Chapter2

Discrete-timeSignalsandSystems

2.1 Discrete-timesignals

Digital signals are discrete in both time (the independent variable)and amplitude(the dependent variable).Signals that are discrete in time but continuous in amplitude are referred to as discrete-time signals.

Discrete- time signals are data sequences. A sequence of data is denoted$\{x[n]\}$ or simply$[n]$when the meaning is clear. The elements of the sequence are called samples. The index n associated with each sample is an integer. If appropriate, the range of n will be specified. Quite often, we are interested in identifying the samplewhere$n=0$.Thisisdonebyputtinganarrowunderthatsample.Forinstance,

$$\{x[n]\}=\{\ldots,0.35,\underset{\uparrow}{1},1.5,-0.6,-2,\ldots\}$$

The arrow is often omitted if it is clear from the context which sample is $x[0]$.Sample values can either be real or complex. In the rest of this book, the terms"discrete-time signals"and"sequences"are used interchangeably.

The time interval between samples is not explicitly shown.It can be assumed to be normalized to1unit of time.So the corresponding normalized sampling frequency is1Hz.If the actual sampling interval is$T$seconds,then the sampling frequency is given by $f_s=\frac{1}{T}$.

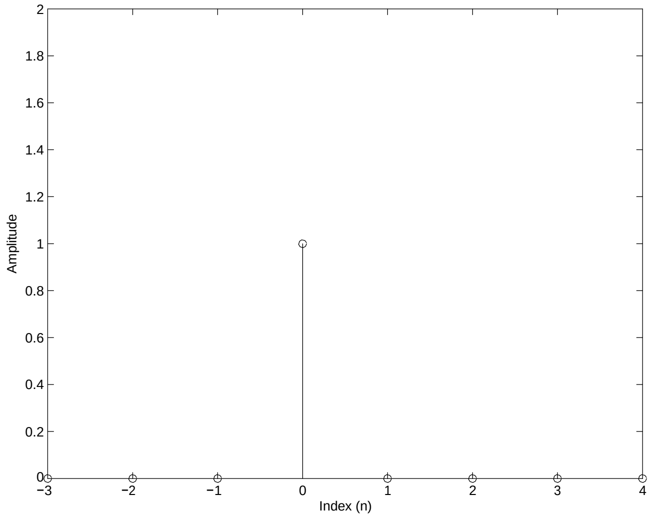


Figure2.1:The Unit Impulse Sequence

2.1.1 SomeElementarySequences

There are some sequences that we shall encounter frequently. They are described here.

2.1.1.1 UnitImpulseSequence

The unit impulse sequence is defined by

$$\delta[n]= \begin{cases} 1, & n=0 \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

This is depicted graphically inFigure2.1.Note that while the continuous-time unit Impulse function is a mathematical object that cannot be physically realized, the unit impulse sequence can easily be generated.

2.1.1.2 UnitStepSequence

The unit step sequence is one that has amplitude of zero for negative indices and amplitude of one for non-negative indices. It is shown in Figure2.2.

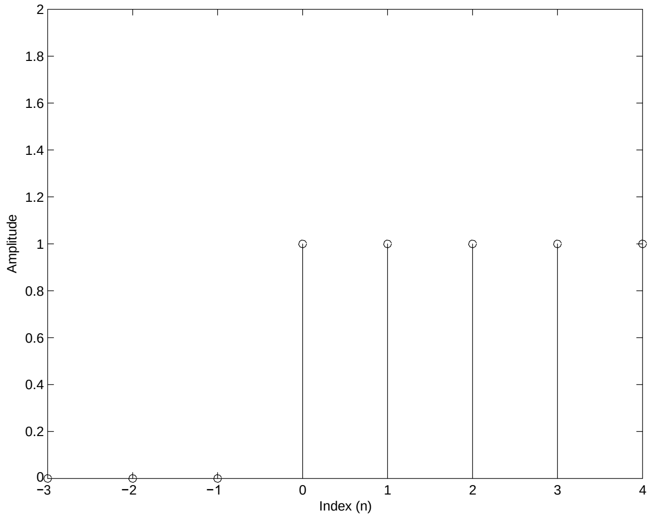


Figure2.2:The Unit Step Sequence

2.1.1.3 Sinusoidal Sequences

A sinusoidal sequence has the form

$$x[n]=A\cos(\omega_0 n+\varphi) \tag{2.2}$$

This function can also be decomposed into It sin-phase$_i[q]$and quadrature $x_q[n]$ components.

$$x[n] = A\cos\varphi\cos\omega_0 n - A\sin\varphi\sin\omega_0 n \tag{2.3}$$

$$= x_i[n] + x_q[n] \tag{2.4}$$

This is a common practice in communications signal processing.

2.1.1.4 Complex Exponential Sequences

Complex exponential sequences are essentially complex sinusoids.

$$x[n] = A\, e^{j(\omega_0 n+\varphi)} \tag{2.5}$$

$$= A\cos(\omega_0 n+\varphi) + jA\sin(\omega_0 n+\varphi) \tag{2.6}$$

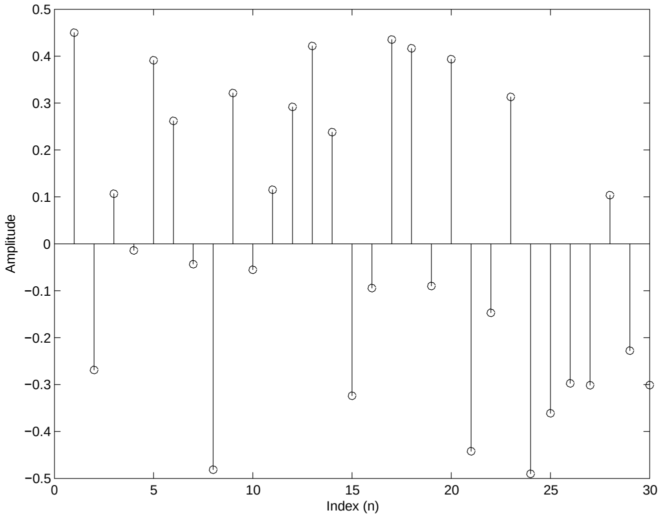

0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
Amplitude
0
5
10
15
20
25
30
Index (n)

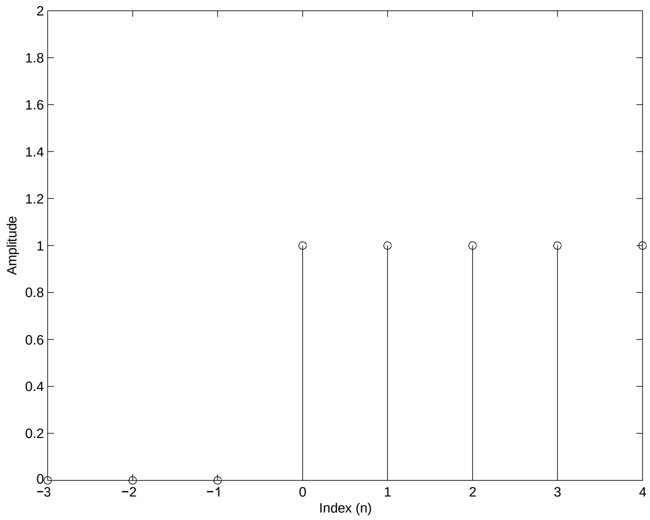
Figure2.3: Uniformly distributed random sequence with amplitudes between-0.5 and 0.5.

2.1.1.5 Random Sequences

The sample values of a random sequence are randomly drawn from a certain probability distribution. They are also called stochastic sequences. The two most common distributions are the Gaussian (normal) distribution and the uniform distribution. The zero mean Gaussian distribution is often used to model noise. Figure2.3 and figure2.4 show examples of uniformly distributed and Gaussian distributed random sequences respectively.

2.1.2 Types of Sequences

The discrete time signals that between counter can be classified in a number of ways.
Some basic classifications that are of interest to us are described below



Real vs. Complex Signals

A sequence is considered complex at least one sample is complex-valued.

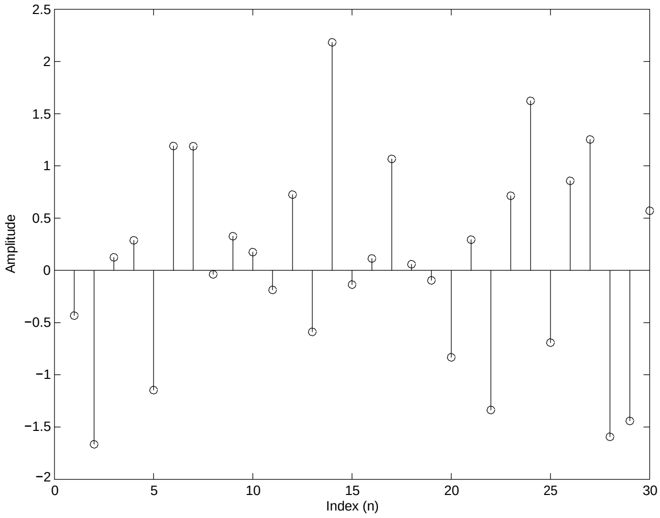


Figure2.4:Gaussian distributed random sequence with zero mean and unit variance.

Finite vs. Infinite Length Signals

Finite length sequences are defined only for arrange of indices, say N_1 to N_2. The length of this finite length sequence is given by $|N_2-N_1+1|$.

Causal Signals A sequence $x[n]$ is a causal sequence if $x[n]=0$ for $n<0$.

Symmetric Signals First consider a real-valued sequence $\{x[n]\}$. Even symmetry implies that $x[n]=x[-n]$ and for odd symmetry $x[n]=-x[n]$ for all n.

Any real-valued sequence can be decomposed into odd and even parts so that

$$x[n]=x_e[n]+x_o[n]$$

Where the even part is given by

$$x_e[n]=(x[n]+x[-n])$$

and the odd part is given by

$$x_o[n]=(x[n]-x[-n])$$

A complex valued sequence is conjugate symmetric If $x[n]=x[-n]$. The sequence has conjugate anti-symmetry if $x[n]=-x[-n]$. Analogous to real valued sequences, any complex-valued sequence can be decomposed into its conjugate symmetric and conjugate anti-symmetric parts:

$$x[n] = x_{cs}[n]+x_{ca}[n] \quad (2.7)$$

$$x_{cs}[n] = \frac{1}{2}(x[n]+x[-n]) \quad (2.8)$$

$$x_{ca}[n] = \frac{1}{2}(x[n]-x[-n]) \quad (2.9)$$

Periodic Signals A discrete-time sequence is periodic with a period of N samples if

$$x[n]=x[n+kN] \quad (2.10)$$

for all integer values of k. Note that N has to be a positive integer. If a sequence is not periodic, it is aperiodic or non-periodic.

We know that continuous-time sinusoidal is periodic. For instance, the continuous-time signal

$$x(t)= \cos(\omega_0 t) \quad (2.11)$$

has a frequency of ω_0 radians per second or $f_0=\omega_0/2\pi$Hz. The period of this sinusoidal signal is $T=1/f_0$ seconds.

Now consider a discrete-time sequence $x[n]$ based on a sinusoidal with angular frequency ω_0:

$$x[n]= \cos(\omega_0 n) \quad (2.12)$$

If this sequence is periodic with a period of N samples, then the following must be true:

$$\cos\omega_0(n+N)= \cos\omega_0 n \quad (2.13)$$

However, the left hand side can be expressed as

$$\cos\omega_0(n+N)= \cos(\omega_0 n+\omega_0 N) \quad (2.14)$$

and the cosine function is periodic with a period of 2π and therefore the right hand side of (2.13) is given by

$$\cos\omega_0 n= \cos(\omega_0 n+2\pi r) \quad (2.15)$$

for integer values of r. Comparing (2.14) with (2.15), we have

$$\begin{aligned}\omega_0 N &= 2\pi r\\ \Rightarrow 2\pi f_0 N &= 2\pi r\\ \Rightarrow f_0 &= \frac{r}{N}\end{aligned} \qquad (2.16)$$

Where $\omega_0=2\pi f_0$. Since both r and N are integers, a discrete time sinusoidal sequence is periodic if its frequency is a rational number. Otherwise, it is non-periodic.

Problem 2.1. Is $x[n]=\cos n\frac{\pi}{8}$ periodic? If so, what is the period?

The sequence can be expressed as

$$x[n]=\cos 2\pi \frac{1}{16} n$$

So in this case, $f_0=1/16$ is a rational number and the sinusoidal sequence is periodic with a period $N=16$.

Problem 2.2. Determine the fundamental period of the following sequence:

$$x[n]=\cos(1.1\pi n)+\sin(0.7\pi n)$$

Solution:

For the cosine function, the angular frequency is

$$\omega_1=1.1\pi=2\pi(0.55)=2\pi f_1$$

Therefore,

$$\begin{aligned}f_1 &= \frac{55}{100}\\ &= \frac{11}{20}\end{aligned}$$

And the period is $N_1=20$.

For the sine function, the angular frequency is

$$\omega_2=0.7\pi= 2\pi(0.35)=2\pi f_2$$

Where

$$f_2 = \frac{35}{100} = \frac{7}{20}$$

And the period is $N_2=20$.

So the period of $x[n]$ is

20.

It is interesting to note that for discrete time sinusoidal sequences, a small change in frequency can lead to a large change in period. For Problem, a certain sequence has frequency $f_1=0.51=51/100$. So its period is 100 samples. Another sinusoidal sequence with frequency $f_2=0.5=50/100$ has a period of only 2 samples since f_2 can be simplified to 1/2. Thus a frequency difference of 0.01 can cause the period of the two sinusoidal sequences to differ by 98 samples.

Another important point to note is that discrete-time sinusoidal sequences with frequencies separated by an integer multiple of 2π are identical.

Energy and Power Signals The energy of a finite length sequence $x[n]$ is defined as

$$E=\sum_{n=N_1}^{N_2} |x[n]|^2 \qquad (2.17)$$

while that for an infinite sequence is

$$E=\sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad (2.18)$$

Note that the energy of an infinite length sequence may not be infinite. A signal with finite energy is usually referred to as an energy signal.

Problem 2.3. Find the energy of the infinite length sequence

$$x[n]= \begin{cases} 2^{-n}, & n\geq 0 \\ 0, & n<0 \end{cases}$$

According to the definition, the energy is given by

$$E = \sum_{n=0}^{\infty} 2^{-2n} = \sum_{n=0}^{\infty} 1/4^{n}$$

To evaluate the finite sum, first consider

$$S_N = \sum_{n=0}^{N-1} a^n = 1 + a + a^2 + \cdots + a^{N-1} \quad (2.19)$$

Multiplying this equation by a, we have

$$aS_N = a + a^2 + \cdots + a^N \quad (2.20)$$

and the difference between these two equations give

$$S_N - aS_N = (1-a)S_N = 1 - a^N \quad (2.21)$$

Hence if $a \neq 1$,

$$S_N = \frac{1-a^N}{1-a} \quad (2.22)$$

For $a=1$, it is obvious that $S_N = N$. For $a<1$, the infinite sum is therefore

$$S_\infty = \frac{1}{1-a} \quad (2.23)$$

Making use of this equation, the energy of the signal is

$$E = \frac{1}{1-1/4} = \frac{4}{3}$$

Equations (2.22) and (2.23) are very useful and we shall be making use of them later. The average power of aperiodic sequence with a period of N samples is defined as

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 \quad (2.24)$$

And for non-periodic sequences, it is defined in terms of the following limit if it exists:

$$P_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2 \quad (2.25)$$

A signal with finite average power is called a power signal.

Problem2.4. Find the average power of the unit step sequence $u[n]$.The unit step sequence is non-periodic,therefore the average power is

$$P = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=0}^{K} u^2[n]$$
$$= \lim_{K\to\infty} \frac{K+1}{2K+1}$$
$$= \frac{1}{2}$$

Therefore the unit step sequence Is a power signal.Note that its energy is infinite and so it is not an energy signal.

Bounded Signals A sequence is bounded if every sample of the sequence has a magnitude which is less than or equal to a finite positive value.That is,

$$|x[n]| \leq B_x < \infty \tag{2.26}$$

SummableSignals A sequence is absolutely summable if the sumof the absolute value of all its samples is finite.

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \tag{2.27}$$

A sequence is square summable if the sum of the magnitude squared of all its samples is finite.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \tag{2.28}$$

2.1.3 Some Basic Operationson Sequences

Scaling Scaling is the multiplication of a scalar constant with each and every sample value of the sequence.This operation is depicted schematically in Figure2.5.

Addition Addition of two sequences usually refers to point-by - point addition as shown in Figure2.6.

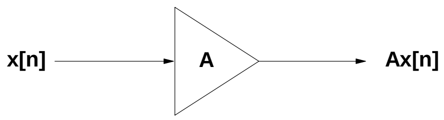


Figure2.5:Scalar Multiplication byA.

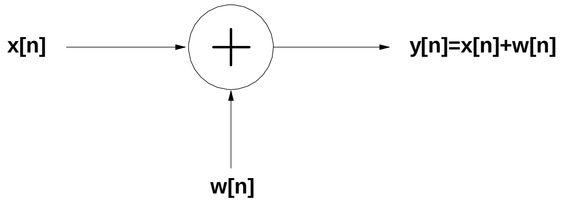


Figure2.6: Point-by-pointAddition

DelayA unit delay shifts as equence by one unit of time as shown in Figure2.7.A sequence $x[n]$ is delayed by N samples to produce $y[n]$ if $y[n]=x[n-N]$.

Up/DownSampling Down-sampling by a factor of L(a positive integer) is an operation by which only one every Lth sample of the original sequence is kept,with therest discarded.The schematic for a down-sampler is shown in Figure2.8.The down-sampled signal $y[n]$ is given by

$$y[n]=x[nM] \tag{2.29}$$

Up-sampling is the opposite operation to down-sampling.It increases the number of samples of the original sequence by a certain factor L(appositive integer).This is done by inserting $L-1$ zeros between a chpair of original samples.Figure shows the schematic

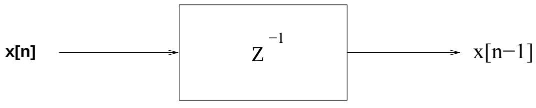


Figure2.7:A Unit Delay

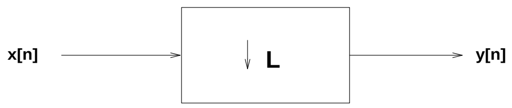


Figure2.8:Down-sampling by a factor of L.

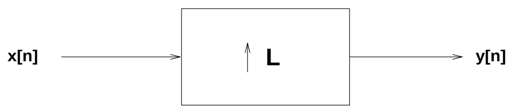


Figure2.9:Up-sample by a factor of L. diagram

of a nup-sampler and the up-sampled sequence $y[n]$ is given by

$$y[n]=\begin{cases} x\left[\frac{n}{L}\right], & n=0,\pm L,\pm 2L,\ldots \\ 0, & \text{otherwise} \end{cases} \tag{2.30}$$

An interpolated signal can be obtained by passing the up-sampled signal through a low-passfilter with an appropriate bandwidth .This process will be discussedinmoredetailinalaterchapter.

Modulation Given two sequences $x[n]$ and $w[n]$, and

$$y[n]=x[n]\cdot w[n] \tag{2.31}$$

Then we say that $y[n]$ is $w[n]$ modulated by $x[n]$.This is analogous to carrier modulation in communication systems.

Correlation The correlation ,or more precisely cross-correlation,between two of infinite legth data sequences $x[n]$ and $w[n]$ is defined by

$$r=\frac{1}{N}\sum_{n=0}^{N} x[n]w[n] \tag{2.32}$$

if each sequence is of length N.The correlation coefficient is of then used as a measure of how similar the two sequences are.If they are very different,then the value of$r$ is low.

The matched filter used in digital communication receivers for optimal detection is also effectively a correlator between the incoming and the template signals.

2.2 Discrete-timeSystems

A discrete-time system is one that processes a discrete-time input sequence to produce a discrete-time output sequence.There are many different kinds of such systems.

2.2.1 Classification of Systems

Discrete-time systems, like continuous-time systems,can be classified in a variety of ways.

Linearity A linear system is one which obeys the superposition principle. For a cer-tain system,let the outputs corresponding to inputs $x_1[n]$ and $x_2[n]$ are $y_1[n]$ and $y_2[n]$ respectively. Now if the input is given by

$$x[n]=Ax_1[n]+Bx_2[n] \tag{2.33}$$

where A and B are arbitrary constants,then the system is linear if its corresponding output is

$$y[n]=Ay_1[n]+By_2[n] \tag{2.34}$$

Superposition is a very nice property which makes analysis much simpler.Although many real systems are not entirely linear throughout its operating region (for instance,the bipolar transistor), they can be considered approximately linear for certain input ranges.Linearization Is a very useful approach to analyzing non linear systems.Almost all the discrete-time systems considered in this book are linear systems.

Problem2.5. Are the down-sampler and up-sampler linear systems?

Consider the down-sampler

$$y[n]=x[nM]$$

For input $x_1[n]$, the corresponding output is $y_1[n]=x_1[nM]$.For input $x_2[n]$,the output is $y_2[n]=x_2[nM]$.Let $x[n]$be a linearcombination of these two input with arbitrary constants A and B so that

$$x[n]=Ax_1[n]+Bx_2[n]$$

The output is given by

$$\begin{aligned}y[n]&=Ax_1[nM]+Bx_2[nM]\\&=Ay_1[n]+By_2[n]\end{aligned}$$

Therefore the down-sampler is a linear system .

Now consider the up-sampler

$$y[n]=\begin{cases} x[n/L], & n=0,\pm L,\pm 2L,\ldots \\ 0, & \text{otherwise} \end{cases}$$

Let $y_1[n]$ and $y_2[n]$ be the outputs for inputs $x_1[n]$ and $x_2[n]$ respectively. For

$$x[n]=Ax_1[n]+Bx_2[n]$$

then the output is

$$y[n]=\begin{cases} Ax_1[n/L]+Bx_2[n/L], & n=0,\pm L,\pm 2L,\ldots \\ 0, & \text{otherwise} \end{cases}$$

$$=Ay_1[n]+By_2[n]$$

Hence the up-sampler is also linear.

Shift Invariance A shift (or time) invariant system is one that doesnot change with time.Let a system response to an input $x[n]$ be $y[n]$. If the input is now shifted by n_0(an integer)samples,

$$x_1[n]=x[n-n_0] \tag{2.35}$$

then the system is shift invariant if its response to $x_1[n]$ is

$$y_1[n]=y[n-n_0] \tag{2.36}$$

In the remainder of this book,we shall use the terms linear time-invariant(LTI) and linear shift-invariant interchangeably.

Problem2.6.A system has input-output relationship given by

$$y[n]=\sum_{k=-\infty}^{n} x[k]$$

Is this system shift - invariant?

If the input is now $x_1[n]=x[n-n_0]$,then the corresponding output is

$$y_1[n]=\sum_{k=-\infty}^{n} x_1[k]$$

$$=\sum_{k=-\infty}^{n} x[k-n_0]$$

$$=\sum_{k=-\infty}^{n-n_0} x[k]$$

$$=y[n-n_0]$$

Therefore the system is shift invariant.

Problem2.7. Is the down-sampler a shift invariant system ?

Le tM (appositive integer)be the down-sampling ratio . So for an inpu t$x[n]$ the output is $y[n]=x[nM]$.Now if $x_1[n]$ is $x[n]$ delayed by n_0 samples ,then

$$x_1[n]=x[n-n_0]$$

and the corresponding output is

$$y_1[n]=x[nM-n_0]$$
$$=x\left[\left(n-\frac{n_0}{M}\right)M\right]$$

If the system is shift invariant , one would expect the output to be

$$y[n-n_0]=x[(n-n_0)M]$$

Since this is not the case,the down-sampler must be shift variant.

Causality The response of a causal system at any time depends only on the input at the current and past instants , not on any “future” samples. In other words, the output sample $y[n_0]$ for any n_0 only depends on $x[n]$ for $n\leq n_0$.

Problem2.8. Determine if the following system is causal:

$$y[n]=\sum_{k=-\infty}^{\infty} (n-k)u[n-k]x[k]$$

Note that $u[n-k]=0$ for $n<k$ be cause the unitstep sequence is zero for negative indices.In otherwords,for a certain n, $u[n-k]=0$ for $k>n$.So the output can be written as

$$y[n]=\sum_{k=-\infty}^{n}(n-k)x[k]$$

So $y[n]$ depends on $x[k]$ for $k\leq n$ and therefore the system is causal.

Stability There are two common criteria for system stability. They are exponential stability and bounded- input bounded- output(BIBO) stability.The first criterion is more stringent.It requires the response of the system to decay exponentially fast for a finite duration input. The second one merely requires that the output be a bounded sequence if the input is a bounded sequence.

Problem2.9. Determine if the system with the following input-output relationship is BIBO stable.

$$y[n]=\sum_{k=-\infty}^{\infty}(n-k)u[n-k]x[k]$$

Consider inpu t$x[n]=\delta[n]$. Then

$$y[n]=\sum_{k=-\infty}^{\infty}(n-k)u[n-k]\delta[k]$$
$$=nu[n]$$

Which is unbounded as It grows linearly with n . Therefore the system is not BIBO stable.

Problem2.10. Determine if the following system is BIBO stable. Note that this system Is an “averager” ,taking the average of the past M samples.

$$y[n]=\frac{1}{}\sum_{k=0}^{M-1}x[n-k]M$$

Let the input $|x[n]| \leq B$ for some finite value B.Consider the magnitude of the output

$$\leq \frac{1}{M} \sum_{k=0}^{M-1} |x[n-k]|$$
$$\leq \frac{1}{M} (MB)$$
$$= B$$

Hence the output is bounded and the system is BIBO stable.

LossyorLossless For a passive system that does not generate any energy internally,the output should have atmost the same energy as the input. So

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \qquad (2.37)$$

A losslesssystem is one which thee quality holds.

2.2.2 LinearShift – Invariant Systems

An discrete-time LTI system, like its continuous-time counter part,is completely characterized by it simpulse response.In otherwords,the impulse response tells us everything we need to know about an LTI system as far as signal processing is concerned.The impulse response is simply the observed system output when the input is an impulse sequence.For continuous-time systems,the impulse function is purely a mathematical entity.However,for discrete-time systems,since we are dealing with sequences of numbers,the impulses equence can realistically (andeasily) be generated.

2.2.2.1 LinearConvolution

Let us consider a discrete-time LTI system with impulse reponse $h[n]$ as shown in Figure 2.10.What would be the output $y[n]$ of the system if the input $x[n]$ is as shown In Figure2.11?

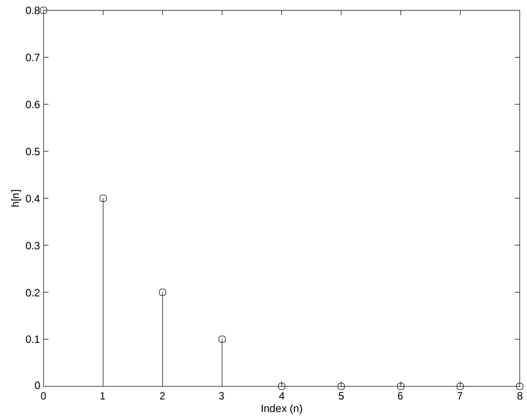


Figure2.10:Impulse Response of the System Consideredin Section 2.2.2.1

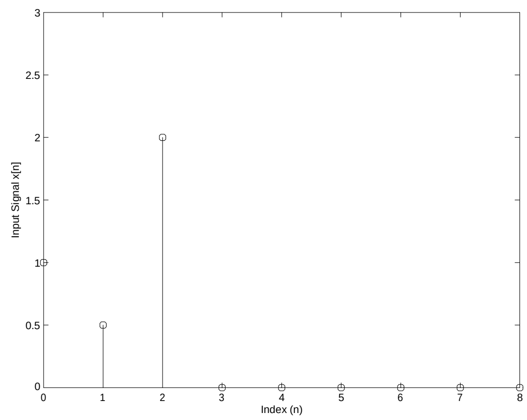


Figure2.11:InputSignalSequence

Since the system is linear and time invariant , we can make use of the superposition prin-ciple to compute the output.The input sequence is composed of three impulses.Mathe-matically,it can be expressed as

$$x[n]=\delta[n]+0.5\delta[n-1]+2\delta[n-2] \quad (2.38)$$

Let

$$\begin{aligned} x_1[n] &= \delta[n] \\ x_2[n] &= 0.5\delta[n-1] \\ x_3[n] &= 2\delta[n-2] \end{aligned}$$

And the system response to each of these inputs are respectively $y_1[n]$, $y_2[n]$ and $y_3[n]$. The sample values of $y_1[n]$ are given by

$$\begin{aligned} y_1[0]=h[0]x_1[0]=0.8 \\ y_1[1]=h[1]x_1[0]=0.4 \\ y_1[2]=h[2]x_1[0]=0.2 \\ y_1[3]=h[3]x_1[0]=0.1 \end{aligned}$$

which is the same as the impulse response since $x_1[n]$ is a unit impulse.Similarly,

$$\begin{aligned} y_2[1]=h[0]x_2[1]=0.4 \\ y_2[2]=h[1]x_2[1]=0.2 \\ y_2[3]=h[2]x_2[1]=0.1 \\ y_2[4]=h[3]x_2[1]=0.05 \end{aligned}$$

and

$$\begin{aligned} y_3[2]=h[0]x_3[2]=1.6 \\ y_3[3]=h[1]x_3[2]=0.8 \\ y_3[4]=h[2]x_3[2]=0.4 \\ y_3[5]=h[3]x_3[2]=0.2 \end{aligned}$$

The system output $y[n]$ in respon set of input $x[n]$ is therefore,through the superposition principle,given by

$$\begin{aligned} y[n] &= y_1[n]+y_2[n]+y_3[n] \\ &=\{0.8,0.8,2,1,0.45,0.2\} \end{aligned}$$

Note that

$$\begin{aligned}
y[0] &= h[0]x[0] \\
y[1] &= h[1]x[0]+h[0]x[1] \\
y[2] &= h[2]x[0]+h[1]x[1]+h[0]x[2] \\
y[3] &= h[3]x[0]+h[2]x[1]+h[1]x[2] \\
y[4] &= h[3]x[1]+h[2]x[2] \\
y[5] &= h[3]x[2]
\end{aligned}$$

In general,we have

$$y[n]=h[n]x[0]+h[n-1]x[1]+...+h[1]x[n-1]+h[0]x[n] \qquad (2.39)$$

or

$$y[n]=\sum_{k=0}^{n} h[k]x[n-k] \qquad (2.40)$$

Alternatively,

$$y[n]= \sum_{k=0}^{n} x[n]h[n-k] \qquad (2.41)$$

Equations 2.40 and 2.41 are the linear convolution equations for finite length sequences.If the length of $h[n]$ is M and the length of $x[n]$ is N,then the length of $y[n]$ is $N+M-1$.

We can further generalize it for infinite length sequences:

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad (2.42)$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad (2.43)$$

These equations are analogous to the linear convolution equation for continuous-time signals.Note that the linear convolution equation comes about as a result of the superposition principle and therefore applies only to LTI systems.

The convolution equation for discrete-time signals is also called the convolution sum.It is denoted by *.So equations 2.42 and 2.43can be written as

$$y[n]=x[n]*h[n] \qquad (2.44)$$

The convolution sum is one of the most important fundamental equations in DSP.

2.2.2.2 Properties of Linear Convolution

The convolution sum has three important properties:

1. Commutative

$$x[n]*y[n]=y[n]*x[n] \tag{2.45}$$

2. Associative

$$(x[n]*w[n])*y[n]=x[n]*(w[n]*y[n]) \tag{2.46}$$

3. Distributive

$$x[n]*(w[n]+y[n])=x[n]*w[n]+x[n]*y[n] \tag{2.47}$$

2.2.2.3 Condition for Stability

Since the impulse response completely characterize an LTI system,we should be able to draw conclusions regarding the stability of a system based on its impulse response.We shall consider BIBO stability here.

Theorem2.1.A discrete –time LTI system is BIBO stable If its impulse response is abso-lutely summable.

Proof. Let the input be bounded, i.e. $|x[n]|<B<\infty$ for some finite value B. The magnitude of the output is given by

$$
\begin{aligned}
|y[n]| &= \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \\
&\leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]| \\
&\leq B \sum_{k=-\infty}^{\infty} |h[k]|
\end{aligned}
$$

So the magnitude of $y[n]$ is bounded if $\sum_{k=-\infty}^{\infty}|h[k]|$ is finite.In otherwords,the impulse response must be absolutely summable. □

2.2.2.4 ConditionforCausality

Theorem2.2.A discrete-time LTI system is causal if and only if its impulse response is a causal sequence.

Proof. Consider an LTI system with impulse response $h[k]$. Two different inputs $x_1[n]$ and $x_2[n]$ are the same upto a certain point in time, that is $x_1[n]=x_2[n]$ for $n\leq n_0$ for some n_0. The outputs $y_1[n]$ and $y_2[n]$ at $n=n_0$ are given by

$$y_1[n_0]= \sum_{k=-\infty}^{\infty} h[k]x_1[n_0-k]$$
$$\sum_{k=-\infty}^{-1} h[k]x_1[n_0-k]+ h[k]x_1[n_0-k]$$

and

$$y_2[n_0]= \sum_{K=-\infty}^{\infty} h[k]x_2[n_0-k]$$

Since $x_1[n]=x_2[n]$ for $n\leq n_0$,if the system is causal,then the outputs $y_1[n]$ and $y_2[n]$ Must be the same for $n\leq n_0$.More specifically,$y_1[n_0]=y_2[n_0]$.Now,

$$\sum_{k=0}^{\infty} h[k]x_1[n_0-k]= \sum_{k=0}^{\infty} h[k]x_2[n_0-k]$$

because $x_1[n_0-k]=x_2[n_0-k]$ for non-negative values of k.Since $x_1[n]$ may not be equal to $x_2[n]$ for $n>n_0$, we must have

$$\sum_{k=-\infty}^{-1} h[k]x_1[n_0-k]= \sum_{k=-\infty}^{-1} h[k]x_2[n_0-k]=0$$

which means that $h[k]=0$ for $k<0$. □

2.2.3 FIR and IIR Systems

In DSP ,finite impulse response (FIR) and infinite impulse response(IIR) systems and usually referred to as FIR and IIR filters.

An causal FIR filter has impulse response

$$h[n]=\{h_0,h_1,h_2,\cdots,h_M,0,0,\cdots\}$$

where $h_0,h_1,h_2,\cdots,h_M$ are called filter coefficients.The length of the impulse response is $M+1$ and It is called an M-th order filter.

Problem2.11. What is the impulse response of a causal LTI system with the following input-output relationship?

$$y[n]=2x[n]+x[n-1]-x[n-3]$$

Impulse response is the system’s output when the input is a unit impulse signal.Therefore,

$$\begin{aligned} h[n]&=2\delta[n]+\delta[n-1]-\delta[n-3] \\ &= \{2,1,0,-1\} \end{aligned}$$

This filter is an FIR filter.

The filtering equation is essentially the linear convolution equation.Since the impulse response has a length of $M+1$, the output can be computed as(assuming a causal filter):

$$y[n]=\sum_{k=0}^{M} h[k]x[n-k] \qquad (2.48)$$

The impulse response of an IIR system,however,has infinite length.So the filtering equation for a causal filter is given by

$$y[n]=\sum_{k=0}^{\infty} h[k]x[n-k] \qquad (2.49)$$

This equation is not computable practice due to the infinite limit.But there is a type of IIR systems that admits recursive computation of $y[n]$ and so it can be computed.This

Type of IIR systems are the only IIR filters that are used in DSP. The general input-output relationship of these IIR filters is

$$y[n]=\sum_{k=1}^{N} a[k]y[n-k]+\sum_{k=0}^{M} b[k]x[n-k] \qquad (2.50)$$

where $\{a[k]\}$ and $\{b[k]\}$ are the filter coefficients.The present output $y[n]$ is a linear combination of previous N outputs and the present and previous M outputs.The order of the filter Is $M-1$ or N, whichever is larger.

Problem2.12.A causal LTI system has impulse response

$$h[n]=ah[n-1]+\delta[n]$$

where a is a constant. What is the input-output equation for the system?

Since $y[n]=h[n]$ for $x[n]=\delta[n]$, it is obvious from the impulse response that the input-output equation for this system is

$$y[n]=ay[n-1]+x[n]$$

Problem2.13. Find the impulse response of the causal system with the following input-output equation:

$$y[n]=0.25y[n-2]+x[n]$$

The impulse response of this system is given by

$$h[n]=0.25h[n-2]+\delta[n]$$

and so

$$h[0]=0.25h[-2]+\delta[0]=1$$
$$h[1]=0.25h[-1]+\delta[1]=0$$
$$h[2]=0.25h[0]+\delta[2]=(0.25)^2$$
$$h[3]=0.25h[1]+\delta[3]=0$$

. .

Hence

$$h[n]=\begin{cases}(0.5)^n, n\geq 0, & \text{and even}\\ 0, & \text{otherwise}\end{cases}$$

2.3 Summary

In this chapter,we have studied the fundamentals of discrete-time signals and systems in the time domain.Discrete-time signals are essentially a sequence of numbers.Some fundamental sequences,such as the unit impulse,unit step and sinusoidal(both real and complex)sequences are examined because more complex sequences can be expressed as a combination of some or all of these fundamental ones.

The most important discrete-time systems are the linear time invariant(LTI)systems.For these systems,the superposition principle applies which leads to the linear convolution sum.This convolution sum is the way by which we derive the output of the system given an input.An LTI system is completely characterized by its impulse response.All the properties of such a system is revealed by its impulse response,including causality and BIBO stability.