UNIT – V

NUMERICAL METHODS

Approximate methods for determining fundamental frequency, Dunkerleys lower bound, Rayleighs upper bound**, Holzer method for closed coupled system and branched system**.

Dunkerlyes equation.

It relates the fundamental frequency of a composite system to the frequencies of its component parts. It is based on the fact that modal frequencies of most system for higher modes are high wth respect to their fundamental frequency. It is an approximate equation and can be derived from an algebraic rule.

Damped Natural frequency.

The damped natural frequency is that frequency of free vibration of a damped linear system. The free vibration of a damped system may be considered periodic in the limited sense that the time interval between zero crossings in the same direction is constant. Even though successive amplitudes decrease progressively.

Dunkerleys lower bound method to determine the frequency of a system.

Consider the eigen value problem in the form of for n degree of freedom system , the frequency equation will be

$$Z^n - C_{n-1} Z^{n-1} + \ldots + C_0 = 0 \qquad (1.1)$$

Where C_{n-1} coefficient from theory of questions represents sum of all the roots of the equation from (6.24), we can write

$$C_{n-1} = \alpha_{11}m_1 + \alpha_{22}m_2 + \ldots + \alpha_{nn} m_r$$

$$= 1/p_1^2 + p_2^2 + \ldots + 1/p_n^2$$

(1.2)

Since influence coefficient $\alpha_{ii} = 1/K_{ii}$ we define

$$p_n^2 = 1/ \alpha_{11}m_1 \qquad (1.3)$$

Where p_{ii} represents the natural frequency of the system with only the i^{th} mass considered

Equation (1.2) now becomes

$$1/p_1^2 + p_2^2 + \ldots + 1/p_n^2 = 1/p_{11}^2 + p_{22}^2 + \ldots + 1/p_{nn}^2$$

(1.4)

Since $p_1 < p_2 < \ldots < p_n$, we can write the above as

$$1/p_n^2 = \sum 1/ p_{ii}^2 \qquad (1.5)$$

The above is Dunkereley's formula and because of the removal of $p_2, \ldots\ldots,$ p_n terms of the left hand side of equation (1.4), p_1 estimated by (1.5) is always less than the exact value . The simplicity of the method lies in the fact that p_1 can be estimated by considering several single freedom systems with masses m1, m2 etc., considered individually, thus reducing the multi degree of freedom system calculations to single degree of freedom system calculations.

Rayleighs upper bound method to determine the frequency of a system.

Consider the multi degree of freedom system with [M] and [K] representing its mass and stiffness matrices as in equation (6.7). Let [X] be a modal vector (as in equation (6.11) with its column representing ith mode shape corresponding to its natural frequency pi) and for the case of harmonic motion with a frequency ω, the maximum kinetic and potential energies are

$$\check{T} = ½\, \omega^2 \{X\}^T \{M\} \{X\}$$
$$\check{U} = ½ \{X\}^T \{K\} \{X\} \qquad (1.6)$$

So,

$$\omega^2 = \{X\}^T \{K\} \{X\}/\{X\}^T \{M\} \{X\} \qquad (1.7)$$

The above equation is known as Rayleigh's quotient. If ω is a natural frequency and {X} is corresponding modal vector, (1.7) will be exactly satisfied. However, neither of them is known at this stage of calculations. Let us assume a modal vector {X} consistent with the kinematics boundary conditions of the system. As in modal expansion, let {X} be expressed in terms of orthonormal modal vectors

$$\{\overline{X}\} = \{\overline{X}^1\}+ \overline{C}_2\{X^2\}+\overline{C}_3\{X^3\}+\ldots \qquad (1.8)$$

Substituting the above for {X} in (1.7) and noting that

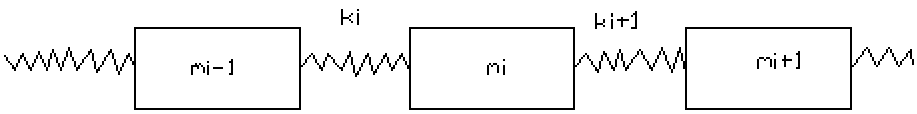
$$\omega^2 = p_1^2+ C_2^2\, p_2^2+\ldots /1+ C_2^{2+\ldots}$$

If $\{\overline{X}\}$ is close to $\{\overline{X}^1\}$, then $C_2 << 1$, $C_3 < C_2$, then

$$\omega^2 = p_1^2 (1+ C_2^2\, p_2^2/ p_1^2+\ldots)$$
$$= p_1^2 \qquad (1.9)$$

ω^2 determined from (1.9) is always greater than the exact value p_1^2. Since Dunkerleys lower bound method gives a lower bound value, the exact frequency of a system lies in between Dunkereley's and Rayleighs approximations.

Holzer method for closed coupled system.



Consider the close coupled system of fig. again, the i^{th} mass and neighboring elements of which are shown. We measure the displacement, velocity and acceleration positive along the outward normal. The displacement X and the force F will define the state vector.

$$\{S\} = \begin{matrix}\{X\} \\ \{F\}\end{matrix}$$

We use a suffix to denote the station number and a superscript R or L to denote the quantities to the right or left of a station respectively. The equation of motion for i^{th} mass is

$$m_i\overline{x_i} = Fi^R - Fi^L$$

The displacement of mass m_i is

$$Xi^R = Xi^L = Xi$$

We combine the equations

$$\{S\}i^R = [P]i\ \{S\}\ i^L$$

[P] is the point matrix which defines the transfer function to obtain the state vector to the right of a station in terms of the state vector to the left of a station.

The point matrix is a function of the mass of the station and the harmonic frequency ω. For an assumed value of ω, the point matrices for all station can be set up.

Now we consider the force field of the spring Ki and observe that

$$\{S\}i^L = [F]_i\ \{S\}_{i-1}{}^R$$

[F] is the field matrix which define the transfer function across field. The field matrix is a function of the stiffness of the system only and can be set up for all stations i.

$$\{S\}i^R = [T]_i\ \{S\}_{i-1}{}^R$$

[T] is the transfer matrix.

It is important to maintain the order of matrix multiplication and the station numbering in using the above transfer matrix. We can use equation for transfer of state vector from station successively to obtain the overall transfer matrix of the systems.

$\{S\}_1^R = [T]_1 \{S\}_0$

$\{S\}_{n+1} = [U] \{S\}_0$

[U] is the overall transfer matrix of the system.

Procedure to find determine a natural frequency, we adopt the following procedure.

1. Assume a value of ω2 representing the desired natural frequency. This may be obtained by making a crude model with few stations or by experience.

2. Set up transfer matrices as in equation for all stations. At the end points determine the required point or field matrices.

3. Determine the overall transfer matrix as in equation.

4. Change ω^2 by a suitable increment and repeat steps 1 to 3.

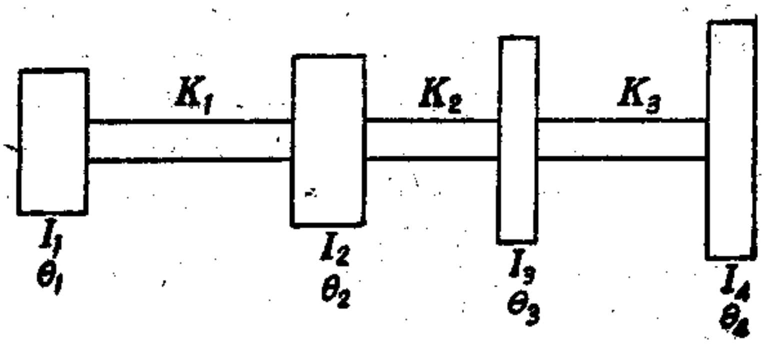
5. Plot u_{12} vs ω^2 and find the value ω^2 for which u_{12} is zero. This value of ω^2 is a natural frequency.

Holzer method for branched system.

In several mechanical systems, like ship propulsion systems, strip steel mill stands, machine tool drives etc.., there may be one or two branch points, as the employ one or two drivers driving one or two driven members.

The Holzer's Method

Consider the system shown in fig.
Equations of motion are



$$I_1\theta + K_1(\theta_1 - \theta_2) = 0$$

$$I_2\theta_2 + K_1(\theta_2 - \theta_1) - K_2(\theta_3 - \theta_2) = 0 \qquad \rightarrow (1)$$

$$I_3\theta_3 + K_2(\theta_3 - \theta_2) - K_3(\theta_4 - \theta_3) = 0$$

$$I_4\theta_4 + K_3(\theta_4 - \theta_3) = 0$$

Assuming

$$\theta_1 = \gamma_1 \cos\ \omega_n t$$

$$\theta_2 = \gamma_2 \cos\ \omega_n t \qquad \rightarrow (2)$$

$$\theta_3 = \gamma_3 \cos\ \omega_n t$$

$$\theta_4 = \gamma_4 \cos\ \omega_n t$$

and substituting in equation (1) and rearranging, we get

$$-I_1\omega_n 2\gamma_1 = K_1(\gamma_2 - \gamma_1)$$

$$-I_2\omega_n 2\gamma_2 = -K_1(\gamma_2 - \gamma_1) + K_2(\gamma_3 - \gamma_2) \qquad \rightarrow (3)$$

$$-I_3\omega_n 2\gamma_3 = -K_2(\gamma_3 - \gamma_2) + K_3(\gamma_4 - \gamma_3)$$

$$-I_4\omega_n 2\gamma_4 = -K_3(\gamma_4 - \gamma_3)$$

Adding these, equation (3) results in the right hand side being zero.

So in general

$$\sum I\omega_n 2\gamma_n = 0 \qquad \rightarrow (4)$$

Where summation is for all the masses.
This means that we can find the natural frequency by trial till equation (4) is satisfied. This is the basis Holzer's Method.

Further assuming that γ_1 = 1 radian (since we are interested only in the relative amplitudes) we get

$$\gamma_2 = \gamma_1 - \frac{I_1\omega_n 2\gamma_1}{K_1}$$

$$\gamma_3 = \gamma_2 - \frac{I_1\omega_n 2\gamma_1 + I_2\omega_n 2\gamma_2}{K_2} \qquad \rightarrow (5)$$

$$\gamma_4 = \gamma_3 - \frac{I_1\omega_n 2\gamma_1 + I_2\omega_n 2\gamma_2 + I_3\omega_n 2\gamma_3}{K_a}$$

Or in general,

$$\gamma_4 = \left(\frac{1}{K_i - 1}\right)\left[K_{i-1}\gamma_{i-1} - \omega_n 2 \sum_{1}^{i-1} I\,\gamma_3\right] \quad \rightarrow (6)$$

The Holzer's method then consists in assuming value for the natural frequency and displacement of one of the rotors. Equation (6) then may be used to find the displacements of any other rotor and the sum of the inertia forces. If the system is free at the ends, equation (4) must hold. If it is fixed at same point, equation (6) which can be used to obtain the displacement of that point should yield zero displacement. If it is not zero, another trial must be made with another frequency. Thus a graph may be plotted for displacement vs assumed frequency. The frequency for zero displacement is then the natural frequency. The mode shapes may then be obtained with the help of equation (6).

The method is equally applicable to translational systems.

Holzer's method can be applied to branched systems. Any end rotor could be given a unit displacement to state with without affecting the final result. All amplitudes and moments have to be proportional to this initially assumed displacements. It may be further seen that the joint must be equal, and that the total; moment at the joint including its inertia moment must equal zero.

A single degree of freedom spring mass system has a natural frequency of 10 cycles per second. Another single degree of freedom spring mass system is attached to it. The latter had a natural frequency of 20 cycles/second. What is the approximate fundamental frequency of composite system?

Solution : from Dunkerley's equation

$$\frac{1}{\omega_{1n^2}} \approx \frac{1}{\omega_{11^2}} + \frac{1}{\omega_{22^2}}$$

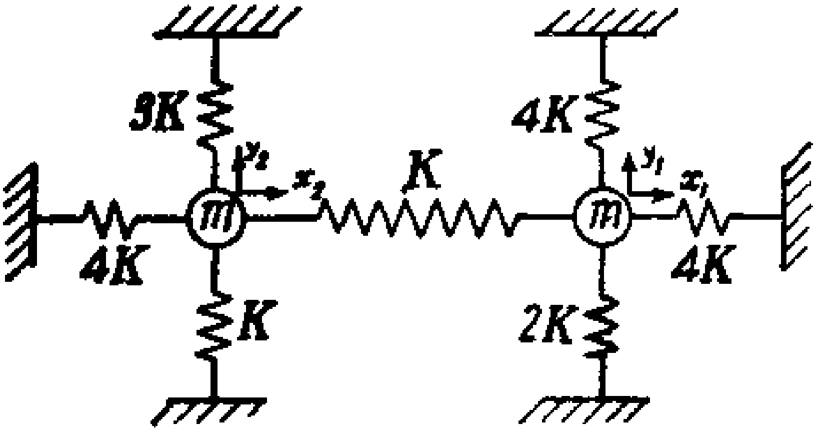
$$= \frac{1}{100} + \frac{1}{400}$$

$$= \frac{5}{400} = \frac{1}{80}$$

or $\omega_{1n^2} = 80$

therefore $\omega = \sqrt{80} =$ 8.95 cycle / second.

6. Find the principal modes of the system shown in fig.



Solution :

Inertia or mass matrix is

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

and the stiffness matrix is K

$$\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$$

So dynamic matrix C is

$$[m]^{-1}[K]$$

$$= \frac{K}{m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$$

$$= \frac{K}{m}\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$$

Eigen values of this matrix are given as

$$\left(\lambda - \frac{5K}{m}\right)^2 - \left(\frac{K}{m}\right)^2 = 0$$

$$\text{or} \quad \begin{vmatrix} \lambda\text{-}\frac{5K}{m} & \frac{K}{m} \\ \frac{K}{m} & \lambda - \frac{5K}{m} \end{vmatrix} = 0 \qquad \to (2)$$

$$\text{so} \quad \lambda = \frac{6K}{m} \text{ or } \frac{4K}{m}$$

Hence

$$\omega_n = \sqrt{\frac{6K}{m}} \text{ and } \sqrt{\frac{4K}{m}}$$

The adjoint of (2) is

$$\begin{bmatrix} \lambda - \frac{5K}{m} & -\frac{K}{m} \\ -\frac{K}{m} & \lambda - \frac{5K}{m} \end{bmatrix}$$

and hence the two principal modes are

$$\begin{Bmatrix} - \\ -1 \end{Bmatrix} \text{ and } \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

For the two cases,

For displacements in vertical direction

$m\ddot{y}_1 + 6Ky_1 = 0$

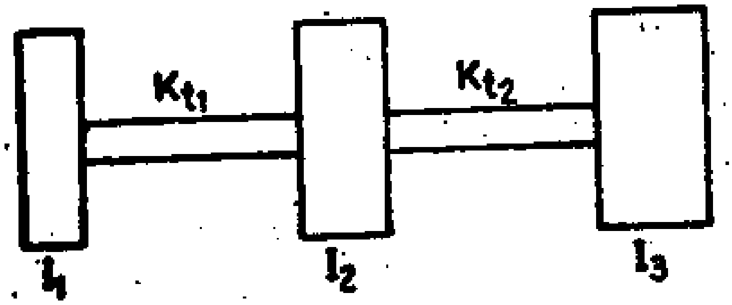
and $m\ddot{y}_2 + 4Ky_2 = 0$ → (3)

and hence the natural frequencies are

$$\sqrt{\frac{6K}{m}} \text{ and } \sqrt{\frac{4K}{m}}$$

in vertical direction. This is a system which has same natural frequencies in both the directions.

Find one natural frequency of the system shown in fig. (a) by the Holzer's Method.



$I_1 = I_2 = I_a$

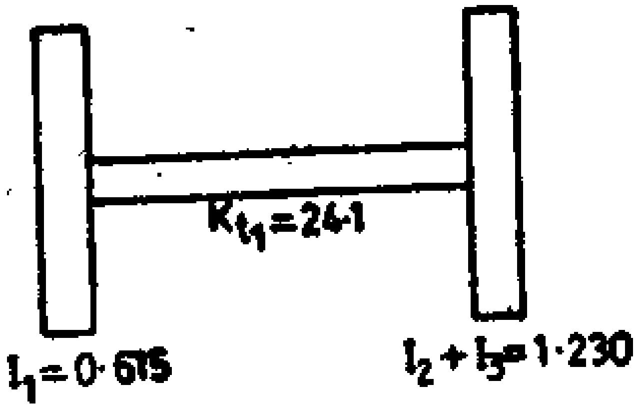
$= 0.615 \text{ kg cm/sec}^2$

$K_{t1} = 24.1 \text{ kg cm/radian}$

$K_{t2} = 25.84 \text{ kg cm/radian}$

Solution : Let us first find approximate natural frequency of the system. For this we will be grouping together discs that have shafts with high relative stiffness between them.

Since here K_{i1} is comparatively greater than K_{t1}, we group together discs 2 and 3, thus the system reduces to a two degree system shown in fig.



$$\omega_n = \sqrt{K_{t1}\left(\frac{1}{I_1}+\frac{1}{I_2}\right)}$$

$$= \sqrt{24.1\left(\frac{1}{0.615}+\frac{1}{1.230}\right)}$$

$$= \sqrt{59}$$

$$\omega^2 = 59$$

So let us start with $\omega^2 = 59$

Trial 1
$\boldsymbol{\omega^2 = 59}$

S.No	I	γ	$I\gamma\omega_{n^2}$	$\sum I\gamma\omega_{n^2}$	Kt	$\frac{1}{Kt}\sum I\gamma\omega^2$
1	0.615	1	36.30	36.30	24.1	1.535
2	0.615	-0.535	-18.15	17.15	25.84	0.664
3	0.615	-1.199	-42.5	-25.35		

The external Torque $\sum I\gamma\omega^2$ should be zero, so choose next approximation so as to achieve this.

Trial 2
$\boldsymbol{\omega_n^2 = 40.3}$

S.No	I	γ	$I\gamma\omega^2$	$\sum I\gamma\omega^2$	Kt	$\frac{1}{Kt}\sum I\gamma\omega^2$
1	0.615	1.0	24.8	24.8	24.1	1.03

2	0.615	-0.03	-0.745	24.055	24.84	0.93
3	0.615	-0.96	-23.1	0.955		

Trial 3
ω^2 = 42

S.No	I	γ	$I\gamma\omega^2$	$\sum I\gamma\omega^2$	Kt	$\frac{1}{Kt}\sum I\gamma\omega^2$
1	0.615	1.0	25.81	25.84	24.1	1.072
2	0.615	-0.072	-1.84	23.94	25.84	0.925
3	0.615	-0.997	-25.7	-1.76		

Trial 4
ω^2 = 41.0

S.No	I	γ	$I\gamma\omega^2$	$\sum I\gamma\omega^2$	Kt	$\frac{1}{Kt}\sum I\gamma\omega^2$
1	0.615	1.0	25.2	25.2	24.1	1.04
2	0.615	-0.047	-1.185	24.015	25.84	0.93
3	0.615	-0.977	-24.6	-0.585		

Trial 5
ω^2 = 40.8

S.No	I	γ	$I\gamma\omega^2$	$\sum I\gamma\omega^2$	Kt	$\frac{1}{Kt}\sum I\gamma\omega^2$
1	0.615	1.0	25.1	25.1	24.1	1.042
2	0.615	-0.042	-1.055	24.045	25.81	0.93
3	0.615	-0.972	-24.39	-0.345		

Trial 6
ω^2 = 40.5

S.No	I	γ	$I\gamma\omega^2$	$\sum I\gamma\omega^2$	Kt	$\frac{1}{Kt}\sum I\gamma\omega^2$
1	0.615	1.0	24.9	24.9	24.1	1.035
2	0.615	-0.035	-0.872	24.028	25.84	0.93
3	0.615	-0.965	-24.028	0		

Hence ω^2 40.5

So exact natural frequency of the system is

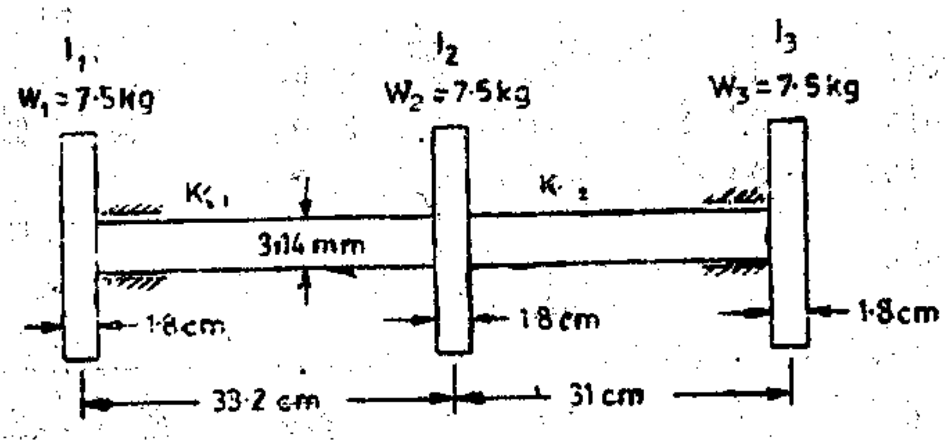
$$\omega_{n^2} = 40.5$$

$$\text{i.e} \quad \omega_n = \sqrt{40.5}$$

$$=6.37 \text{ rad/sec.}$$

Find the natural frequencies and mode shapes of the torsional three rotor system show in fig. diameter of each rotor is 25.4 cm. weight of each rotor and its thickness are shown in the diagram. G for the material = 8.5 × 10^5 kg/cm^2.

Solution : If θ_1, θ_2, θ_3 are the displacements of the three rotors, the equations of motion of the three rotors are,



$$I_1\theta_1 = -K_{t1}(\theta_1 - \theta_2)$$

$$I_2\theta_2 = -K_{t1}(\theta_2 - \theta_1) - K_{t2}(\theta_2 - \theta_3) \qquad \rightarrow (1)$$

$$I_a\theta_3 = -K_{t2}(\theta_3 - \theta_2)$$

Assuming simple harmonic motion with γ_1, γ_2, γ_3, as amplitudes and frequency ω_n.

$$\theta_1 = \gamma_1 \text{ Sin } \omega_{nt} \qquad \theta_1 = -\gamma_1\omega_{n^2} \text{ Sin } \omega_{nt}$$

$$\theta_2 = \gamma_2 \text{ Sin } \omega_{nt} \qquad \theta_2 = -\gamma_2\omega_{n^2} \text{ Sin } \omega_{nt} \qquad \rightarrow (2)$$

$$\theta_3 = \gamma_3 \text{ Sin } \omega_{nt} \qquad \theta_3 = -\gamma_3\omega_{n^2} \text{ Sin } \omega_{nt}$$

From (1) and (2)

$$\text{-}I_1\omega_{n^2}\gamma_1 + K_{t1}(\gamma_1 - \gamma_2)$$

or $\gamma_1\{K_{t1} - I_1\omega_n\} - K_{t1}\theta_2 = 0 \qquad \rightarrow (a)$

$$\text{-}I_2\omega n^2\gamma_2 + K_{t1}(\gamma_2 - \gamma_1) + K_{t2}(\gamma_2 - \gamma_3) = 0$$

or $K_{t1}\gamma_1 + \{K_{t1} + K_{t2} - I_2\omega_{n^2}\}\gamma_2 - K_{t2}\gamma_3 = 0 \qquad \rightarrow (b)$

and $\text{-}I_3\omega_{n^2}\gamma_3 + K_{t2}(\gamma_3 - \gamma_2) = 0$

or $\text{-}\gamma_2K_{t2} + \gamma_3(K_{t2} - I_3\omega_{n^3}) = 0 \qquad \rightarrow (c)$

This is a set of homogeneous equations. It will have a non-zero solution only if determinant formed out of coefficients of γ_1, γ_2 and γ_3 vanishers.

Or

$$\begin{vmatrix} (K_{t1} - I_1\omega_{n^2}) & -K_{t1} & 0 \\ K_{t1} & (K_{t1} + K_{t2} - I_2\omega_{n^2}) & -K_{t2} \\ 0 & -K_{t2} & (K_{t2} - I_3\omega_{n^2} \end{vmatrix} = 0$$

on expanding

$$\left(K_{t1} - I_1\omega_{n^2}\right)\left\{\left(K_{t1} + K_{t2} - I_2\omega_{n^2}\right)\left(k_{t2} - I_3\omega_{n^2}\right) - K_{t2}\right\} + K_{t1}\left\{-K_{t1}\left(K_{t2} - I_3\omega_{n^2}\right)\right\}$$

or

$$\omega_{n^2}\left\{w_{n^4} - \left[K_{t1}\left(\frac{1}{I_1} + \frac{1}{I_2}\right) + K_{t2}\left(\frac{1}{I_2} + \frac{1}{I_3}\right)\right]\omega^3 + K_{t1}K_{t2}\frac{I_1 + I_2 + I_3}{I_1I_2I_3}\right\} = 0 \qquad \rightarrow (3)$$

Mode Shapes

The amplitude ratio of principal modes of vibration can be obtained from equation I and are found to be

$$\frac{\gamma_1}{\gamma_2} = \frac{K_{t1}}{K_{t1} - I_1\omega_{n^2}}$$

and $$\frac{\gamma_2}{\gamma_3} = \frac{K_{t2} - I_3\omega_{n^2}}{K_{t2}}$$

When ω_n^2 is zero, amplitude ratios of the discs are

$$\frac{\gamma_1}{\gamma_2} = \frac{\gamma_2}{\gamma_3} = 1$$

This indicates that whole assembly rotates as a rigid body when

$$\omega_n = 0$$

Since one of the frequencies of this system is zero, this system is a semidefinite system.

Equation (3) is cubic in ω_n^2. One root may be $\omega_{n1}^2 = 0$. The two other natural frequencies can be obtained by solving fourth power equation in ω_n in equation (3)

$$\omega_{n2^2},\ \omega_{n3^2} = \frac{1}{2}\left[\left(\frac{K_{t1}}{I_1} + \frac{K_{t1} + K_{t2}}{I_2} + \frac{K_{t2}}{I_3}\right)\right]$$

$$\pm\sqrt{\left[\left(\frac{K_{t1}}{I_1} + \frac{K_{t1} + K_{t2}}{I_2} + \frac{K_{t2}}{I_3}\right) - 4K_{t1}K_{t2}\frac{I_1 + I_2 + I_3}{I_1 I_2 I_3}\right]}$$

$$I_1 = \frac{1}{2}\ mr^2 = \frac{1}{2}\ \frac{w}{g}\left(\frac{25.4}{4}\right)^2 = \frac{1}{2}.\frac{7.5}{981}(12.7)^2$$

$$=0.616 \text{ kg cm sec}^2$$

$$K_{t1} = \frac{\pi d_4 G}{32 l_1} = \frac{8.5\times10^4\times\pi\times(0.314)^4}{32\times33.2} = 24.7 \text{ kg cm.rad}$$

$$K_{t2} = \frac{8.5\times10^5\times\pi\times\left(\frac{3.14}{10}\right)^4}{}$$

$$=26.5 \text{ kg cm/rad}$$

$$\omega_{n_2^2}, \omega_{n_3^2} = \frac{1}{2}\left[\left(\frac{24.7}{0.616} + \frac{24.7 \times 26.5}{0.616} + \frac{26.5}{0.616}\right)\right]$$

$$\pm\sqrt{\left(\frac{24.7}{0.616} + \frac{24.7 + 26.5}{0.616} + \frac{26.5}{0.616}\right)^2 - \frac{4 \times 24.7 \times 26.5(0.616 \times 3)}{0.616 \times 0.616 \times 0.616}}$$

$$= \frac{1}{2}[166 \pm 83.8]$$

$$\omega_{r2^2} = \frac{1}{2}[166 - 83.8]41.4$$

or $\quad \omega_{n2} = \sqrt{41.4} = 6.4$ rad/sec

$$\omega_{n3^2} = \frac{1}{2}(166 + 83.8) = 124.9$$

or $\quad \omega_{n3} = \sqrt{124.9} = 11.2$ rad/sec

a) Mode shape for $\omega_{n1} = 0$

$$\frac{\gamma_1}{\gamma_2} = \frac{K_{t1}}{K_{t1} - I_1\omega_{n1}^2} = \frac{K_{t1}}{K_{t2}} = 1$$

$$\frac{\gamma_2}{\gamma_3} = \frac{K_{t2} - I_3\omega_{n1^2}}{K_{t2}} = \frac{K_{t2}}{K_{t2}} = 1$$

b) Mode shape for ω_{n2} = 6.4 rad/sec

$$\frac{\gamma_1}{\gamma_2} = \frac{24.7}{24.7 - 0.616 \times 41.1} = -38$$

$$\frac{\gamma_2}{\gamma_3} = \frac{26.5 - 0.616 \times 41.1}{26.5} = 0.0434$$

c) Mode shape for ω_{n2} = 11.2 rad/sec

$$\frac{\gamma_1}{\gamma_2} = \frac{24.7}{24.7 - 0.616 \times 124.9} = -0.472$$

$$\frac{\gamma_2}{\gamma_3} = \frac{26.5 - 0.616 \times 124.9}{26.5} = -1.905$$

Mode shapes are plotted in fig.

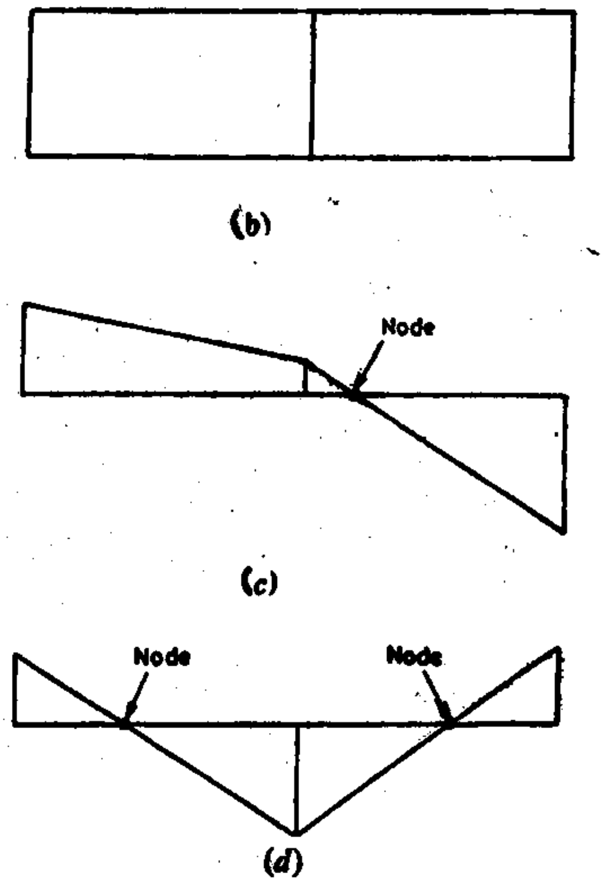
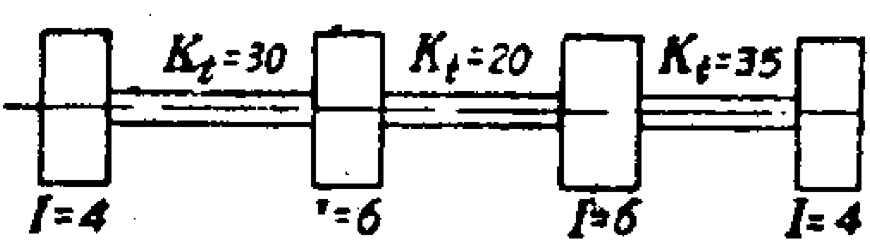


Fig. E-4·16

Natural frequencies of the torsional system shown in fig.



Solution :

$$\gamma_2 = \gamma_1\left(1-\frac{4\omega_{n^2}}{30}\right)$$

$$=\left(1\text{-}\frac{\omega_{n^2}}{7.5}\right)\gamma_1$$

$$\gamma_3 = \gamma_2 - \frac{4\gamma_1\omega_{n^2} + 6\gamma_2\omega_{n^2}}{20}$$

$$=\left(1-\frac{\omega_{n^2}}{7.5}\right)\gamma_1 - \frac{4\omega_{n^2} + 6\left(1-\frac{\omega_{n^2}}{7.5}\right)\gamma_1\omega_{n^2}}{20}$$

$$=\left(1-\frac{\omega_{n^2}}{7.5}-\frac{\omega_{n^2}}{7.5}+\frac{\omega_{n^4}}{25}\right)\gamma_1$$

$$\gamma_4 = \left(\frac{\omega_{n^2}}{2}+\frac{\omega_{n^4}}{25}-\frac{\omega_{n^2}}{7.5}+1\right)\gamma_1 - \frac{(10\omega_{n^2} - 8\omega_{n^4})\gamma_1}{35} + \frac{6\omega_{n^2}\left(1-\frac{\omega_{n^2}}{2}+\frac{\omega_{n^4}}{25}-\frac{\omega_{n^2}}{7}\right)}{35}$$

$$= 0.5 + 0.04\omega_{n^4} - 0.505\omega_{n^2} - 0.0066\omega_{n^6})\gamma_1$$

Let us now construct a Holzer's table

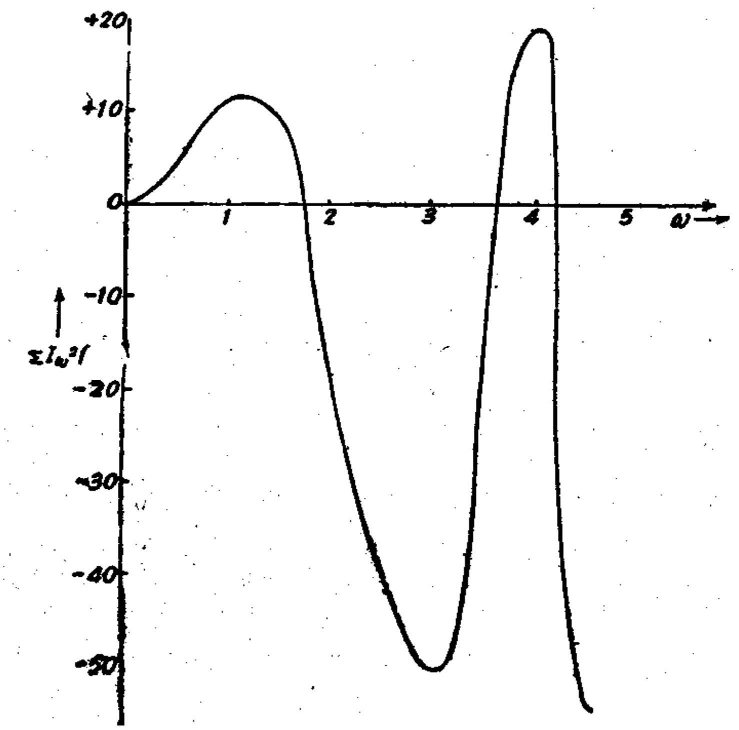
	N_0	I	γ_n	$I\omega^2\gamma_n$	$\sum I\omega^2\gamma_n$
ω_n=2.1	1	4	1	17.64	17.64
	2	6	0.413	10.9	28.54
	3	6	-1.012	-26.7	2.84
	4	4	-0.96	-16.9	-1500
ω_n=1	1	4	1	4	4
	2	6	0.867	5.2	9.2
	3	6	0.407	2.442	11.64
	4	4	0.074	0.296	11.94
ω_n=1.5	1	4	1	9	9

	2	6	0.7	9.45	18.45
	3	6	-0.225	-3.04	15.41
	4	4	-0.666	-6.00	9.41
ω_n=1.9	1	4	1	15.24	15.24
	2	6	0.492	11.29	26.49
	3	6	-0.833	-19.1	7.39
	4	4	-1.4	-21.3	-13.97
ω_n=1.7	1	4	1	11.55	11.55
	2	6	0.615	10.70	22.25
	3	6	-0.50	-8.67	13.58
	4	4	-0.888	-10.25	+2.33
ω_n=3	1	4	1	36	36
	2	6	-0.2	-10.8	25.2
	3	6	-1.46	-78.8	-53.6
	4	4	-0.07	2.52	-51.08
ω_n=4	1	4	1	64	64
	2	6	-1.13	-108.5	-44.5
	3	6	1.095	105	60.5
	4	4	-0.63	-44.2	19.3
ω_n=3.8	1	4	1	57.75	57.75
	2	6	-0.925	-80.0	-22.25
	3	6	0.187	16.1	-6.25
	4	4	0.357	20.6	14.35
ω_n=3.6	1	4	1	52	52
	2	6	-0.733	-57.2	-5.2
	3	6	0.473	-36.9	-42.1
	4	4	0.727	37.8	-4.3
ω_n=5	1	4	1	100	100
	2	6	-2.33	-350	-250

	3	6	10.17	1520	1270
	4	4	-26.33	-2613	-1343
ω_n=4.5	1	4	1	81	81
	2	6	-1.7	-206.5	-125
	3	6	4.57	544.0	419.0
	4	4	17.33	-583.0	-164.0

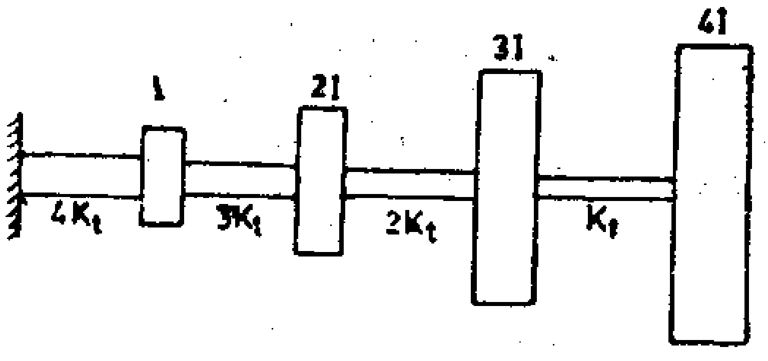
ω_n=4.2	1	4	1	81	81
	2	6	-1.7	-206.5	-125
	3	6	4.57	544.0	419.0
	4	4	-17.33	-583.0	-164.0
ω_n=4.2	1	4	1	70.4	70.4
	2	6	-1.35	-143	-72.6
	3	6	2.28	242	169.4
	4	4	-2.55	-180	-10.6

Now we can plot a graph between ω and $\sum I\omega\gamma$. fig (b) shows this graph. From this natural frequencies re found to be 1.73, 3.64, 4.17.



Using Holzer's method, determine the natural frequencies of the system shown in fig.

K_t = 1 kg-cm/radian
I = 1 kg-cm^2
Both in consistent S.I units.



Solution : **I**n this example we shall use the Holzer's method but the criterion applied to determine the natural frequency will be that when the trial is made with the natural frequency, the displacement at the left hand support will be zero.

One can estimate first natural frequency by Dunkerley's equation but the trials are made without that help.

Different trials are recorded in the following table.

Item	**I**	**$I\omega^2$**	**γ**	**$I\omega^2\gamma$**	**$\sum I\omega^2\gamma$**	**K_t**	$\frac{\sum I\omega^{2\gamma}}{K_t}$
Trial with ω = 0.2							
1	4	0.16	1	0.16	0.16	1	0.16
2	3	0.12	0.84	0.101	0.261	2	0.13
3	2	0.08	0.71	0.056	0.317	3	0.105
4	1	0.04	0.605	0.025	0.342	4	0.0855
5	∞	∞	0.5195				
Trial with ω = 0.3							
1	4	0.36	1	0.36	0.36	1	0.36
2	3	0.27	0.64	0.173	0.533	2	0.267
3	2	0.18	0.373	0.067	0.600	3	0.200
4	1	0.09	0.173	0.0155	0.6155	4	0.1539
5	∞	∞	0.0192				
Trial with ω = 0.4							
1	4	0.64	1	0.64	0.64	1	0.64
2	3	0.48	0.36	0.173	0.813	2	0.406
3	2	0.32	-0.046	0.0147	0.798	3	0.266
4	1	0.16	-0.312	0.049	0.748	4	0.187
5	∞	∞	-0.499				
Trial with ω = 0.6							
1	4	1.44	1	1.41	1.44	1	1.44
2	3	1.08	-0.44	-0.475	0.965	2	0.482
3	2	0.72	-0.922	-0.664	0.301	3	0.100

4	1	0.36	-1.023	-0.368	-0.067	4	0.017
5	∞	∞	-1.006				

Trial with ω = 0.8

1	4	2.56	1	2.56	2.56	1	2.56
2	3	1.92	-1.56	-3.00	0.44	2	-0.22
3	2	1.28	-1.34	-1.72	2.16	3	-0.73
4	1	0.64	0.61	0.39	2.55	4	0.64
5	∞	∞	0.03				

Trial with ω = 1.0

1	4	4	1	4	4	1	4
2	3	3	-3	9	-5	2	-2.5
3	2	2	-0.5	-1	-6	3	-2.0
4	1	1	1.5	1.5	4.5	4	-1.13
5	∞	∞	2.63				

Trial with ω = 1.5

1	4	9	1	9	9	1	9
2	3	6.75	-8	-54	-45	2	-22.5
3	2	4.5	14.5	65.3	20.3	3	6.77
4	1	2.25	7.73	17.4	37.7	4	9.43
5	∞	∞	-1.70				

Trial with ω = 1.8							
1	4	12.96	1	12.96	12.96	1	12.96
2	3	9.72	-11.96	-116.4	-103.44	2	-51.72
3	2	6.48	-39.76	257.7	134.26	3	51.42
4	1	3.24	-11.66	-37.8	116.46	4	29.12
5	∞	∞	-40.78				

Trial with ω = 2.0

1	4	16	1	16	16	1	16
2	3	12	-15	-180	-164	2	-82
3	2	8	+67	536	372	3	124
4	1	4	-57	-228	144	4	36
5	∞	∞	-93				

Trial with ω = 2.5

1	4	25	1	25	25	1	25
2	3	18.75	-25	-450	-425	2	-212.5
3	2	12.5	188.5	2360	1935	3	645
4	1	6.25	-456.5	-2860	-1925	4	-231
5	∞	∞	-225.5				

Trial with ω = 3.0

1	4	36	1	36	36	1	36
2	3	27	-35	-945	-909	2	-455
3	2	18	420	7560	6651	3	2220
4	1	9	-1800	-16200	-9550	4	-2388
5	∞	∞	588				

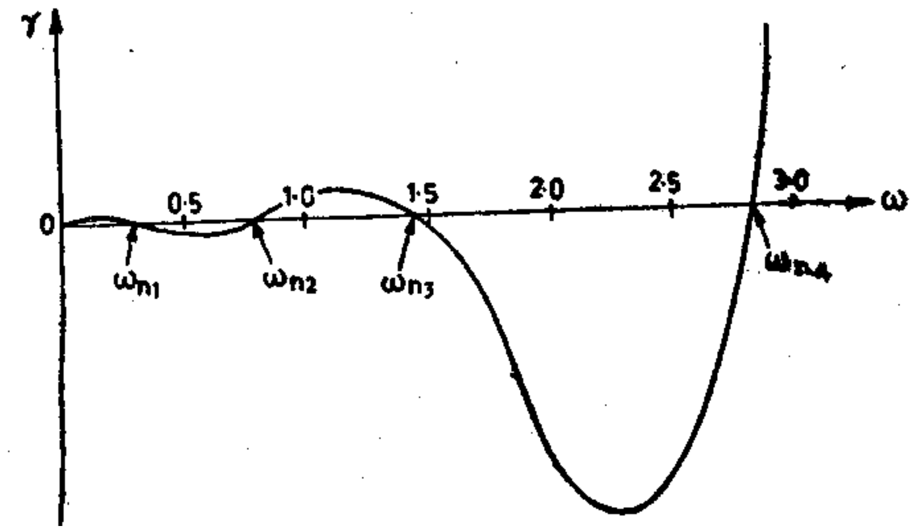


Fig.

Now plotting displacement of the fixed end γ versus ω we get natural frequencies where the curve intersects the x-axis. Natural frequencies are the frequencies which make the displacement of the support zero. They are red off the graph fig.(b) as

ω_{n1} = 0.30 rad./sec

ω_{n2} = 0.81 rsd/sec

ω_{n3} = 1.45 rad./sec

ω_{n4} = 2.83 rad./sec

Student should check the validity of the Dunkerley's equation.

Find the fundamental natural frequency of the system shown in fig(a). The gear ratio for both the branches is $\sqrt{10}$.

V 10.

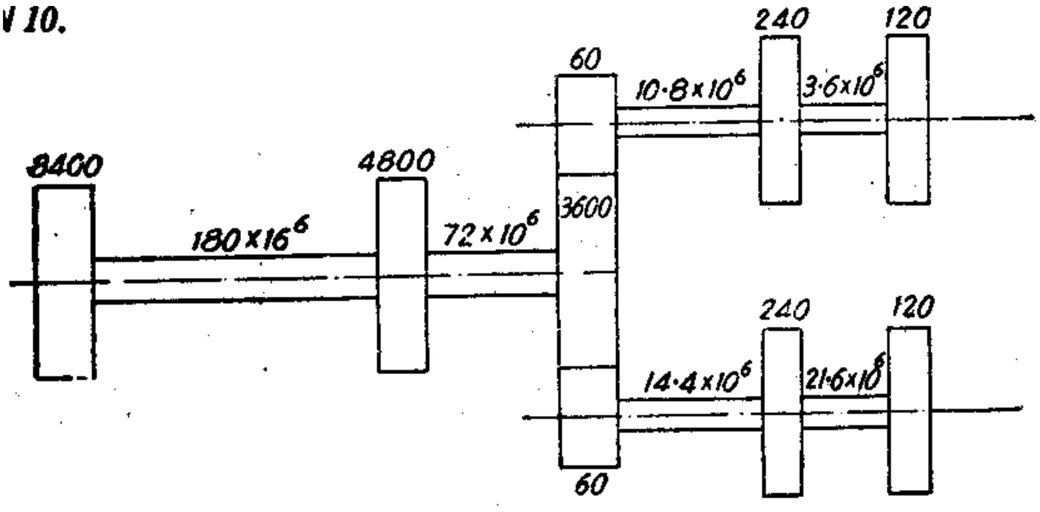


Fig.

All in consistent SI units.

Solution : The equivalent system is given in fig.(b).

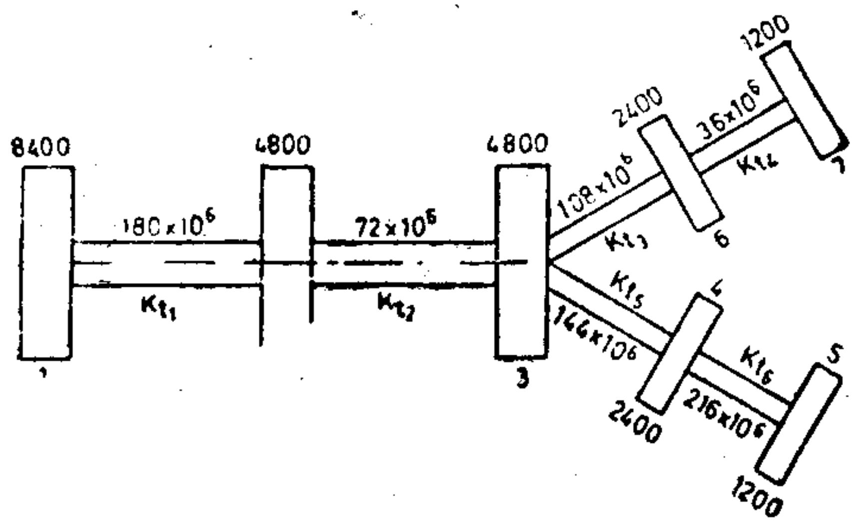


Fig.

Let us take 1200 as a unit for inertia and 36×10^6 as a unit for K_t. since K_{t3}, K_{t4}, and K_{t6} are all too large, we can lump

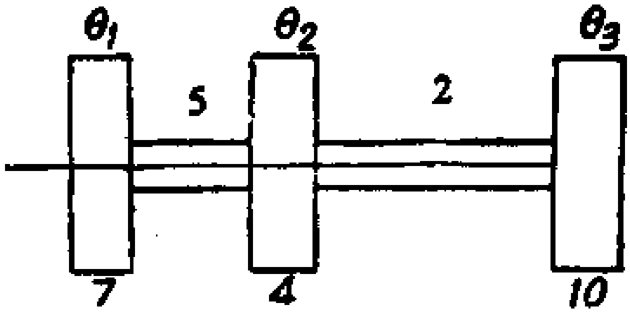


Fig.

3, 4, 5, 6, 7 together. So our equivalent approximate system becomes fig. (c)

We can use $7\theta_1+5(\theta_1-\theta_3)=0$

$4\theta_2+5(\theta_2-\theta_1)+4(\theta_2-\theta_3)=0$

$10\theta_3+2(\theta_3-\theta_2=0$

$$\omega^2\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}=\begin{bmatrix}\frac{5}{7}-\frac{5}{7} & 0\\ -\frac{5}{4}\ \frac{7}{4} & -0.5\\ 0-0.2 & 0.2\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}$$

The above model however is a crude one. We shall therefore treat the problem as that of a branched system as shown in fig.(b). we shall use the Holzer's method to obtain the natural frequency.

Let us assume a unit displacement for mass 7. the displacement of masses 6 and 3 can than be determined from equation, If unit displacement for mass 3 will result. These two values of displacement for mass 3 must be made the same by suitably modifying proportionately the displacement of one of the branches. After this is done, displacement of masses 2 and 1 are calculated and sum of all inertias is found. This process is repeated for each assumed frequency. Frequency corresponding to which the sum zero is naturally frequency. Let us start with ω = 1.

Item	**I**	**Iω²**	**γ**	**Iω²γ**	**ΣIω²γ**	**K_t**	$\frac{\sum I\omega^2\gamma}{K_t}$
7	1	1	1	1	1	1	1
6	2	2	0	0	1	3	0.33
3	4	4	-0.33	-1.33	-0.33		
5	1	1	1	1	1	6	0.167
4	2	2	0.833	1.67	2.67	4	0.67

3	4	4	+0.167

Hence $\gamma_5 = \frac{-0.33}{0.167} = -2$

and we can re-construct the table for masses 5, 4, 3 and add for rotors 1 and 2.

Item	I	$I\omega^2$	γ	$I\omega^2\gamma$	$\sum I\omega^2\gamma$	K_t	$\frac{\sum I\omega^2\gamma}{K_t}$
5	1	1	-2	-2	-2	6	-0.333
4	2	2	-1.67	-3.33	-5.33	4	-1.33
3	4	4	-0.33	-1.33	-6.66	2	-2.83
Torque acting on mass K_{t2} = 1-5.30-1.33=-5.66							
2	4	4	2.5	10.0	4.34	5	0.87
1	7	7	1.63	11.41	15.75		

Trial with ω = 1.5

Item	I	$I\omega^2$	γ	$I\omega^2\gamma$	$\sum I\omega^2\gamma$	K_t	$\frac{\sum I\omega^2\gamma}{K_t}$
7	1	2.25	1	2.25	2.25	1	2.25
6	2	1.5	-1.25	-5.6	-3.35	3	-1.12
3	1	9.0	-0.13	-1.17	-4.52		
5	1	2.25	1	2.25	2.25	6	0.39
4	2	4.5	0.61	2.75	5.0	4	1.25
3	4	9.0	-0.64				

$$\gamma_5 = \left(\frac{0.64}{0.13}\right)^{-1} = 0.2$$

Let us reconstruct the table for masses 5, 4, 3 and add for masses 1 and 2.

Item	I	$I\omega^2$	γ	$I\omega^2\gamma$	$\sum I\omega^2\gamma$	K_t	$\frac{\sum I\omega^2\gamma}{K_t}$
5	1	2.25	0.2	0.450	0.45	6	0.0715
4	2	4.5	0.125	0.56	1.01	4	0.25
2	4	9.0	-0.13	-1.17	-3.51	2	-1.75

Torque acting in K_{t2} = -3.35+1.01-1.17 = -3.51							
2	4	9.0	1.62	13.58	10.07	5	2.01
1	7	15.8	-0.39	-6.15	3.82		

Trial with ω = 2.0

Item	**I**	**$I\omega^2$**	**γ**	**$I\omega^2\gamma$**	**$\sum I\omega^2\gamma$**	**K_t**	$\frac{\sum I\omega^2\gamma}{K_t}$
7	1	4	1	4	4	1	4
6	2	8	-3	-24	-20	3	-6.67
3	4	16	3.37				
5	1	4	1	4	4	6	0.67
4	2	8	0.33	2.64	6.64	4	1.66
3	4	16	-133				

$$\gamma_5 = \frac{-3.37}{1.33} = -2.53$$

Let us now re-construct table for masses 5, 4, 3 and add for masses 1 and 2.

Item	**I**	**$I\omega^2$**	**γ**	**$I\omega^2\gamma$**	**$\sum I\omega^2\gamma$**	**K_t**	$\frac{\sum I\omega^2\gamma}{K_t}$
5	1	4	-2.53	-10.12	-10.12	6	-1.68
4	2	8	-0.85	-6.75	-17	4	-4.21
3	4	16	3.37	53.9	16.9*	2	8.45
2	4	16	-5.03	-80.48	63.6	5	-12.7
1	7	28	-7.67	-2.5	-278.6		

Now we plot a graph between $\sum I\omega^2\gamma$ and ω. Natural frequency equals ω when curve intersects the ω axis.

- Torque acting in K_{t2} = -20-17+53.9 = 16.9

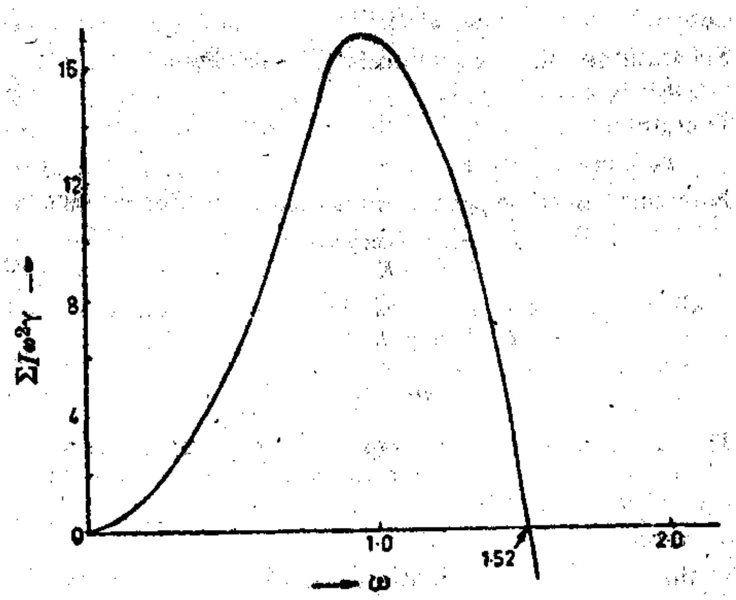


Fig.

From graph fig. ω_n = 1.52, But this must be modified because of our units for inertias and stiffnesses taken to facilitates calculations. Thus ω_n, the natural frequency is given by

$$\omega_n = 1.52 \times \sqrt{\frac{36 \times 10^6}{1200}}$$

= 283 radians/sec.