,G(s)\,H(s) = \infty$$

$$K_a = \lim_{s \to 0} s^2 G(s)\,H(s) = \frac{K}{600}$$

Due to unit step i/p

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

Ramp $\quad e_{ss} = \dfrac{10}{K_v} = \dfrac{10}{\infty} = 0$

Parabolic $\quad e_{ss} = \dfrac{20}{K_a} = \dfrac{20 \times 600}{K} = \dfrac{12000}{K}$

Total error $= 0 + 0 + \frac{12000}{k}$..

$$\frac{12000}{k} = 10$$

$$k = 1200$$

A unity feedback system has the T.f $G(s) = \frac{K_1(2s+1)}{s(5s+1)(1+s)^2}$

The i/p $r(t) = 1+6t$ is applied to the system. Determine the minimum value of K_1 if the steady state error is less than 0.1

$$r(t) = 1+6t$$

$$R(s) = \frac{1}{s} + \frac{6}{s^2} \quad ; \quad H(s) = 1$$

$$E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\frac{1}{s} + \frac{6}{s^2}}{1 + \frac{K_1(2s+1)}{s(5s+1)(1+s)^2}}$$

$$E(s) = \frac{1}{s}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2 + K_1(2s+1)}\right] + \frac{6}{s^2}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2 + K_1(2s+1)}\right]$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$e_{ss} = 0 + \frac{6}{K_1} = \frac{6}{K_1}$$

Given that $e_{ss} < 0.1$

$$0.1 = \frac{6}{K_1}$$

$$K_1 = \frac{6}{0.1} = 60 \qquad \boxed{K_1 = 60}$$

For servomechanism with open loop T.F is given below, explain what type of I/p signal give rise to constant steady state error & calculate their values.

(i) $G(s) = \frac{20(s+2)}{s(s+1)(s+3)}$ $H(s)=1$ Assume unity feed back system

Poles at origin, Type 1 system, velocity (Ramp) I/p will give a constant steady state error.

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} sG(s)$$

$$K_v = \lim_{s\to 0} s\frac{20(s+2)}{s(s+1)(s+3)} = 40/3$$

$$e_{ss} = 1/K_v = 3/40 = 0.075$$

(ii) $G(s) = \frac{10}{(s+2)(s+3)}$ $H(s)=1$

No pole at open loop system, type 0 system, constant steaddy state error

$$K_p = \lim_{s\to 0} G(s)H(s) = 5/3$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+5/3} = 0.375$$

(iii) $G(s) = \frac{10}{s^2(s+1)(s+2)}$

Two Poles, Type 2, $e_{ss} = 1/K_a$

$K_a = 5$ $e_{ss} = \frac{1}{5} = 0.2$

A closed loop T.F servo is represented by differential equation $\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64e$

where c is the displacement of the o/p shaft

r is the displacement of the I/p shaft

$e = r - c$

Determine undamped natural frequency, damping ratio, % Mp for unit step I/p

$$\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64e$$

$$e = r - c$$

$$\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64(r-c)$$

Take L.T

$$s^2C(s) + 8sC(s) = 64R(s) - 64C(s)$$

$$s^2C(s) + 8sC(s) + 64C(s) = 64R(s)$$

$$\frac{C(s)}{R(s)} = \frac{64}{s^2+8s+64}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$2\xi\omega_n = 8 \qquad \omega_n^2 = 64$$

$$\xi = \frac{8}{2\times 8} = 0.5 \qquad \omega_n = 8$$

$$\%\,Mp = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100$$

$$= e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} \times 100$$

$$= 16.3\,\%$$

The open loop transfer function of a servo system with unity feed back is $G(s) = \frac{10}{s(0.1s+1)}$. Evaluate the static error constants of the system. Obtain the steady state error of the system, when subjected to an input given by the polynomial $r(t) = a_0 + a_1 t + \frac{a_2}{2}t^2$

To find static error constant

$$H(s) = 1 \qquad G(s)H(s) = G(s)$$

$$K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{10}{s(0.1s+1)} = \infty$$

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\,\frac{10}{s(0.1s+1)} = 10$$

$$K_a = \lim s^2G(s) = \lim s^2 \quad 10$$

To find steady state error

Method 1

The error signal $e(t) = r(t)\, c_0 + \dot{r}(t)\, c_1 + \ddot{r}(t)\, \frac{c_2}{2!} + \cdots$

$$r(t) = a_0 + a_1 t + \frac{a_2}{2} t^2$$

$$\dot{r}(t) = \frac{d}{dt} r(t) = \frac{d}{dt}\left[a_0 + a_1 t + \frac{a_2}{2} t^2\right] = a_1 + a_2 t$$

$$\ddot{r}(t) = \frac{d^2}{dt^2} r(t) = \frac{d}{dt}\left[\frac{d}{dt} r(t)\right] = a_2$$

$$\dddot{r}(t) = 0$$

$$C_0 = \lim_{s\to 0} F(s) \qquad C_1 = \lim_{s\to 0} \frac{d}{ds} F(s) \qquad C_2 = \lim_{s\to 0} \frac{d^2}{ds^2} F(s)$$

$$F(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+G(s)} = \frac{1}{1+\frac{10}{s(0.1s+1)}} = \frac{0.1s^2+s}{0.1s^2+s+10}$$

$$C_0 = \lim_{s\to 0} F(s) = 0$$

$$C_1 = \lim_{s\to 0} \frac{d}{ds} F(s) = \lim_{s\to 0} \frac{d}{ds}\left[\frac{0.1s^2+s}{0.1s^2+s+10}\right]$$

$$= \lim_{s\to 0}\left[\frac{(0.1s^2+s+10)(0.2s+1) - (0.1s^2+s)(0.2s+1)}{(0.1s^2+s+10)^2}\right]$$

$$= \lim_{s\to 0} \frac{2s+10}{(0.1s^2+s+10)^2} = 0.1$$

$$C_2 = \lim_{s\to 0} \frac{d}{ds^2} F(s) = \lim_{s\to 0} \frac{d}{ds}\left[\frac{2s+10}{(0.1s^2+s+10)^2}\right] = 0$$

$$e(t) = r(t)\, c_0 + \dot{r}(t)\, c_1 + \ddot{r}(t)\, \frac{c_2}{2!} = (a_1 + a_2 t)\, 0.1$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty}\left[(a_1 + a_2 t)\, 0.1\right] = \infty$$

Method 2

$$E(s) = \frac{R(s)}{1+G(s)H(s)} \qquad r(t) = a_0 + a_1 t + \frac{a_2}{2} t^2$$

$$G(s) = \frac{10}{s(0.1s+1)} \qquad H(s) = 1$$

$$R(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{2}\frac{2!}{s^3}$$

$$= \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}$$

$$E(s) = \frac{\frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}}{1 + \frac{10}{s(0.1s+1)}} = \frac{\frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}}{\frac{s(0.1s+1)+10}{s(0.1s+1)}}$$

$$= \frac{a_0}{s}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} s\left\{\frac{a_0}{s}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right\} + \frac{a_1}{s^2}\left(\frac{s(0.1s+1)}{s(0.1s+1)+10}\right\} + \frac{a_2}{s^3}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]\right.$$

$$= 0 + \frac{a_1}{10} + \infty = \infty$$

iii method.

Error Signal in s-domain, $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)} \qquad G(s) = \frac{10(s+2)}{s^2(s+1)} \qquad H(s) = 1$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+\frac{10(s+2)}{s^2(s+1)}} = \frac{s^2(s+1)}{s^2(s+1)+10(s+2)}$$

$$= \frac{s^3+s^2}{s^3+s^2+10s+20} = \frac{s^2+s^3}{20+10s+s^2+s^3}$$

$$20+10s+s^2+s^3 \overline{\Big) \, s^2+s^3} \quad \frac{s^2}{20}+\frac{s^3}{40}+\cdots$$

$$\frac{(-)}{s^2} + \frac{s^3}{2} + \frac{s^4}{20} + \frac{s^5}{20}$$

$$\frac{s^3}{2} - \frac{s^4}{20} - \frac{s^5}{20}$$

$$\frac{s^3}{2} + \frac{s^4}{4} + \frac{s^5}{40} + \frac{s^6}{40}$$

$$-\frac{3s^4}{10} - \frac{3}{40}s^5 - \frac{s^6}{40}$$

$10s \times \frac{s^2}{20} = \frac{s^3}{2}$

$s^3 - \frac{s^3}{2}$

$\frac{2s^3-s^3}{2} = \frac{s^3}{2}$

$20 \times \frac{s^3}{40} = \frac{s^3}{2}$

$10s \times \frac{s^3}{40} = \frac{10s^4}{40}$

$s^2 \times \frac{s^3}{40} = \frac{s^5}{40}$

$s^3 \times \frac{s^3}{40} = \frac{s^6}{40}$

$-\frac{s^4}{20} - \frac{s^4}{4} =$

$$E(s) = R(s)\left[\frac{s^2}{20} + \frac{s^3}{40} + \cdots\right]$$

$$= \frac{1}{20}s^2R(s) + \frac{1}{40}s^3R(s) + \cdots$$

On taking inverse laplace transform

$$e(t) = \frac{1}{20}\ddot{r}(t) + \frac{1}{40}\dddot{r}(t) + \cdots$$

Given that $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

$$r(t) = \mathcal{L}^{-1}[R(s)] = \mathcal{L}^{-1}\left[\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}\right]$$

$$= 3 - 2t + \frac{t^2}{6}$$

$$\dot{r}(t) = \frac{d}{dt}r(t) = -2 + \frac{1}{6}2t = -2 + \frac{t}{3}$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = \frac{1}{3}$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\ddot{r}(t) = 0$$

$\therefore$ Error signal in time domain, $e(t) = \frac{1}{20}\frac{d}{dt}\ddot{r}(t)$

$$= \frac{1}{20}\left(\frac{1}{3}\right) = \frac{1}{60}$$

Steady state error,

$$e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{t\to\infty}{Lt}\, \frac{1}{\cdot} = \frac{1}{\cdot}$$

(ii) method

Error Signal in s-domain, $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

Given that $G(s) = \dfrac{10}{s(0.1s+1)}$ $\quad H(s) = 1$

$$\frac{E(s)}{R(s)} = \frac{1}{1+\dfrac{10}{s(0.1s+1)}} = \frac{s(0.1s+1)}{s(0.1s+1)+10} = \frac{0.1s^2+s}{0.1s^2+s+10} = \frac{s+0.1s^2}{10+s+0.1s^2}$$

$$10+s+0.1s^2 \;\big)\; s+0.1s^2 \;\big(\; \frac{s}{10} - \frac{s^3}{1000}$$

$$\overset{(-)}{s} \overset{(-)}{+} \frac{s^2}{10} \overset{(-)}{+} \frac{s^3}{100}$$

$$-\frac{s^3}{100}$$

$$\overset{(+)}{-}\frac{s^3}{100} \overset{(+)}{-} \frac{s^4}{1000} \overset{(+)}{-} \frac{s^5}{10000}$$

$$\frac{s^4}{1000} + \frac{s^5}{10000}$$

$$\frac{E(s)}{R(s)} = \frac{s}{10} - \frac{s^3}{1000} + \cdots$$

$$E(s) = \frac{s}{10}R(s) - \frac{s^3}{1000}R(s) + \cdots$$

On taking inverse laplace transform

$$e(t) = \frac{1}{10}\dot{r}(t) - \frac{1}{1000}\dddot{r}(t)$$

Given that $r(t) = a_0 + a_1 t + \dfrac{a_2}{2}t^2$

$$\dot{r}(t) = \frac{d}{dt}r(t) = a_1 + a_2 t$$

$$\ddot{r}(t) = \frac{d}{dt}\dot{r}(t) = a_2$$

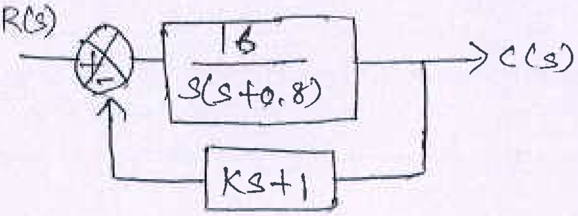
$$\dddot{r}(t) = \frac{d}{dt}\ddot{r}(t) = 0$$

$\therefore$ Error Signal in time domain, $e(t) = \dfrac{1}{10}\dot{r}(t)$

$$= \frac{1}{10}(a_1 + a_2 t)$$

Steady state error, $e_{ss} = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} \frac{1}{10}(a_1 + a_2 t)$
$= \infty$

A Positional control system with velocity feedback. What is the response $c(t)$ to the unit step input. Given that $\xi = 0.5$ Also calculate rise time, peak time, maximum overshoot and settling time.



The closed loop t/f

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Given that $G(s) = \frac{16}{s(s+0.8)}$ $\quad H(s) = Ks+1$

$$\frac{C(s)}{R(s)} = \frac{\frac{16}{s(s+0.8)}}{1+\frac{16}{s(s+0.8)}(Ks+1)} = \frac{16}{s(s+0.8)+16(Ks+1)}$$

$$= \frac{16}{s^2+0.8s+16Ks+16}$$

$$= \frac{16}{s^2+(0.8+16K)s+16}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{16}{s^2+(0.8+16K)s+16}$$

$\omega_n^2 = 16$ $\qquad 0.8+16K = 2\xi\omega_n$

$\omega_n = 4$ rad/sec $\qquad K = \frac{2\xi\omega_n - 0.8}{16}$

$$= \frac{2\times0.5\times4-0.8}{16}$$

$$= 0.2$$

$$\frac{C(s)}{R(s)} = \frac{16}{s^2+(0.8+16\times0.2)s+16}$$

$$= \frac{16}{s^2+4s+16}$$

$\xi = 0.5$ $\quad \xi < 1$ so it is underdamped

$$C(s) = R(s)\frac{16}{s^2+4s+16} = \frac{1}{s}\,\frac{16}{s^2+4s+16}$$

$$= \frac{16}{s(s^2+4s+16)}$$

$$C(s) = \frac{16}{s(s^2+4s+16)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+16}$$

$$A = C(s)\,s\Big|_{s=0} = \frac{16}{s^2+4s+16}\Big|_{s=0} = \frac{16}{16} = 1$$

$$\frac{16}{s(s^2+4s+16)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+16}$$

$$16 = A(s^2+4s+16)+(Bs+C)s$$

$$16 = As^2+4As+16A+Bs^2+Cs$$

On equating the coefficients of s^2 we get $0 = A+B$

$\therefore\ B = -A = -1$

On equating the coefficients of s we get $0 = 4A+C$

$\therefore\ C = -4A = -4$

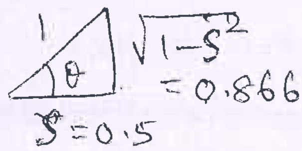
$$\therefore\ C(s) = \frac{1}{s} + \frac{-s-4}{s^2+4s+16} = \frac{1}{s} - \frac{s+4}{s^2+4s+4+12}$$

$$= \frac{1}{s} - \left(\frac{s+2+2}{(s+2)^2+\sqrt{12}^2}\right) = \frac{1}{s} - \frac{s+2}{(s+2)^2+\sqrt{12}^2} - \frac{2}{\sqrt{12}}\,\frac{\sqrt{12}}{(s+2)^2+\sqrt{12}^2}$$

$$c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s} - \frac{s+2}{(s+2)^2+\sqrt{12}^2} - \frac{2}{\sqrt{12}}\,\frac{\sqrt{12}}{(s+2)^2+\sqrt{12}^2}\right]$$

$$= 1 - e^{-2t}\cos\sqrt{12}t - \frac{2}{2\sqrt{3}}\,e^{-2t}\sin\sqrt{12}t$$

$$= 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin\sqrt{12}t + \cos\sqrt{12}t\right]$$



$\sin\theta = 0.866 = \frac{\sqrt{3}}{2}$ $\quad \tan\theta = 1.732$

$\cos\theta = 0.5 = 1/2$ $\quad \theta = 1.047$ rad

$$c(t) = 1 - e^{-2t}\left[\frac{1}{\sqrt{3}} \times 2 \times \sin\sqrt{12}t \times \frac{1}{2} + \frac{2}{\sqrt{3}} \times \cos\sqrt{12}t \times \frac{\sqrt{3}}{2}\right]$$

$$= 1 - e^{-2t}\,\frac{2}{\sqrt{3}}\left[\sin\sqrt{12}t\cos\theta + \cos\sqrt{12}t \times \sin\theta\right]$$

$$= 1 - \frac{2}{\sqrt{3}} e^{-2t} \left[\sin(\sqrt{12}t + \theta) \right]$$

$$= 1 - \frac{2}{\sqrt{3}} e^{-2t} \left[\sin(\sqrt{12}t + 1.047) \right]$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464 \text{ rad/sec}$$

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - 1.047}{3.464} = 0.6046 \text{ sec}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.464} = 0.907 \text{ sec}$$

$$\% M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} \times 100 = 0.163 \times 100$$

$$= 16.3\%$$

$t_s = 3T$ for 5% error

$= 4T$ for 2% error

$$T = \frac{1}{\zeta \omega_n} = \frac{1}{0.5 \times 4} = 0.5 \text{ sec}$$

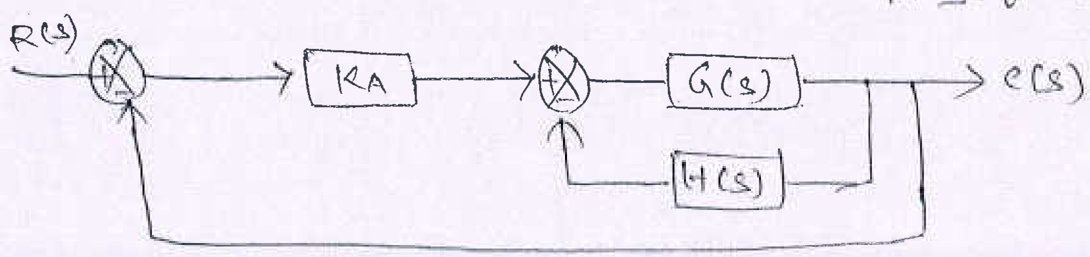
$$t_s = 3T = 3 \times 0.5 = 1.5 \text{ sec}$$

$$t_s = 4T = 4 \times 0.5 = 2 \text{ sec}$$

A unity feed back control syste

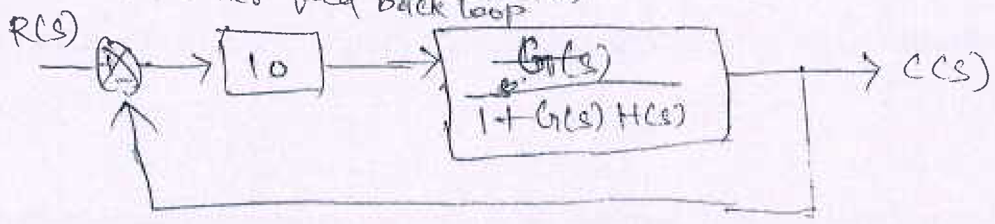
gain $K_A = 10$ and gain ratio $G(s) =$

path. A derivative feed back $H(s) = sK_o$ is introduced as a minor loop around $G(s)$. Determine the derivative feed back constant K_o so that the system damping factor is 0.6



$K_A = 10 \quad G(s) = \frac{1}{s(s+2)} \quad H(s) = sK_o$

Reducing the inner feed back loop



$$\frac{G(s)}{1+G(s)H(s)} = \frac{\frac{1}{s(s+2)}}{1+\frac{1}{s(s+2)} \times s K_o} = \frac{1}{s^2+2s+s K_o} \Rightarrow$$ Cascade with 10 & $H(s)=1$

$$\frac{C(s)}{R(s)} = \frac{10}{s^2+(2+K_o)s+10}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$\omega_n^2 = 10$ $\qquad$ $2+K_o = 2\zeta\omega_n$

$\omega_n = \sqrt{10} = 3.162$ $\qquad$ $K_o = 2\zeta\omega_n - 2$

$= (2 \times 0.6 \times 3.162) - 2$

$= 1.7944$

Correlation b/w static & Dynamic Error Coefficients

$$C_0 = \frac{1}{1+K_p}$$

$$C_1 = \frac{1}{K_v}$$

$$C_2 = \frac{1}{K_a}$$

Proof

$$C_0 = \lim_{s\to 0} F(s) = \lim_{s\to 0} \frac{1}{1+G(s)H(s)} = \frac{1}{1+\lim_{s\to 0} G(s)H(s)} = \frac{1}{1+K_p}$$

2014 (M)

Generalized Error Coefficient

The drawback in static error coefficients is that it does not show the variation of error with time and input should be a standard input. The generalized coefficients gives the steady state error as a fn of time. Also using the generalized error coefficients, the steady state error can be found for any type of input.

The error signal in s-domain, E(s) can be expressed as a product of two s-domain function

where $F(s) = \frac{1}{1+G(s)H(s)}$

error signal $E(s) = \frac{R(s)}{1+G(s)H(s)}$ input signal $= \frac{1}{1+G(s)H(s)} R(s) = F(s)\,R(s)$

Steady state error when the input is unit step signal

Steady State error, $e_{ss} = \underset{s\to 0}{Lt} \dfrac{sR(s)}{1+G(s)H(s)}$

When the input is unit step, $R(s) = \dfrac{1}{s}$

$$\therefore e_{ss} = \underset{s\to 0}{Lt} \frac{s\frac{1}{s}}{1+G(s)H(s)} = \underset{s\to 0}{Lt} \frac{1}{1+G(s)H(s)} = \frac{1}{1+\underset{s\to 0}{Lt}\, G(s)H(s)}$$

where, $K_p = \underset{s\to 0}{Lt}\, G(s)H(s)$ $\qquad = \dfrac{1}{1+K_p}$

The constant K_p is called Positional error Constant.

Type-0 system

$$K_p = \underset{s\to 0}{Lt}\, G(s)H(s) = \underset{s\to 0}{Lt}\, \frac{K(s+z_1)(s+z_2)(s+z_3)\cdots}{(s+P_1)(s+P_2)(s+P_3)\cdots}$$

$$= \frac{K z_1 z_2 z_3}{P_1 P_2 P_3} = \text{Constant}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \text{constant}$$

Hence in type-0 systems when the input is unit step there will be a constant steady state error.

Type-1 System

$$K_p = \underset{s\to 0}{Lt}\, G(s)H(s) = \underset{s\to 0}{Lt}\, \frac{K(s+z_1)(s+z_2)(s+z_3)\cdots}{s(s+P_1)(s+P_2)(s+P_3)} = \infty$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

In systems with type number 1 and above, for unit step input the value of K_p is infinity and so the steady state error is zero.

Steady state error when the input is unit ramp signal

Steady state error, $e_{ss} = \underset{s\to 0}{Lt} \dfrac{sR(s)}{1+G(s)H(s)}$

when the input is unit ramp, $R(s) = \frac{1}{s^2}$

$$e_{ss} = \underset{s \to 0}{Lt} \frac{s \frac{1}{s^2}}{1 + G(s)H(s)} = \underset{s \to 0}{Lt} \frac{1}{s + sG(s)H(s)}$$

$$= \frac{1}{\underset{s \to 0}{Lt}\; sG(s)H(s)} = \frac{1}{K_v}$$

where, $K_v = \underset{s \to 0}{Lt}\; sG(s)H(s)$

The constant K_v is called velocity error constant

Type-0 system

$$K_v = \underset{s \to 0}{Lt}\; sG(s)H(s) = \underset{s \to 0}{Lt} \frac{sK(s+z_1)(s+z_2)(s+z_3)}{(s+P_1)(s+P_2)(s+P_3)} = 0$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$

Hence in type 0 systems when the input is unit ramp, the Steady state error is infinity.

Type-1 system

$$K_v = \underset{s \to 0}{Lt}\; sG(s)H(s) = \underset{s \to 0}{Lt} \frac{sK(s+z_1)(s+z_2)(s+z_3)}{s(s+P_1)(s+P_2)(s+P_3)}$$

$$= K \frac{z_1 z_2 z_3}{P_1 P_2 P_3} = \text{Constant}$$

$$\therefore e_{ss} = \frac{1}{K_v} = \text{constant}$$

Hence in type-1 system when the i/p is unit ramp there will be a constant steady state error.

Type-2 system

$$K_v = \underset{s \to 0}{Lt}\; sG(s)H(s) = \underset{s \to 0}{Lt} \frac{sK(s+z_1)(s+z_2)(s+z_3)}{s^2(s+P_1)(s+P_2)(s+P_3)} = \infty$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

In systems with type number 2 and above, for unit ramp input, the value of K_v is infinity so the steady state error is zero.

Steady state error when the input is Unit Parabolic Signal

Steady state error, $e_{ss} = \underset{s\to 0}{Lt} \dfrac{sR(s)}{1+G(s)H(s)}$

When the input is unit Parabola, $R(s) = \dfrac{1}{s^3}$

$$\therefore\ e_{ss} = \underset{s\to 0}{Lt} \frac{s\frac{1}{s^3}}{1+G(s)H(s)} = \underset{s\to 0}{Lt} \frac{1}{s^2+s^2G(s)H(s)} = \frac{1}{\underset{s\to 0}{Lt}\, s^2G(s)H(s)}$$

$$\text{where, } K_a = \underset{s\to 0}{Lt}\, s^2G(s)H(s) \qquad = \frac{1}{K_a}$$

The constant K_a is called acceleration error constant

Type-0 System

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)H(s) = \underset{s\to 0}{Lt} \frac{s^2K(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)} = 0$$

$$\therefore\ e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type 0 systems for unit Parabolic input, the steady state error is infinity.

Type-1 System

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)H(s) = \underset{s\to 0}{Lt} \frac{s^2K(s+z_1)(s+z_2)(s+z_3)}{s(s+p_1)(s+p_2)(s+p_3)} = 0$$

$$\therefore\ e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-1 systems for unit parabolic input, the steady state error is infinity.

Type-2 System

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)H(s) = \underset{s\to 0}{Lt} \frac{s^2K(s+z_1)(s+z_2)(s+z_3)}{s^2(s+p_1)(s+p_2)(s+p_3)} =$$

$$e_{ss} = \frac{1}{K_a} = \text{constant} \qquad \frac{K z_1 z_2 z_3}{p_1 p_2 p_3} = \text{constant}$$

Hence in type-2 system when the input is unit Parabolic Signal there will be a constant steady state error.

Type -3 system

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \frac{s^2 K(s+z_1)(s+z_2)(s+z_3)}{s^3(s+p_1)(s+p_2)(s+p_3)} = \infty$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

In systems with type number 3 and above for unit Parabolic input the value of Ka is infinity and so the steady state error is zero.

Static error constant for various type no of systems

Error Constant	Type no of system 0	1	2	3
K_p	c	∞	∞	∞
K_v	0	c	∞	∞
K_a	0	0	c	∞

Steady state error for various types of inputs

Input signal	Type no of system 0	1	2	3
Unit step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

Generalized Error Coefficient.

$$e(t) = C_0 r(t) + C_1 \dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) + \cdots \frac{C_n}{n!} r^{n}(t) -$$

Steady state error

$$e_{ss} = \lim_{t\to\infty} r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} + \dddot{r}(t)\frac{C_3}{3!} + \cdots$$

$$= C_0 r(t) + C_1 \dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) -$$

Evaluation of Generalized error coefficients

The generalised error coefficient is given by

$$C_n = (-1)^n \int^{t} T^n f(T)\, dT$$

where $F(s) = \frac{1}{1+G(s)H(s)}$

We know that $L\{F(t)\} = F(s)$, hence by the definition of Laplace transform

$$F(s) = \int_0^t f(\tau) e^{-s\tau} d\tau$$

on taking $\underset{s\to 0}{Lt}$ on both sides we get

$$\underset{s\to 0}{Lt} F(s) = \underset{s\to 0}{Lt} \int_0^t f(\tau) e^{-s\tau} d\tau$$

$$= \int_0^t f(\tau) \underset{s\to 0}{Lt} e^{-s\tau} d\tau = \int_0^t f(\tau) d\tau = C_0$$

$$\boxed{C_0 = \underset{s\to 0}{Lt} F(s)}$$

Differentiating equation

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^t f(\tau) e^{-s\tau} d\tau$$

$$= \int_0^t f(\tau) \frac{d}{ds} e^{-s\tau} d\tau = \int_0^t f(\tau)(-\tau) e^{-s\tau} d\tau$$

$$= -\int_0^t \tau f(\tau) e^{-s\tau} d\tau$$

$$\underset{s\to 0}{Lt} \frac{d}{ds} F(s) = \underset{s\to 0}{Lt} -\int_0^t \tau f(\tau) e^{-s\tau} d\tau$$

$$= -\int_0^t \tau f(t) \underset{s\to 0}{Lt} e^{-s\tau} d\tau = -\int_0^t \tau f(t) d\tau = C_1$$

$$\boxed{C_1 = \underset{s\to 0}{Lt} \frac{d}{ds} F(s)}$$

Differentiating equation

$$\frac{d}{ds}\left[\frac{d}{ds}(F(s))\right] = \frac{d}{ds}\left[-\int_0^t \tau f(t) e^{-s\tau} d\tau\right]$$

$$\frac{d^2}{ds^2}F(s) = \left[-\int_0^t T f(T)\frac{d}{ds}(e^{-sT})dT\right] = -\int_0^t T f(T)(-T)$$

$$\frac{d^2 F(s)}{ds^2} = \int_0^t T^2 f(T) e^{-sT} dT$$

$$\underset{s\to 0}{Lt} \frac{d^2}{ds^2}F(s) = \underset{s\to 0}{Lt} \int_0^t T^2 f(T) e^{-sT} dT$$

$$= \int_0^t T^2 f(T) \cdot \underset{s\to 0}{Lt}\, e^{-sT} dT = \int_0^t T^2 f(T)\, dT = C_2$$

$$C_2 = \underset{s\to 0}{Lt} \frac{d^2}{ds^2}F(s)$$

$$C_n = \underset{s\to 0}{Lt} \frac{d^n}{ds^n}F(s)$$

Correlation b/w static and dynamic error coefficients

$$C_0 = \frac{1}{1+K_p}$$

$$C_1 = \frac{1}{K_v}$$

$$C_2 = \frac{1}{K_a}$$

$$= \frac{1}{1 + \underset{s\to 0}{Lt}\, G(s)H(s)} = \frac{1}{1+K_p}$$

Type Number of Control Systems

$$G(s)\,H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{s^N (s+p_1)(s+p_2)(s+p_3)\ldots}$$

z_1, z_2 – zeros

p_1, p_2 – Poles

K – constant

N – no of poles at the origin

If $N=0$, then the system is type-0 system

If $N=1$, then the system is type-1 system

If $N=2$, then the system is type-2 system

If $N=3$, then the system is type-3 system &