UNIT I MOMENTUM TRANSPORT

CONTENTS

1.1 Introduction to Momentum Transport

Momentum transport deals with the transport of momentum which is responsible for flow in fluids. Momentum transport describes the science of fluid flow also called fluid dynamics. A few basic assumptions are involved in fluid flow and these are discussed below.

No slip boundary condition

This is the first basic assumption used in momentum transport. It deals with the fluid flowing over a solid surface, and states that whenever a fluid comes in contact with any solid boundary, the adjacent layer of the fluid in contact with the solid surface has the same velocity as the solid surface. Hence, we assumed that there is no slip between the solid surface and the fluid or the relative velocity is zero at the fluid–solid interface. For example, consider a fluid flowing inside a stationary tube of radius R as shown in Fig 7.1. Since the wall of the tube at $r=R$ is stationary, according to the no-slip condition implies that the fluid velocity at $r=R$ is also zero.

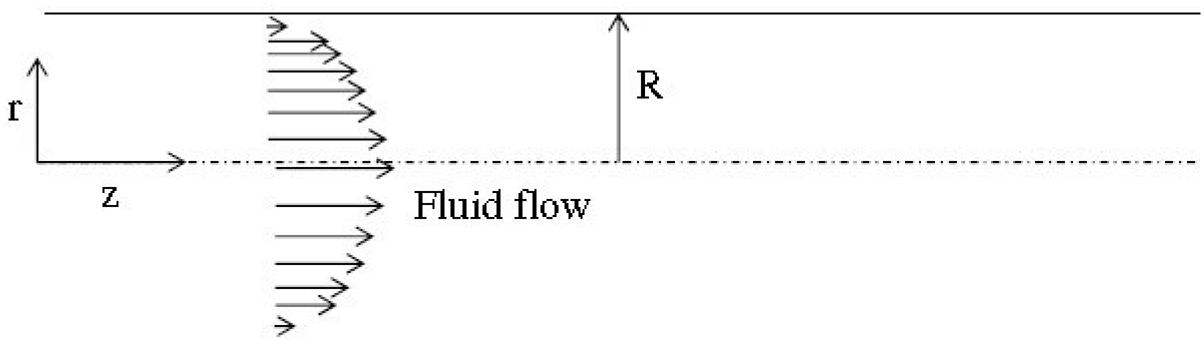

Fig 7.1 Fluid flow in a circular tube of radius R

In the second example as shown in Fig. 7.2, there are two plates which are separated by a distance h, and some fluid is present between these plates. If the lower plate is forced to move with a velocity V in x direction and the upper plate is held stationary, no-slip boundary conditions may be written as follows

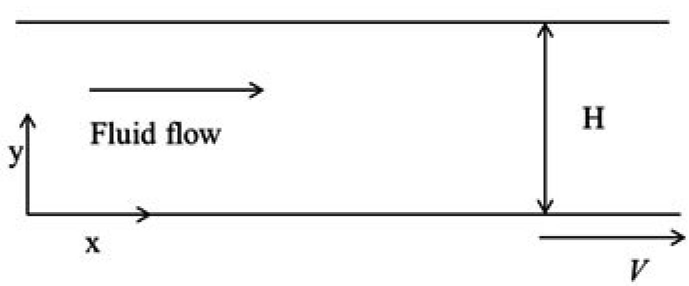

Fig 7.2 Two parallel plates at stationary condition

$$y = 0,\ v_x = v$$

$$y = H,\ v_x = 0$$

Thus, every layer of fluid is moving at a different velocity. This leads to shear forces which are described in the next section.

1.2 Newton's Law of Viscosity

Newton's law of viscosity may be used for solving problem for Newtonian fluids. For many fluids in chemical engineering the assumption of Newtonian fluid is reasonably acceptable. To understand Newtonian fluid, let us consider a hypothetical experiment, in which there are two infinitely large plates situated parallel to each other, separated by a distance h. A fluid is present between these two plates and the contact area between the fluid and the plates is *A*.
A constant force *F1* is now applied on the lower plate while the upper plate is held stationary. After steady state has reached, the velocity achieved by the lower plate is measured as V_1. The force is then changed, and the new velocity of the plate associated with this force is measured. The experiment is then repeated to take sufficiently large readings as shown in the following table.

Table No 7.1 Applied force vs. velocity

F	V	F/A	v/h
F_1	v_1	F_1/A	v_1/h
F_2	v_2	F_2/A	v_2/h
F_3	v_3	F_3/A	v_3/h
F_4	v_4	F_4/A	v_4/h
F_5	v_5	F_5/A	v_5/h
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F_n	v_n	F_n/A	v_n/h

If the *F/A* is plotted against *V/h*, we may observe that they lie on a straight line passing through the origin.

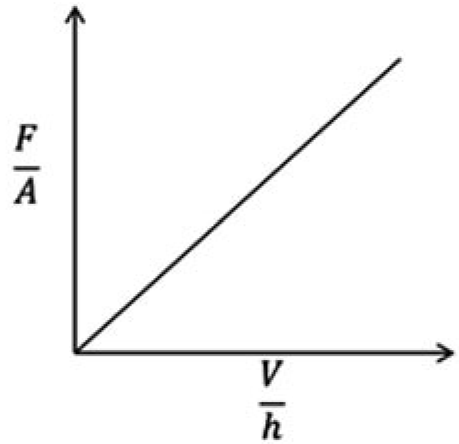

Fig 7.4 Shear stress vs. shear stain

Thus, it may be said that *F/A* is proportional to *v/h* for a Newtonian fluid.

$$\frac{F}{A} \propto \frac{v}{h}$$

It may be noted that it is the velocity gradient which leads to the development of shear forces. The above equation may be re-written as

$$\Rightarrow \frac{F_1}{A} \propto \frac{v_1 - 0}{h - 0} = \frac{\Delta v_x}{\Delta y}$$

In the limiting case, as $h \rightarrow 0$, we have

$$\frac{F}{A} \propto \frac{dv_x}{dy}$$

$$\frac{F}{A} = \pm \mu \frac{dv_x}{dy}$$

where, μ is a constant of proportionality, and is called as the viscosity of the fluid. The quantity *F/A* represents the shear forces/stress. It may be represented as τ_{yx}. , where the subscript *x* indicates the direction of force and subscript *y* indicates the direction of outward normal of the surface on which this force is acting. The quantity $\frac{dv_x}{dy}$ or the velocity gradient is also called the shear rate. μ is a property of the fluid and is measured the resistance offered by the fluid to flow. Viscosity may be constant for many Newtonian fluids and may change only with temperature.

Thus, the Newton's law of viscosity, in its most basic form is given as

$$\tau_{yx} = \pm\mu \frac{dv_x}{dy}$$

Here, both '+' or '–' sign are valid. The positive sign is used in many fluid mechanics books whereas the negative sign may be found in transport phenomena books. If the positive sign is used then τ_{yx}. may be called the shear force while if the negative sign is used τ_{yx}. may be referred to as the momentum flux which flows from a higher value to a lower value.

$$\tau_{yx} = +\mu \frac{dv_x}{dy} \quad \rightarrow \text{Shear force}$$

$$\tau_{yx} = -\mu \frac{dv_x}{dy} \quad \rightarrow \text{Momentum flux}$$

The reason for having a negative sign for momentum flux in the transport phenomena is to have similarities with Fourier's law of heat conduction in heat transport and Ficks law of diffusion in mass transport. For example, in heat transport, heat flows from higher temperature to lower temperature indicating that heat flux is positive when the temperature gradient is negative. Thus, a minus sign is required in the Fourier's law of heat conduction. The interpretation of τ_{yx}. as the momentum flux is that *x* directed momentum flows from higher value to lower value in *y* direction.

The dimensions of viscosity are as follows:

$$\mu = \frac{Force/Area}{\left[\dfrac{dv_x}{dy}\right]} = \frac{MLT^{-1}L^{-2}}{LT^{-1}L^{-1}} = ML^{-1}T^{-1}$$

The SI unit of viscosity is *kg/m.s* or *Pa.s*. In CGS unit is *g/cm.s* and is commonly known as poise (*P*). where *1 P = 0.1 kg/m.s*. The unit poise is also used with the prefix *centi*-, which refers to one-hundredth of a poise, i.e. *1 cP = 0.01 P*. The viscosity of air at 25^oC is *0.018 cP,* water at 25^oC is *1 cP* and for many polymer melts it may range from *1000* to *100,000 cP*, thus showing a long range of viscosity.

1.3 Laminar and turbulent flow

Fluid flow can broadly be categorized into two kinds: laminar and turbulent. In laminar flow, the fluid layers do not inter-mix, and flow separately. This is the flow encountered when a tap is just opened and water is allowed to flow very slowly. As the flow increases, it becomes much more irregular and the different fluid layers start mixing with each other leading to turbulent flow. Osborne Reynolds tried to distinguish between the two kinds of flow using an ingenious experiment and known as the Reynolds's experiment. The basic idea behind this experiment is described below.

1.4 Reynolds's experiment

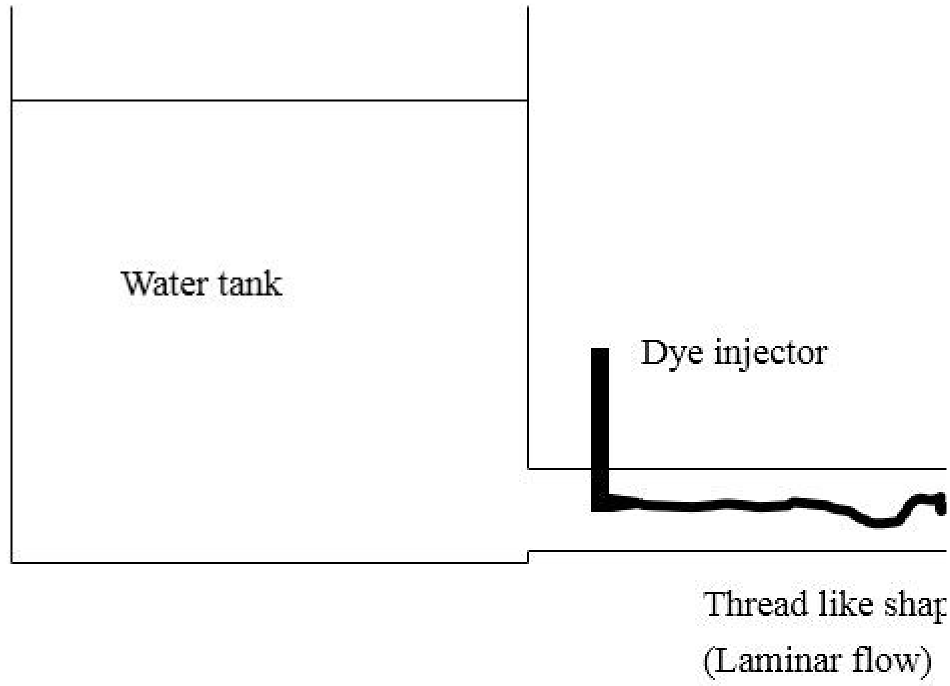

Fig 7.5 Reynolds's experiments

The experiment setup used for performing the Reynolds's experiment is shown in Fig. 7.5. The average velocity of fluid flow through the pipe diameter can be varied. Also, there is an arrangement to inject a colored dye at the center of the pipe. The profile of the dye is observed along the length of the pipe for different velocities for different fluids. If this experiment is performed, it may be seen that for certain cases the dye shows a regular thread type profile, which is seen at low fluid velocity and flow is called laminar flow. when the fluid velocity is increased the dye starts to mixed with the fluid and for larger velocities simply disappears. At this point fluid flow becomes turbulent.
For the variables average velocity of fluid $v_{z\ avg}$, pipe diameter D, fluid density ρ, and the fluid viscosity μ, Reynolds found a dimensionless group which could be used to characterize the type of fluid flow in the tube. This dimensionless quantity is known as the Reynolds number. From the experiment, It was observed that if $Re > 2100$, the dye simply disappeared and the flow has changed to laminar to turbulent flow.

$$\mathrm{Re} = \frac{\rho v_{z,avg} D}{\mu}$$

Thus, for Re <2100, we have laminar flow, i.e., no mixing in the radial direction leading to a thread like flow and for *Re >2100*, we have the turbulent flow, i.e., mixing in the radial direction between layers of fluid.

In laminar flow, the fluid flows as a stream line flow with no mixing between layers. In turbulent flow, the fluid is unstable and mixes rapidly due to fluctuations and disturbances in the flow. The disturbance might be present due to pumps, friction of the solid surface or any type of noise present in the system. This makes solving fluid flow problem much more difficult. To understand the difference in the velocity profile in two kinds of fluid flows, we consider a fluid flowing to a horizontal tube in *z* direction under steady state condition. Then, we can intuitively see the velocity profile may be shown below

For laminar flow, it is observed that fluid flows as smooth stream line and all other components of velocity are zero. Thus

$$v_z = v_z(r)$$

$$v_r = 0$$

$$v_\theta = 0$$

For turbulent flow, if we observe the fluid flows at a local point. It is observed that fluid flows in very random manner in all directions where these local velocities may be the function of any dimensions.

$$v_z = v_z(r, z, \theta, t)$$

$$v_r = v_r(r, z, \theta, t)$$

$$v_\theta = v_\theta(r, z, \theta, t)$$

Thus, we see that for laminar flow there is only one component of velocity present and it depends only on one coordinate whereas the solution of turbulent flow may be vary complex. For turbulent flow, one can ask the question that if the fluid is flowing in the *z* direction then why are the velocity components in *r* and *θ* direction non-zero? The mathematical answer for this question can be deciphered from the equation of motion. The equation of motion is a non-linear partial differential equation. This non-linear nature of the equation causes instability in the system which produces flow in other directions. The instability in the system may occur due to any disturbances or noise present in the environment. On the other hand, if the velocity of fluid is very low the deviation due to disturbances may decay with time, and becomes negligible after that. Thus the flow remains in laminar region. Consider a practical example in which some cars are moving on the highway in the same direction but in the different lanes at different speeds. If suddenly, some obstacle comes on the road, then if the car's speed is sufficiently low, it can move on to other lane smoothly and come back to its original lane after the obstacle is crossed. This is the regular laminar case. On the other hand, if the car is moving at a high speed and suddenly encounters an obstacle, then the driver may lose control, and this car may move haphazardly and hit other cars and after that traffic may never return to normal traffic conditions. This is the turbulent case.

1.4.1 Internal and external flows

Depending on how the fluid and the solid boundaries contact each other, the flow may be classified as internal flow or external flow. In internal flows, the fluid moves between solid boundaries. As is the case when fluid flows in a pipe or a duct. In external flows, however, the fluid is flowing over an external solid surface, the example may be sited is the flow of fluid over a sphere as shown in Fig. 8.1.

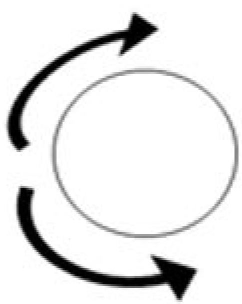

Fig 8.1 External flow around a sphere

Boundary layers and fully developed regions

Let us now consider the example of fluid flowing over a horizontal flat plate as shown in Fig. 8.2. The velocity of the fluid is V_∞ before it encounters the plate. As the fluid touches the plate, the velocity of the fluid layer just adjacent to the plate surface becomes zero due to the no slip boundary condition. This layer of fluid tries to drag the next fluid layer above it and reduces its velocity. As the fluid proceeds along the length of the plate (in x-direction), each layer starts to drag adjacent fluid layer but the effect of drag reduces as we go further away from the plate in y-direction. Finally, at some distance from the plate this drag effect disappears or becomes insignificant. This region where the velocity is changing or where the velocity gradients exists, is called the boundary layer region. The region beyond boundary layer where the velocity gradients are insignificant is called the potential flow region.

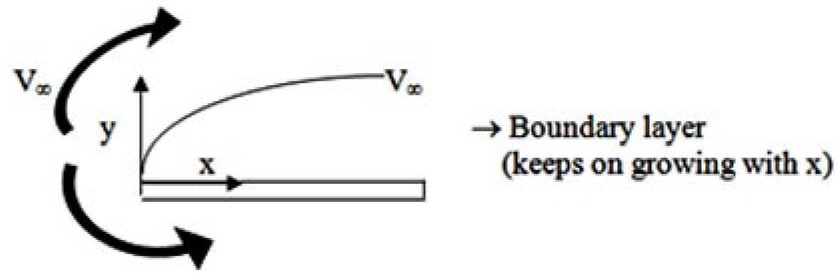

Fig 8.2 External flow over a flat plate

As depicted in Fig. 8.2, the boundary layer keeps growing along the x-direction, and may be referred to as the developing flow region. In internal flows (e.g. fluid flow through a pipe), the boundary layers finally merge after flow over a distance as shown in Fig. 8.3 below.

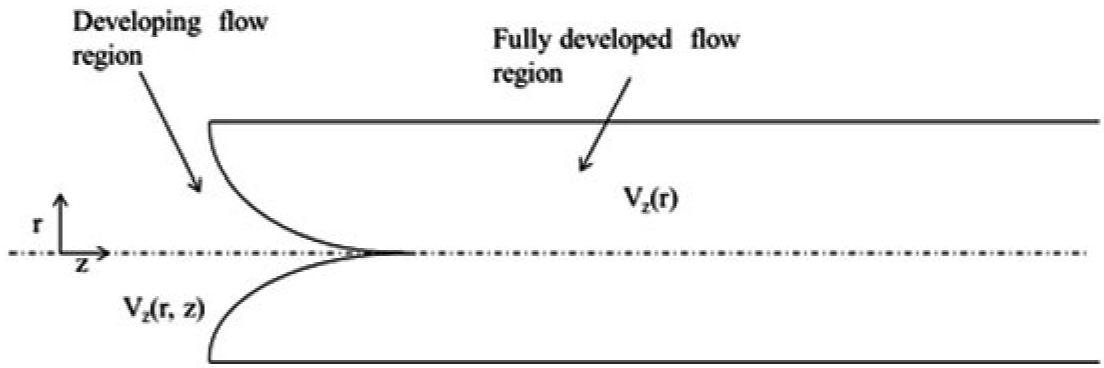

Fig 8.3 Developing flow and fully developed flow region

The region after the point at which the layers merge is called the fully developed flow region and before this it is called the developing flow region. In fact, fully developed flow is another important assumption which is taken for finding solution for varity of fluid flow problem. In the fully developed flow region (as shown in Figure 8.3), the velocity vz is a function of r direction only. However, the developing flow region, velocity vz is also changing in the z direction.

Main axioms of transport phenomena

The basic equations of transport phenomena are derived based on following five axioms.

- Mass is conserved, which leads to the equation of continuity.
- Momentum is conserved, which leads to the equation of motion.
- Moment of momentum is conserved leads to an important result that the 2nd order stress tensor $\underset{\sim}{\tau}$ is symmetric.

- Energy is conserved, which leads to equation of thermal energy.

- Mass of component i in a multi-component system is conserved, which leads to the convective diffusion equation.

The solution of equations, resulting from axiom 2, 4 and 5 leads to the solution of velocity, temperature and concentration profiles. Ones these profiles are known, all other important information needed can be determined. We first take the axiom -1. Other axioms will be taken up one by one letter on.
There are three types of control volumes (CV) which may be chosen for deriving the equations based these axioms.

- Rectangular shaped control volume fixed in space

In this case, the control volume is rectangular volume element and is fixed in space. This method is the easiest to understand but requires more number of steps.

- Irregular shaped control volume element fixed in space

In this case, the control volume can be of any shape, but it is again fixed in space. This method is somewhat more difficult than the previous method as it requires little better understanding vector analysis and surface and volume integrals.

- Material volume approach

In this case, the control volume can be of any shape but moves with the velocity of the flowing fluid. This method is most difficult in terms of mathematics, but requires least number of steps for deriving the equations.

All three approaches when applied to above axiom, lead to the same equations. In this web course, we follow the first approach. Other approaches may be found elsewhere.
Axioms-1

Mass is conserved

Consider a fluid of density ρ flowing with velocity $\underset{\sim}{v}$ as shown in Fig. 8.4. Here, ρ and $\underset{\sim}{v}$ are functions of space (x,y,z) and time (t). For conversion of mass, the rate of mass entering and leaving from the control volume (net rate of inflow) has to be evaluated and this should be equal to the rate of accumulation of mass in the control volume (CV). Thus, conservation of mass may be written in words as given below

$$\begin{pmatrix}\text{rate of accumulation of}\\ \text{mass in control volume}\end{pmatrix} = \begin{pmatrix}\text{rate of inflow of mass}\\ \text{into control volume}\end{pmatrix} - \begin{pmatrix}\text{rate}\\ \text{out}\end{pmatrix}$$

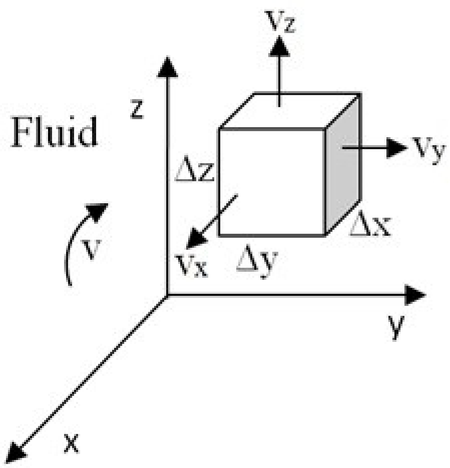

Fig 8.4 Fixed rectangular volume element through which fluid is flowing

The equation is then divided by the volume of the CV and converted into a partial differential equation by taking the limit as all dimensions go to zero. This limit effectively means that CV collapses to a point, thereby making the equation valid at every point in the system.

Let m and m+Δm be the mass of the control volume at time t and t+Δt respectively. Then, the rate of accumulation,

$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

In order to evaluate the rate of inflow of mass into the control volume, we need to inspect how mass enters the control volume. Since the fluid velocity has three components vx, vy and vz, we need to identify the components which cause the inflow or the outflow at each of the six faces of the rectangular CV. For example, it is the component vx which forces the fluid to flow in the x direction, and thus it makes the fluid enter or exit through the faces having area ΔyΔz at x = x and x = x+Δx respectively. The component vy forces the fluid in y direction, and thus it makes the fluid enter or exit through the faces having area ΔxΔz at y = y and y = y+Δy respectively. Similarly, the component vz forces the fluid to flow in z direction, and thus it makes fluid enter or exit through the faces having area ΔxΔy at z = z and z = z+Δ z respectively.

The rate mass entering in x direction through the surface ΔyΔz is (ρvxΔyΔz|x), the rate of mass entering in y direction through the surface ΔxΔz is (ρvyΔxΔz|y) and the rate of mass entering from z direction through the surface ΔxΔy is (ρvzΔxΔy|z). In a similar manner, expressions for the rate of mass leaving from the control volume may be written.

Thus, the conservation of mass leads to the following expressions

$$\frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = \Big[\rho v_x \Delta y \Delta z \big|_x - \rho v_x \Delta y \Delta z \big|_{x+\Delta x} + \rho v_y \Delta x \Delta z \big|_y - \rho v_y \Delta x \Delta z \big|_{y+\Delta y} + \rho v_z$$

Dividing the Equation (8.1) by the volume ΔxΔyΔz, we obtain

$$\frac{\partial \rho}{\partial t} = \left[\frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x} \right] + \left[\frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y} \right] + \left[\frac{(\rho v_z)|_z - (\rho v}{\Delta z} \right.$$

Note that each term in Equation (8.2) has the unit of mass per unit volume per unit time. Now, taking the limits $\Delta x \to 0$, $\Delta y \to 0$ and $\Delta z \to 0$, we get

$$\frac{\partial \rho}{\partial t} = \lim_{\Delta x \to 0} \left[\frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x} \right] + \lim_{\Delta y \to 0} \left[\frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y} \right] + \lim_{\Delta z \to 0} \Big[$$

and using the definition of derivative, we finally obtain

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} \right]$$

Equation (8.4) is applicable to each point of the fluid. Rearranging the terms, we get the equation of continuity, may be written as given below.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

We need not to derive the equation of continuity again and again in other coordinate system (that is, spherical or cylindrical). The idea is to rewrite Equation (8.5) in vector and tensor form. Once it is written in this form, the same equation may be applied to other coordinate system as well. Thus, the Equation (8.5) may be rewritten in vector and tensor form as shown below.

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} . (\rho \underset{\sim}{v}) = 0$$

Vector and tensor analysis of cylindrical and spherical coordinate systems is not done here, and can be looked up elsewhere. Thus, the final expressions in cylindrical and spherical coordinates are given as below.

cylindrical coordinates (r, θ, z)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

Spherical coordinates (r, θ, φ)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(\rho v_\varphi)=0$$

Equation of continuity in terms of substantial derivative

The second term in Equation (8.6) may be broken into two parts as shown below. Partial derivative present in the Equation (8.6) can be converted into substantial derivative using vector and tensor identities.

$$\frac{\partial \rho}{\partial t}+\underset{\sim}{v}.\underset{\sim}{\nabla}\rho+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0$$

In the above equation, the first two terms may be combined using the definition of substantial derivative to obtain the following equation.

$$\frac{D\rho}{Dt}+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0$$

In some cases, the fluid may be incompressible, i.e. density ρ is a constant with time as well as space coordinates. For example, water may be assumed as an incompressible fluid under isothermal conditions. In fact, all liquids may be assumed as incompressible fluids under isothermal conditions. For this special case, the equation of continuity may be further simplified as shown below

$$\underset{\sim}{\nabla}.\underset{\sim}{v}=0 \quad (\rho \text{ is constant})$$

The above equation for an incompressible fluid does not mean that the system is under steady state conditions. The velocity of the fluid may still be a function of time. It only implies that if the velocity of the fluid changes in a particular direction (x, y or z) then it should also change in the other directions such that mass is conserved without changing its density. The equation of continuity provides additional information about the velocity profile and helps in solution of

1.5 EQUATION OF MOTION.

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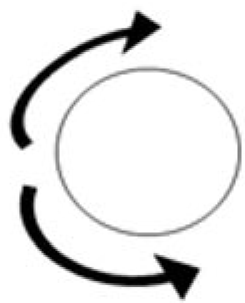

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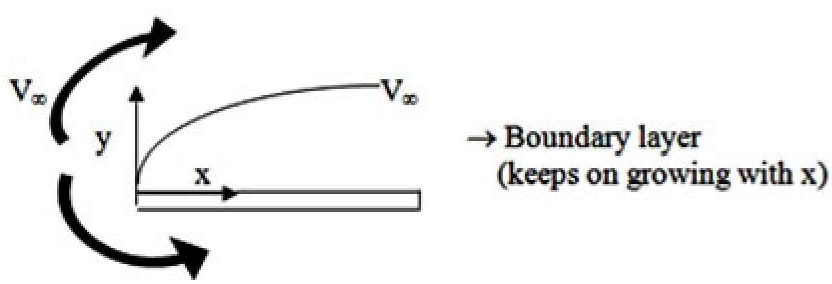

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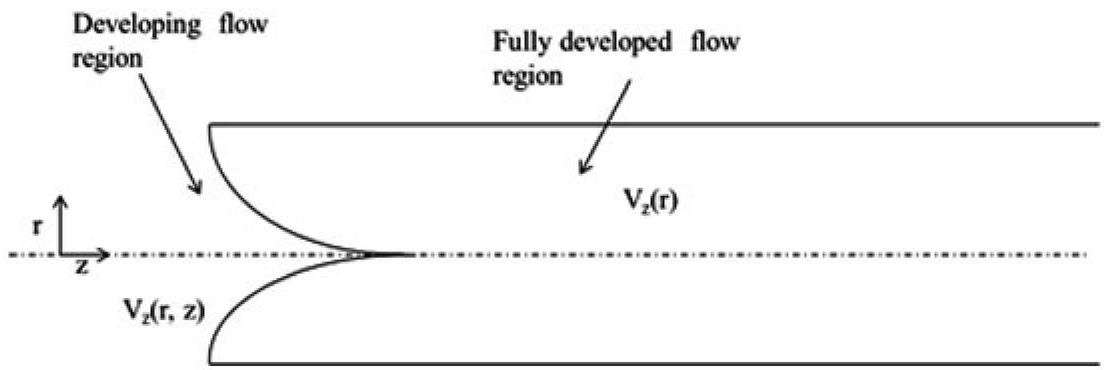

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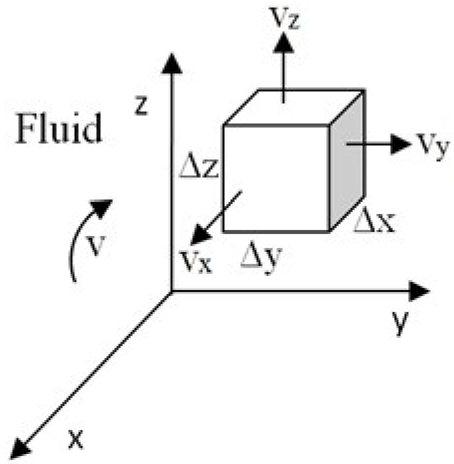

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$$\frac{\partial m}{\partial t} = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

In order to evaluate the rate of inflow of mass into the control volume, we need to inspect how mass enters the control volume. Since the fluid velocity has three components vx, vy and vz, we need to identify the components which cause the inflow or the outflow at each of the six faces of the rectangular CV. For example, it is the component vx which forces the fluid to flow in the x direction, and thus it makes the fluid enter or exit through the faces having area ΔyΔz at x = x and x = x+Δx respectively. The component vy forces the fluid in y direction, and thus it makes the fluid enter or exit through the faces having area ΔxΔz at y = y and y = y+Δy respectively. Similarly, the component vz forces the fluid to flow in z direction, and thus it makes fluid enter or exit through the faces having area ΔxΔy at z = z and z = z+Δ z respectively.

The rate mass entering in x direction through the surface ΔyΔz is (ρvxΔyΔz|x), the rate of mass entering in y direction through the surface ΔxΔz is (ρvyΔxΔz|y) and the rate of mass entering from z direction through the surface ΔxΔy is (ρvzΔxΔy|z). In a similar manner, expressions for the rate of mass leaving from the control volume may be written.

Thus, the conservation of mass leads to the following expressions

$$\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z) = \Big[\rho v_x \Delta y\Delta z\big|_x - \rho v_x \Delta y\Delta z\big|_{x+\Delta x} + \rho v_y \Delta x\Delta z\big|_y - \rho v_y \Delta x\Delta z\big|_{y+\Delta y} + \rho v_z$$

Dividing the Equation (8.1) by the volume ΔxΔyΔz, we obtain

$$\frac{\partial \rho}{\partial t} = \left[\frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x}\right] + \left[\frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y}\right] + \left[\frac{(\rho v_z)|_z - (\rho v}{\Delta z}\right.$$

Note that each term in Equation (8.2) has the unit of mass per unit volume per unit time. Now, taking the limits $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$ and $\Delta z \rightarrow 0$, we get

$$\frac{\partial \rho}{\partial t} = \lim_{\Delta x \to 0}\left[\frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x}\right] + \lim_{\Delta y \to 0}\left[\frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y}\right] + \lim_{\Delta z \to 0}\Big[$$

and using the definition of derivative, we finally obtain

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right]$$

Equation (8.4) is applicable to each point of the fluid. Rearranging the terms, we get the equation of continuity, may be written as given below.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

We need not to derive the equation of continuity again and again in other coordinate system (that is, spherical or cylindrical). The idea is to rewrite Equation (8.5) in vector and tensor form. Once it is written in this form, the same equation may be applied to other coordinate system as well. Thus, the Equation (8.5) may be rewritten in vector and tensor form as shown below.

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}.(\rho \underset{\sim}{v}) = 0$$

Vector and tensor analysis of cylindrical and spherical coordinate systems is not done here, and can be looked up elsewhere. Thus, the final expressions in cylindrical and spherical coordinates are given as below.

cylindrical coordinates (r, θ, z)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

Spherical coordinates (r, θ, φ)

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}(\rho v_\varphi)=0$$

1.6 Equation of continuity in terms of substantial derivative

The second term in Equation (8.6) may be broken into two parts as shown below. Partial derivative present in the Equation (8.6) can be converted into substantial derivative using vector and tensor identities.

$$\frac{\partial \rho}{\partial t}+\underset{\sim}{v}.\underset{\sim}{\nabla}\rho+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0$$

In the above equation, the first two terms may be combined using the definition of substantial derivative to obtain the following equation.

$$\frac{D\rho}{Dt}+\rho\underset{\sim}{\nabla}.\underset{\sim}{v}=0$$

In some cases, the fluid may be incompressible, i.e. density ρ is a constant with time as well as space coordinates. For example, water may be assumed as an incompressible fluid under isothermal conditions. In fact, all liquids may be assumed as incompressible fluids under isothermal conditions. For this special case, the equation of continuity may be further simplified as shown below

$$\underset{\sim}{\nabla}.\underset{\sim}{v}=0 \quad (\rho \text{ is constant})$$

The above equation for an incompressible fluid does not mean that the system is under steady state conditions. The velocity of the fluid may still be a function of time. It only implies that if the velocity of the fluid changes in a particular direction (x, y or z) then it should also change in the other directions such that mass is conserved without changing its density. The equation of continuity provides additional information about the velocity profile and helps in solution of equation of motion.

Solution of momentum transport problem by shell momentum balances

Here, we solve few simple problems of fluid mechanics with simple geometries by using the shell momentum balance approach. This will lead to greater understanding of various terms involved in the application of conservation of momentum in fluid given in Equation (9.1)

Flow through circular tube

Flow of fluids through a circular tube is a common problem, encountered frequently in different fields of engineering. Consider an incompressible, Newtonian fluid, flowing through a horizontal circular tube as shown in Fig. (10.1). Assume that the fluid flow is laminar and under steady state conditions. Determine the velocity profile and average velocity of the fluid using shell momentum balance approach.

solution procedure

Assumptions

- Fluid density and viscosity are constants.
- System is in steady state.
- Laminar flow (simple shear flow).
- Newton's law of viscosity is applicable.
- Fully developed flow.

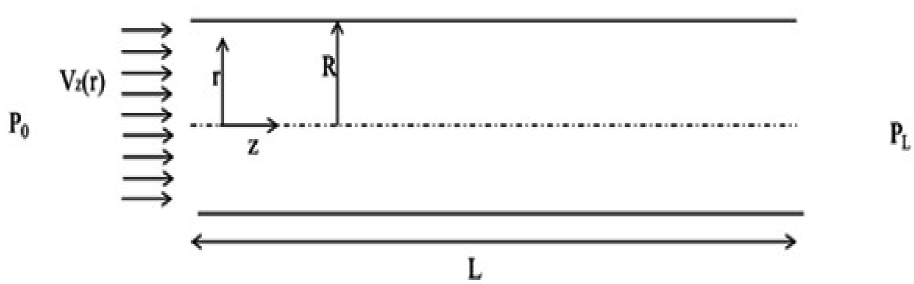


Fig Laminar flow in a horizontal pipe

Intuitively guess the velocity profile

Since the flow is steady and laminar, we may intuitively say that the velocities in r direction and θ direction are zero. Due to steady state conditions, the fluid velocity in z direction, vz, is not dependent on time t. Furthermore, due to the axisymmetric geometry fluid flow the velocity vz is independent of θ. Thus,

$$v_r = 0,\ v_\theta = 0,\ v_z = v_z(r, z, \cancel{\theta}, \cancel{t})$$

$$v_z = v_z(r, z)$$

By applying the equation of continuity in cylindrical coordinates

$$\frac{\partial v_z}{\partial z} = 0$$

Hence,

$$v_z = v(r)$$

Since the fluid is flowing in z direction, we may conclude the following.

- Since $v_r=0$, r directional momentum balance is not important.
- Since $v_\theta=0$, θ directional momentum balance is again not important.
- Since $v_z \neq 0$, z directional momentum balance is most important.

1.7 Equation for circular pipe:

The control volume should be decided very carefully. The geometry and size of the control volume should be taken according to the geometry of the system and based on the conditions given in the problem. In this case, the geometry of the pipe is cylindrical, hence we use the cylindrical control volume. The fluid is flowing in the z direction but velocity is changing only in r direction. Therefore, the control volume is taken in such a way that the variable thickness of the control volume is in the r direction. As the flow is not dependent on z and θ coordinates, we may choose any dimension in z or θ directions. This means that z may be any length. It may be L/4, L/2 or L. In a similar manner, any value of θ may be taken. It may be 2π or π or $\pi/2$ or $\pi/4$. However, in the r direction, we need to take the differential thickness dr. These arguments leads us to a control volume as shown in Fig. (10.2). The length of the cylindrical shell is L which is equal to length of pipe and thickness is dr.

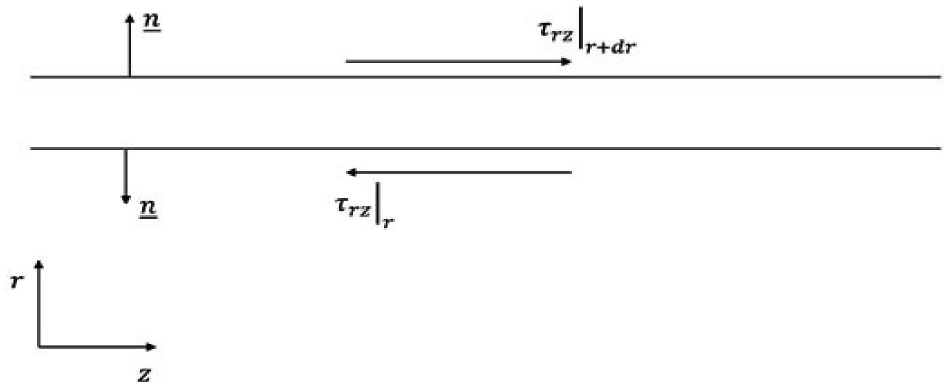


Fig 10.2 Control volume for flow through pipe

Momentum balance

As discussed earlier, the shear stress/forces may be written in two ways:

- Taking shear stress as actual shear forces.
- Taking shear stress as momentum flux.

Here, we show that both methods lead to the same final results for velocity profile.

Momentum balance using shear stress as shear force

Momentum flux entering the control volume by convection
=

$$(\rho v_z^2 2\pi r dr)\Big|_{z=0}$$

Momentum flux leaving the control volume by convection
=

$$(\rho v_z^2 2\pi r dr)\Big|_{z=L}$$

Since the pipe is horizontal, the force due to gravity is zero. No other body forces are acting on the control volume.

Surface forces

- Pressure force: Fluid is flowing in z direction only. So pressure forces which are working on the surface normal to z direction are

Pressure force at z=0 is

$$P_0 2\pi r dr\big|_{z=0} \qquad (10.5)$$

Pressure force at z=L is

$$- P_L 2\pi r dr\big|_{z=L} \qquad (10.6)$$

- Shear forces: The shear stress tensor in cylindrical coordinate is given below.

$$\begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix}$$

Among all 9 components the first column of stresses are important for r directional flow, the second column of stresses are important for θ directional flow, and the third column are important for z directional flow. Since the fluid is flowing in the z direction, only the third column needs to be considered. Since the Velocity gradient is present only in the r-direction, only τ_{rz} needs to be considered, the remaining two terms are not significant. Now, we need to decide the direction in which the shear forces are acting. Recall

$$\underset{\sim}{T_n} = \underset{\sim}{\delta_n}.\underset{\sim}{\tau}$$

Where the unit vector $\underset{\sim}{\delta_n}$ is the outer normal of a surface and if it is in positive direction then $\underset{\sim}{T_n}$ is also positive while if it is in negative direction then $\underset{\sim}{T_n}$ is shown as negative direction. Therefore, τ_{rz} (as a force) is positive at r+dr and negative at r as shown in Fig. 10.2.(Note: the first index, z, in τ_{rz} from right to left indicates the direction of force and second index, r, indicates the surface on which it acts).

Accumulation term: Due to steady state system, the rate of accumulation of momentum equals to zero .

General momentum balance is given below

$$\begin{pmatrix} \text{rate of accummulation} \\ \text{of momentum in CV} \end{pmatrix} = \begin{pmatrix} \text{rate of momentum} \\ \text{entering CV} \\ \text{by convection} \end{pmatrix} - \begin{pmatrix} \text{rate of momentum} \\ \text{leaving CV} \\ \text{by convection} \end{pmatrix} + \left(\sum \text{applied forces}\right)$$

or in this case

$$0 = (rv_z^2 2\pi r dr)|_{z=0} - (rv_z^2 2\pi r dr)|_{z=L} + 0 + P_0 2\pi r dr$$
$$- P_L 2\pi r dr + (\tau_{rz} 2\pi r L)|_{r+dr} - (\tau_{rz} 2\pi r L)|_r$$

Since the velocity is constant along the axial direction as shown in Equation (10.2), the first two terms in Equation (10.8) are cancel out and we are left with following Equation.

$$0 = 2\pi L\left[\left(\tau_{rz} r\right)|_{r+dr} - \left(\tau_{rz} r\right)|_r\right] + P_0 2\pi r dr - P_L 2\pi r dr$$

Dividing by $2\pi r dr$, we have

$$0 = \frac{P_0 - P_L}{L} + \frac{(\tau_{rz} r)|_{r+dr} - (\tau_{rz} r)|_r}{r dr}$$

As dr→0, the Equation (10.10) may be rewritten as given below.

(Note that, τ_{rz} is a function of r only which means we get the total derivative instead of the partial derivative.)

$$\frac{d(\tau_{rz} r)}{dr} = \frac{-r\,(P_0 - P_L)}{L}$$

Further integrating the Equation (10.11) once with respect to the variable r, we obtain

$$\tau_{rz} r = -\frac{r^2}{2}\frac{(P_0 - P_L)}{L} + c_1$$

or

$$\tau_{rz} = -\frac{r}{2}\frac{(P_0 - P_L)}{L} + \frac{c_1}{r}$$

Here, c1 is a constant of integration. Equation (10.12) shows that if r=0, the value of τ_{rz} will be infinite, which is physically not possible. Therefore, c1 must be zero. Hence,

$$\tau_{rz} = -\frac{r}{2}\frac{(P_0 - P_L)}{L}$$

Now, by applying Newton's law of viscosity, and taking τ_{rz} as force, we obtain

$$\tau_{rz} = +\mu\frac{dv_z}{dr} = -\frac{r}{2}\frac{(P_0 - P_L)}{L}$$

Momentum balance using shear stress as momentum flux

Now, we will employ the second method where shear force are considered as momentum flux. To indicate the direction of momentum flux, we draw the arrow in r direction and find where this arrow enters the control volume and also leaves the control volume as shown in Fig (10.3). Thus, the momentum flux enters the control volume through the surface 2π rL at r=r and leaves through the surface 2π rL at r=r+dr.

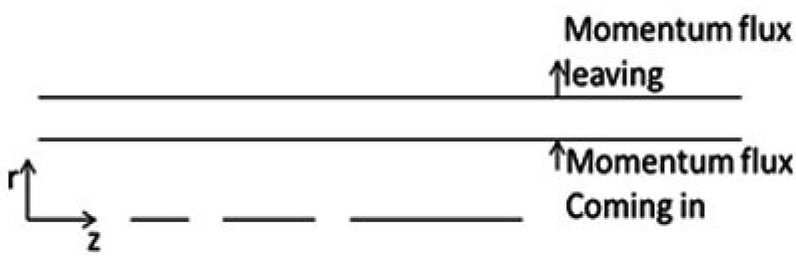


Fig 10.3 Momentum flux applied on control volume

Thus,
Momentum flux at r = r is

$$(\tau_{rz} 2\pi rL)\big|_r \tag{10.15}$$

Momentum flux at r = r +Δr is

$$(\tau_{rz} 2\pi rL)\big|_{r+dr} \tag{10.16}$$

(Note: when we consider τ_{rz} as the momentum flux, first index, z, indicates the direction of momentum flux, while the second index, r, indicates the direction of flow of momentum flux from higher to lower value. Subsequently, it will become clear that if we follow the coordinate system's directions and assume momentum is flowing in this direction, the sign convention for momentum flux is automatically taken place.)

In this case, momentum balance in Equation (9.2) may be modified as shown below

$$\begin{pmatrix}\text{rate of accummulation}\\ \text{of momentum in CV}\end{pmatrix} = \begin{pmatrix}\text{rate of momentum}\\ \text{entering CV}\\ \text{by convection}\end{pmatrix} - \begin{pmatrix}\text{rate of momentum}\\ \text{leaving CV}\\ \text{by convection}\end{pmatrix} + (\sum$$

$$+\begin{pmatrix}\text{rate of viscous}\\ \text{momentum}\\ \text{entering CV}\end{pmatrix} - \begin{pmatrix}\text{rate of viscous}\\ \text{momentum}\\ \text{leaving CV}\end{pmatrix}$$

Here, the shear stress are taken into account as momentum flux. The pressure and gravity are the only applied forces.

Substituting various terms in above equation, we obtain

$$0 = 0 + (P_0 - P_L)\ 2\pi r dr + (\tau_{rz} 2\pi r L)\big|_r - (\tau_{rz} 2\pi r L)\big|_{r+dr}$$

Dividing by $2\pi r dr$, we obtain

$$0 = (\tau_{rz} r)\big|_r - (\tau_{rz} r)\big|_{r+dr} + \frac{(P_0 - P_L)}{L}$$

Again as dr→0 Equation (10.17) leads to

$$0 = -\frac{d(\tau_{rz} r)}{r dr} + \frac{(P_0 - P_L)}{L}$$

or

$$\frac{d(\tau_{rz} r)}{dr} = \frac{(P_0 - P_L)}{L} r$$

By integrating the Equation (10.18), we have

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r + \frac{c_1}{r}$$

As we discussed earlier, c1 should be zero. Therefore,

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

Now applying Newton's law of viscosity where shear stress is taken as momentum flux, we obtain

$$\mu \frac{dv_z}{dr} = -\frac{(P_0 - P_L)}{2L} r$$

Equation (10.14) and (10.20) are identical and hence show that both methods finally lead to the same result.
To obtain velocity profile we further integrating the Equation (10.21)

$$v_z = -\frac{(P_0 - P_L)}{2\mu L} \frac{r^2}{2} + c_2$$

Here c2 is the second constant of integration which may be determined by using appropriate boundary condition.

Boundary condition

By no-slip boundary condition
vz=0 at r=R

$$0 = -\frac{(P_0 - P_L)}{4\mu L} R^2 + c_2$$

$$c_2 = \frac{(P_0 - P_L)}{4\mu L} R^2$$

Substituting the value of c2 in Equation (10.22), we finally get

$$v_z = \frac{(P_0 - P_L)}{4\mu L}(R^2 - r^2)$$

Note: c1 can also be calculated by using the boundary condition in terms of velocity vz: i.e., vz is finite at r=0

or $\left.\frac{dv_z}{dr}\right|_{r=0} = 0$ (since the velocity profile is symmetric about r=0).

Thus, the velocity profile for flow through pipe is given by the following expression

$$v_z = \frac{(P_0 - P_L)}{4\mu L}R^2\left(1 - \frac{r^2}{R^2}\right)$$

The maximum velocity of the fluid will be exhibited at the centre of the pipe and is given by

$$v_{z\max} = v_z\,|_{r=0} = \frac{(P_0 - P_L)}{4\mu L}R^2$$

Alternatively, the velocity profile may also be expressed in terms of the maximum velocity as

$$v_z = v_{z,\max}\left(1 - \frac{r^2}{R^2}\right)$$

The average velocity of the fluid in the pipe is the average of all local velocities. Thus, this may be calculated by estimating the volumetric flow rate through the pipe and then dividing it by the cross sectional area of the pipe. The total volumetric flow in the system is

$$Q = \int dQ$$

where, dQ is the volumetric flow rate from small cylindrical strip of thickness dr.

$$= \int v_z 2\pi r dr$$

By substituting the value of v z from equation (10.27), we have

$$= \int v_{z\,max}\left[1-\frac{r^2}{R^2}\right]2\pi r dr$$

By integrating the equation (10.30) from r=0 to r=R, we obtain

$$= \int_0^R 2\pi\, v_{z\,max}\left[r-\frac{r^3}{R^2}\right]dr$$

or

$$= 2\pi v_{z\,max}\left[\frac{r^2}{2}-\frac{r^4}{4R^2}\right]_0^R$$

Thus,

$$Q = \pi v_{z\,max}\frac{R^2}{2}$$

and average velocity is

$$v_{z,avg} = \frac{Q}{A_c}$$

$$= \frac{\pi\, v_{z,max}\,\frac{R^2}{2}}{\pi R^2}$$

or

$$v_{z,avg} = \frac{v_{z,max}}{2}$$

The velocity profile for laminar flow in a circular tube is shown in Fig. 10.5.

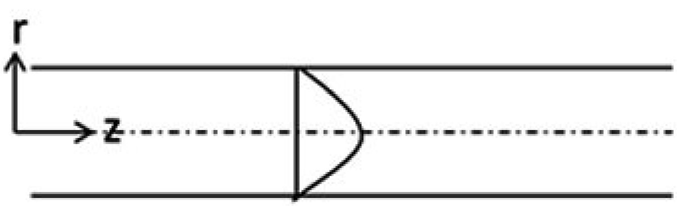


Fig 10.5 Velocity profile in horizontal pipe

We can also find the radial distance at which the local velocity of fluid flow equals the average velocity. For this, substitute $v_z = v_{z,avg} = \frac{v_{z,\max}}{2}$ into Equation (10.26), we obtain

$$1 - \frac{r^2}{R^2} = \frac{1}{2}$$

$$\frac{r^2}{R^2} = \frac{1}{2}$$

$$\text{or, } r = \frac{R}{\sqrt{2}} \tag{10.34}$$

Finally , the volumetric flow rate in terms of pressure drop is as follows

$$Q = v_{zavg} \pi R^2 = \frac{\pi (P_0 - P_L)}{8 \mu L} R^4 \tag{10.35}$$

$$Q = \frac{\pi (P_0 - P_L)}{128 \mu L} D^4 \tag{10.36}$$

Equation (10.36) is known as the Hagen – Poiseuille equation. Thus, if the pressure drop is given, we can calculate the volumetric flow rate in the pipe and vice-versa. This equation can also be used for the calculation of viscosity in capillary flow viscometer. However, it may be noted that Hagen – Poiseuille equation is valid only for fully developed laminar flow. Therefore, when this equation is used for various calculations there may be some errors due to developing and exiting flow at both ends of the pipe. Hence, this equation has to be modified for real situations.

Friction factor

The friction factor is a dimensionless number, which provides an idea about the magnitude of shear stress produced by a solid boundary as fluid flows. This is defined as the ratio of shear stress at the wall and the kinetic energy head of the fluid, $\frac{1}{2} \rho v^2_{zavg}$. Here, ρ is the density and v_{zavg} is the average velocity of fluid. The friction factor is thereby defined as

$$f = \frac{\tau_w}{\frac{1}{2}\rho v^2_{zavg}}$$

where, τ_w is the shear force per unit area on the wall of the tube. This may be calculated as shown below

$$\tau_w = -\left(-\tau_{rz}\big|_{r=R}\right)$$

Here, first minus sign is used as the inside surface of the tube wall has outer normal in the negative r direction and second minus sign is used because τ_{rz} is treated here as momentum flux. If τ_{rz} is treated as actual shear force then positive sign would have to be taken. For fully developed laminar flow, the velocity profile is parabolic and is given by

$$v_z = v_{z\,max}\left[1-\frac{r^2}{R^2}\right]$$

Evaluating the velocity gradient at the wall (r=R), we have

$$\left.\frac{dv_z}{dr}\right|_{r=R} = -\frac{2v_{z\,max}}{R}$$

Thus, the shear stress considered as momentum flux is given by

$$\tau_{rz}\big|_{r=R} = -\mu\left.\frac{dv_z}{dr}\right|_{r=R} = +\frac{2\mu v_{z\,max}}{R}$$

or

$$\tau_w = \tau_{rz}\big|_{r=R} = \frac{2\mu v_{zmax}}{R} = \frac{2\mu(P_0 - P_L)}{4\mu L}\frac{R^2}{R} = \frac{(P_0 - P_L)}{2L}R$$

The friction factor may now be calculated as shown below

$$f = \frac{\tau_w}{\frac{1}{2}\rho v^2_{zavg}} = \frac{\frac{(P_0 - P_L)}{2L}R}{\frac{1}{2}\frac{\rho(P_0 - P_L)}{8\mu L}R^2} = \frac{8\mu}{\rho R v_{zavg}}$$

or

$$f = \frac{16\,\mu}{\rho D v_{zavg}} = \frac{16}{Re}$$

Equation (11.7) shows that the friction factor in laminar flow region depends only on the Reynolds number. Clearly, the friction factor is also a dimensionless number.

Friction factor in turbulent flow

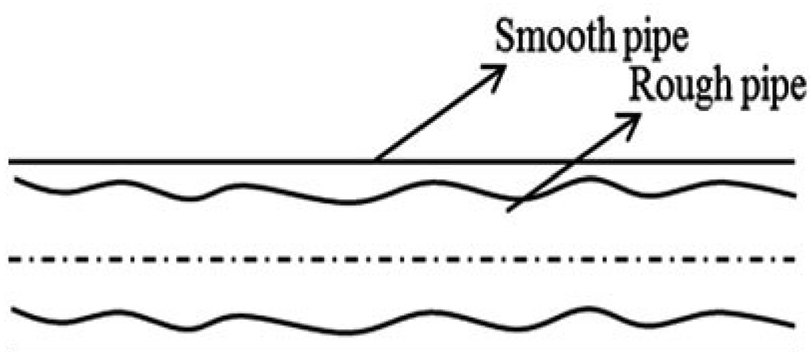
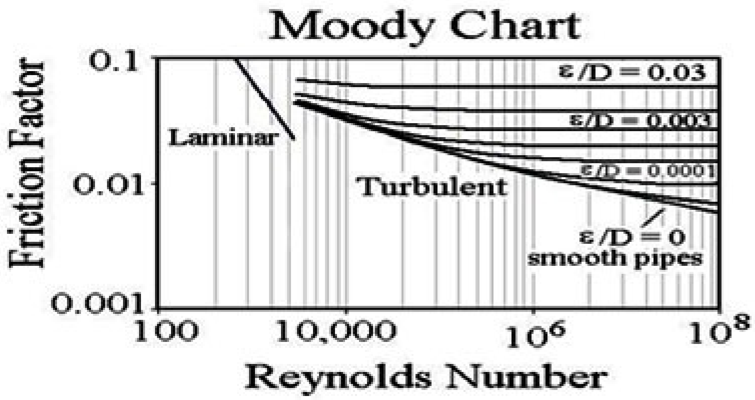


Fig 11.1 Smooth and rough surface of pipe

In turbulent flow, the friction factor also depends on the surface of the pipe. A rough pipe leads to higher turbulence than a smoother pipe, so that the friction factor for smoother pipes is less than that for rougher pipes. The ratio of surface roughness height (∈) to pipe diameter (D) is used to quantify the “roughness” of the pipe surface. In practice, the shear stress on the wall may be calculated by measuring the pressure drop across the pipe for a given flow rate. Thus, friction factor may be calculated as the function of Reynolds number and plotted on a log-log plot for a given surface roughness. The curves are different for different surface roughness as shown in figure. (11.2). The collection of these f-Re plots is called Moody Chart as shown in figure below, and can be used for estimating the friction factor for given flow parameters.



Ref: http://www.brighthub.com/engineering/civil/articles

Solution of some more fluid flow problems by shell momentum balance approach

In this section, we solve a few more fluid mechanics problems in simple geometries using the shell momentum balance approach. The detail procedure, which was also used in previous example, is outlined below.

1) Make a diagram of the flow geometry with the appropriate coordinate system

2) Specify all necessary assumptions

3) Intuitively assume the velocity profile

This is an important step for solving these problems. In laminar flow, the fluid flows in parallel layer without mixing. Thus, it is easy to guess the non-zero components of velocities by intuition.

4) Apply of the equation of continuity to modify the velocity profile

5) Determine the non-zero shear stress component(s)

Since the shear stress components depend on the velocity profile, the non-zero shear stress components may now be determine.

6) Determine control volume and make shell momentum balance for the control volume

Draw control volume in system diagram according to system shape, size and problem statement. The selection of proper control volume is very important to solve problem correctly. The control volume should be select in such way that it can be easily integrated for whole system. The differential length of control volume should be taken in direction of changing velocity.
Write momentum balance equation for the control volume. The shear stress may be considered as shear force or as momentum flux, both provide the same results as shown in previous example . Write down all surface and body force acting on the fluid carefully. Finally obtain a appropriate

differential equation and integrate.

7) Boundary conditions

Use appropriate boundary conditions which help us to determine the constant of integration in above step.

1.8 Falling film on an inclined flat surface

An inclined surface of length L and width W is situated at an angle B to the vertical direction as shown in Fig. (11.3). A Newtonian fluid is freely falling on the surface as a film of thickness δ. Assuming the flow to be laminar, determine the velocity profile, flow rate and shear force on the surface by the fluid.

Solution

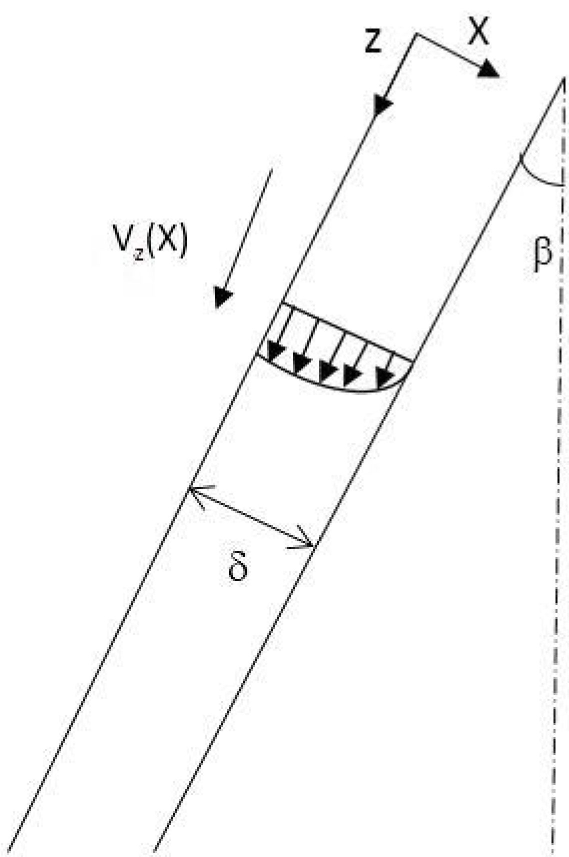


Fig 11.3 Laminar flow on an inclined surface

Assumptions

- Constant density, viscosity
- Steady state
- Laminar flow (simple shear flow)
- Fully developed flow
- Newton's law of viscosity is applicable

Assume velocity profile
The fluid is flowing in the z direction, hence only the z component of velocity is non-zero. Thus, we may assume

$$v_x = 0,\quad v_y = 0 \quad \text{and}$$
$$v_z = v_z(x, y, z, t)$$

We may further assume that vz does not depends upon y coordinate. Since the flow is steady, vz does not depend on time. Thus,

$$v_z = v_z(x, z)$$

Using the equation of continuity in the cartesian coordinates for constant fluid density, we have

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

which reduces to

$$\frac{\partial v_z}{\partial z} = 0$$

Equation (11.11) indicates that vz does not depend on the z coordinate. Thus,

$$v_z = v_z(x)$$

There are nine components of the shear stress as shear force or momentum flux, namely

$\tau_{xx}\,\tau_{xy}\,\tau_{xz} \rightarrow$ acting on x surface

$\tau_{yx}\,\tau_{yy}\,\tau_{yz} \rightarrow$ acting on y surface

$\tau_{zx}\,\tau_{zy}\,\tau_{zz} \rightarrow$ acting on z surface

Since vz is only the non-zero velocity, and also it is the function of x coordinate , τ_{zx} is the only significant component of shear stress and we need to write momentum balance only in z direction. Because the pressure is same at both ends of the inclined plane, there is no pressure force on the fluid. Now, we can solve this problem by assuming shear stress as a shear force or shear stress as momentum flux.

Assuming τ_{xz} as momentum flux
Draw a control volume of length L, width W and differential thickness dx.

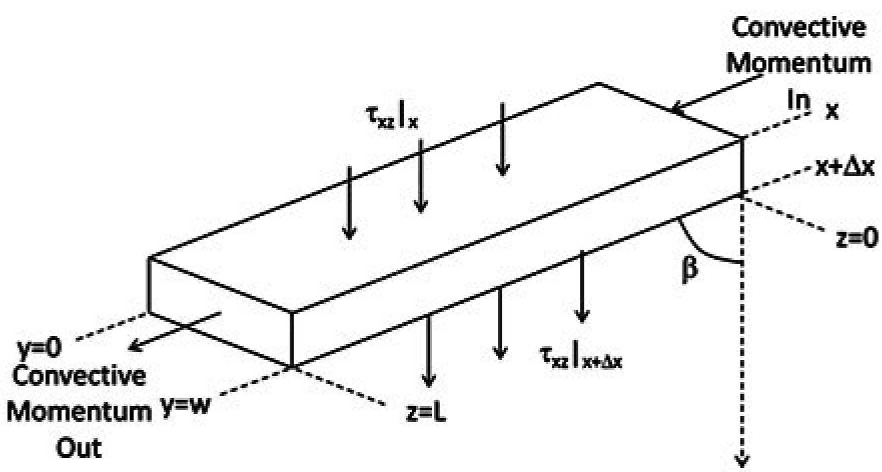


Fig 11.3 Control volume for falling film problem

Momentum balance in x direction

Rate of momentum flux entering CV due to viscous transport at

$$x = LW\tau_{xz}\Big|_x$$

Rate of momentum flux leaving CV due to viscous transport at

$$x+\Delta x = LW\tau_{xz}\Big|_{x+\Delta x}$$

Gravity force acting on fluid in z direction

$$= (LW\Delta x)(\rho g \cos\beta)$$

Rate of momentum flux entering in CV due to convective transport

$$= (\rho v_z W dx\ v_z)\big|_{z=0}$$

Rate of momentum flux leaving from CV due to convective transport

$$= (\rho v_z W dx\ v_z)\big|_{z=L}$$

Now, when above terms are substituted for z-momentum balance, we obtain

$$(\rho v_z w dx v_z)\big|_{z=0} - (\rho v_z w dx\, v_z)\big|_{z=L} + LW(\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x}) + (LW\Delta x)(\rho g \cos\beta) = 0$$

Since the velocity vz does not depends on z coordinate, the first two terms cancel out and we obtain

$$LW(\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x}) + (LW\Delta x)(\rho g \cos\beta) = 0$$

Dividing Equation (11.19) by volume of the control volume (LWΔx), we have

$$\frac{(\tau_{xz}\big|_x - \tau_{xz}\big|_{x+dx})}{\Delta x} + (\rho g \cos\beta) = 0$$

As Δx→0 , The Equation (11.20)simplified to

$$\frac{d\tau_{xz}}{dx} = \rho g \cos\beta$$

The Newton's law of viscosity (here, shear stress is defined as momentum flux) is given by

$$\tau_{xz} = -\mu \frac{dv_z}{dx}$$

$$\frac{d}{dx}\left(-\mu \frac{dv_z}{dx}\right) = \rho g \cos\beta$$

or

$$-\mu \frac{d^2 v_z}{dx^2} = \rho g \cos\beta$$

or

$$\frac{d^2 v_z}{dx^2} = -\left(\frac{\rho g \cos\beta}{\mu}\right)$$

By integrating the Equation (11.25), we have

$$\frac{dv_z}{dx} = -\left(\frac{\rho g \cos\beta}{\mu}\right) x + c_1$$

or

$$v_z = -\left(\frac{\rho g \cos\beta}{\mu}\right)\frac{x^2}{2} + c_1 x + c_2$$

The above equation requires two boundary conditions for determining c1 and c2.

Boundary conditions

1 At x=0 the liquid surface is in contact with air where the shear stresses at both gas liquid phases should be equal. Thus,

$$\tau_{xz\,(air)}\Big|_{x=0} = \tau_{xz\,(liquid)}\Big|_{x=0}$$

Since both may be assumed Newtonian fluids, we have

$$\mu_g \rho_g \frac{dv_{z(air)}}{dx}\bigg|_{x=0} = \mu\rho \frac{dv_z}{dx}\bigg|_{x=0}$$

where ρg is the density and µg is the viscosity of air. Thus

$$\frac{dv_z}{dx}\bigg|_{x=0} = \frac{\mu_g \rho_g}{\mu\rho} \frac{dv_{z(air)}}{dx}\bigg|_{x=0}$$

Since, µg and ρg is much smaller than µ and ρ, and Equation (11.30) may be approximately written as

$$\frac{dv_z}{dx}\bigg|_{x=0} = 0$$

Substituting above boundary condition in Equation (11.26), we obtain

$$c_1 = 0$$

2. At x=δ no slip boundary condition may be applied, i.e.,

at

$$x = \delta \ , v_z = 0$$

Thus, from Equation (11.27), we get

$$0 = -\left(\frac{\rho g \cos\beta}{\mu}\right)\frac{\delta^2}{2} x + c_2$$

or

$$c_2 = \left(\frac{\rho g \cos\beta}{\mu}\right)\frac{\delta^2}{2}$$

Finally the velocity profile is obtained as

$$v_z = -\left(\frac{\rho g \cos\beta}{\mu}\right)\frac{x^2}{2} + \left(\frac{\rho g \cos\beta}{\mu}\right)\frac{\delta^2}{2}$$

or

$$v_z = \frac{\rho g \delta^2 \cos\beta}{2\mu}\left(1 - \left(\frac{x}{\delta}\right)^2\right)$$

Falling film "Assuming τ_{xz} as shear force"

Now, we again solve the same problem (falling film over an inclined plane) by treating shear stress as a shear force. For this purpose, we take the same control volume as before.
For momentum balance in z direction, all terms are same as before except the terms for shear forces. Here, τ_{xz} represents the force in z direction acting on the surfaces which have normal in x direction. Shear force is positive if the outward normal is in positive direction and negative if normal is in negative direction. Thus,

shear force at x=x is
$$-LW\tau_{xz}\big|_x$$

Shear force at x=x+Δx is
$$+LW\tau_{xz}\big|_{x+\Delta x}$$
;

The z momentum balance for this case is as follows

$$LW(\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x) + (LW\Delta x)(\rho g \cos\beta) = 0$$

Dividing Equation (12.3) by the volume of control volume WLΔx, we have

$$\frac{(\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x)}{\Delta x} + (\rho g \cos\beta) = 0$$

As Δx→0 Equation (12.4) leads to

$$\frac{d\tau_{xz}}{dx} = -\rho g \cos\beta$$

Now, substituting the Newton's law of viscosity for shear stress as a force

$$\tau_{xz} = \mu \frac{dv_z}{dx}$$

Therefore,

$$\mu \frac{d^2 v_z}{dx^2} = -\rho g \cos\beta$$

Equations (11.24) and (12.7) are the same, which show that both approaches provide the same answer.

Maximum velocity

It is clear from Equation (11.37) that the maximum velocity is given by

$$v_{z,max} = \frac{\rho g \delta^2 \cos\beta}{2\mu}$$

Average velocity and volumetric flow rate of falling film

vz is the linear velocity in z direction. Hence, the volumetric flow rate can be determined by integrating it over the cross section of flow (Wδ).Thus ,

$$Q = \int_0^W \int_0^\delta v_z \, dx\, dy$$

From Equation (11.37), we get

$$Q = \int_0^W \int_0^\delta \frac{\rho g \delta^2 \cos\beta}{2\mu} \left(1 - \left(\frac{x}{\delta} \right)^2 \right) dx dy$$

By integrating Equation (12.10), we find

$$Q = \frac{W \rho g \delta^3 \cos\beta}{3\mu}$$

To obtain the average velocity, we divide the volumetric flow rate by the cross sectional area.

$$\langle v_z \rangle_{avg} = \frac{Q}{\int_0^W \int_0^\delta dx\, dy}$$

or

$$\langle v_z \rangle_{avg} = \frac{\rho g \delta^2 \cos\beta}{3\mu}$$

Equation (12.12) may also be written as

$$\langle v_z \rangle_{avg} = \frac{2}{3} v_{z,max}$$

Force acting on solid surface due to the fluid

$$F = \int_0^L \int_0^W + \left(+ \tau_{xz}\big|_{x=\delta} \right) dy\, dz$$

(Note: in Equation (12.14), first '+' sign shows the direction of the normal of the inclined surface and second '+' sign is taken since shear stress is defined as shear force). Thus,

$$F = \rho g \delta L W \cos\beta$$

In this lecture, we have once again seen that the shear stress tensor may be assumed as a shear force or as a momentum flux. In either case, we finally obtain the same expression for the velocity profile. The only difference is that when we treat shear stress as a shear force, it is included in the summation of all forces term in the momentum balance equation, while when we treat shear stress as momentum flux, it is written as momentum entering and leaving by the viscous transport. From now onwards, we will treat shear stress as momentum flux as it is more consistent with what we see in heat transfer as Fourier's law of heat conduction and in mass transfer as Fick's law of diffusion. Thus, in transport phenomena (Momentum transport, Heat transport, and Mass transport) for the basic transport laws we have minus sign in front the relevant gradient implying fluxes flow from higher values to lower values.

Falling film on the outside of a circular shell

In an experiment, a fluid flows upward through a small circular shell and then flows downward out side the tube under laminar conditions as shown in Fig. 12.2. We need to set up a relevant momentum balance and determine the velocity profile, mass flow rate and the force acting on outer surface of the tube.

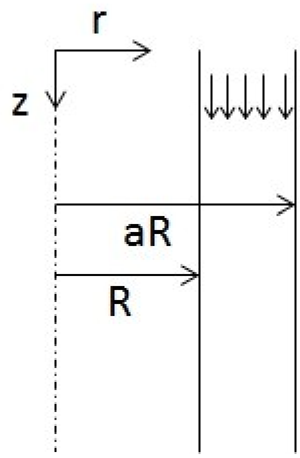


Fig 12.2 Falling film outside the circular tube

Assumptions

- Density and viscosity are constants.
- Steady state.
- Fully developed laminar flow.
- Newton's law of viscosity is applicable.

Non-zero velocities
Fluid is flowing in the z direction due to gravity. There is no driving force in the θ direction and a solid surface is present in the r direction. Therefore, we may intuitively assume that

$$v_z = v_z(r,z)$$

$$v_\theta = 0$$

$$v_r = 0$$

Now, using the equation of continuity in cylindrical coordinate system, we have

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

or

$$\rho \frac{\partial}{\partial z}(v_z) = 0$$

From Equation (12.18), we obtain that vz is independent to z. Therefore,

$$v_z = v_z(r)$$

Choose a control volume in the film of differential thickness dr and length L (it is a cylindrical shell).

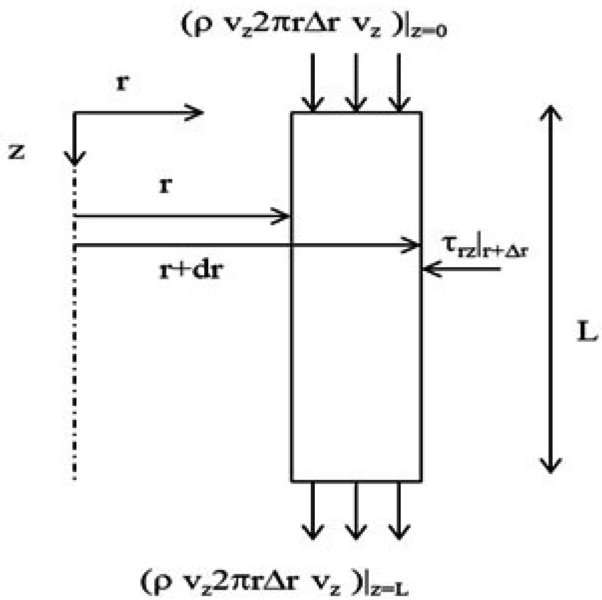


Fig 12.3 Control volume for falling film outside the circular tube

There are nine components of shear stress tensor. Since the fluid is flowing in z direction and it is a function of r only, we may argue as before τ_{rz} is the only important component of the shear stress tensor. The other components are insignificant for momentum balance in z direction. The momentum balance in z-direction is given below.

Momentum balance for control volume

Convective momentum entering the control volume at z=0 is

$$(\rho v_z 2\pi r \Delta r v_z)\big|_{z=0} \qquad \textbf{(12.20)}$$

Convective momentum leaving the control volume at z=L is

$$(\rho v_z 2\pi r \Delta r v_z)\big|_{z=L} \qquad \textbf{(12.21)}$$

Shear stress as momentum flux entering the control volume at r= r is

$$(2\pi rL\tau_{rz})\Big|_{r=r} \qquad \textbf{(12.22)}$$

Shear stress as momentum flux entering the control volume at r= r + Δr is

$$(2\pi rL\tau_{rz})\Big|_{r=r+\Delta r} \qquad \textbf{(12.23)}$$

{Note: If you consider shear stress as momentum flux, then it always flows in the positive direction of axes}

Fluid is flowing only due to gravity and may be written as

$$(2\pi r\Delta rL\rho g) \qquad \textbf{(12.24)}$$

Substituting above terms, we obtain

$$(\rho v_z 2\pi rDrv_z)\Big|_{z=0} - (\rho v_z 2\pi r\Delta rv_z)\Big|_{z=L} + (2\pi rL\tau_{rz})\Big|_{r=r} - (2\pi rL\tau_{rz})\Big|_{r=r+\Delta r} + 2\pi r\Delta rL\rho g = 0$$

Since velocity, vz, is not dependent on the z, the first two terms in above equation are equal and cancel out, leaving the following equation for momentum balance.

$$(2\pi rL\tau_{rz})\Big|_{r=r} - (2\pi rL\tau_{rz})\Big|_{r=r+\Delta r} + 2\pi r\Delta rL\rho g = 0$$

Dividing Equation (12.26) by volume of control volume $2\pi r\Delta rL$, we obtain

$$\frac{(r\tau_{rz}|_r - r\tau_{rz}|_{r+\Delta r})}{r\Delta r} = -\rho g$$

As dr→0, Equation (12.27) reduces to

$$-\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = -\rho g$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(-\mu\frac{dv_z}{dr}\right)\right) = \rho g$$

After integration we obtain

$$\frac{dv_z}{dr} = -\frac{\rho g r}{2\mu} + \frac{c_1}{r}$$

and

$$v_z = -\frac{\rho g r^2}{4\mu} + c_1 \ln r + c_2$$

Here, c_1 and c_2 are the constants of integration.

Boundary conditions

1. r=aR we have the air water interface where we may assume that

$$\frac{dv_z}{dr} = 0$$

(The explanation is given earlier in Lecture 11.)
Substituting the above boundary condition in Equation (12.31), we obtain

$$c_1 = \frac{\rho g a^2 R^2}{2\mu}$$

2. At r = R, no slip boundary condition is applicable. Thus,

$$v_z\big|_{r=R} = 0$$

Using this boundary condition, we obtain

$$c_2 = \frac{\rho g R^2}{4\mu} - c_1 \ln R$$

or,

$$c_2 = \frac{\rho g R^2}{4\mu} - \frac{\rho g a^2 R^2}{2\mu} \ln R$$

Therefore, the velocity profile is given by

$$v_z = -\frac{\rho g r^2}{4\mu} + \frac{\rho g a^2 R^2}{2\mu} \ln r + \frac{\rho g a^2 R^2}{4\mu} - \frac{\rho g a^2 R^2}{2\mu} \ln R$$

or

$$v_z = \frac{\rho g R^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2 + 2a^2 \ln\left(\frac{r}{R}\right)\right]$$

Maximum velocity

At r = aR, the velocity is maximum. Thus,

$$v_{z,max} = \frac{\rho g R^2}{4\mu}\left[1 - a^2 + 2a^2 \ln a\right]$$

1.9 Flow through Annulus

A Newtonian fluid is flowing in a narrow slit (B<<W<<L), formed by two parallel plates as shown in Fig. (13.1), due to the combined effect of both gravity and pressure. Determine the velocity profile, average velocity, and mass flow rate for laminar and steady flow.

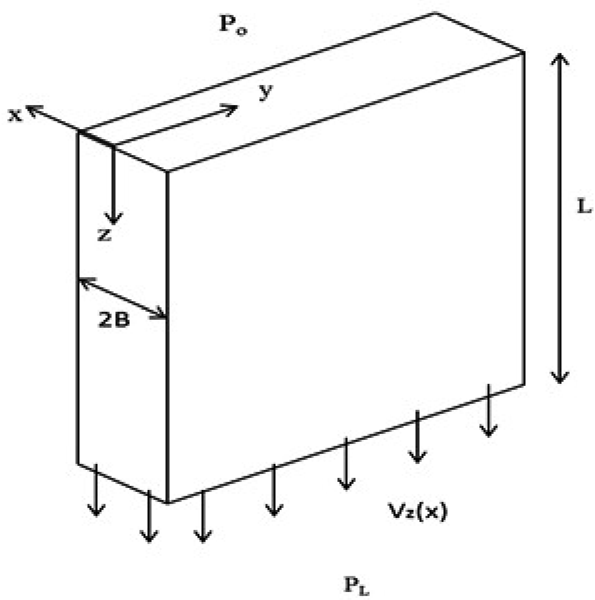


Fig 13.1 Laminar flow in narrow slit

Assumptions

- Density and viscosity are constant.
- Steady state.
- Laminar Flow(simple shear flow).

- Newton's law of viscosity is applicable.

Fluid is flowing in the z direction due to both gravity and pressure difference. Therefore, vz is the only important velocity component. As the slit is very narrow (B<<W<<L), we may assume that end effects are negligible in y direction and vz is not a function of y.

Thus, intuitively we assume the velocity profile as,

$$v_z = v_z(x,z)$$

$$v_x = 0$$

$$v_y = 0$$

Now, using the equation of continuity in cartesian coordinate system

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

or

$$\rho \frac{\partial v_z}{\partial z} = 0$$

Therefore,

$$v_z = v_z(x)$$

From above velocity profile, we may conclude that τ_{xz} is the only important shear stress component. We now select a cuboidal control volume of dimensions L, W, Δx, as shown in Fig. 13.2 (Note: differential thickness is chosen in x direction)

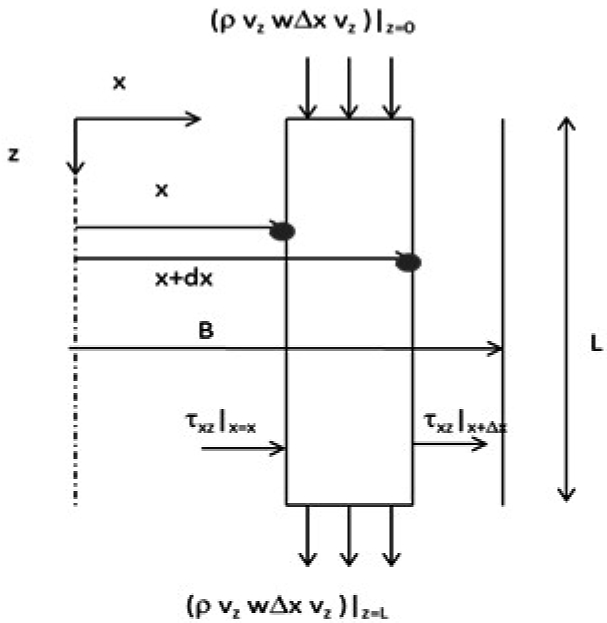


Fig 13.2 Control volume for laminar flow in narrow slit.

Momentum balance in z direction

Convective momentum entering the CV at z=0 is

$$(\rho v_z W \Delta x v_z)\big|_{z=0} \qquad \textbf{(13.5)}$$

Convective momentum leaving the CV at z=L is

$$(\rho v_z W \Delta x v_z)\big|_{z=L} \qquad \textbf{(13.6)}$$

Momentum entering CV by viscous transport at x=x is

$$(LW\tau_{xz})\big|_{x=x} \qquad \textbf{(13.7)}$$

Momentum leaving the CV by viscous transport at x=x+Δx is

$$(LW\tau_{xz})\big|_{x=x+\Delta x} \qquad \textbf{(13.8)}$$

Pressure force at z=0 is

$$P_0 W \Delta x \qquad \textbf{(13.9)}$$

Pressure force at z=L is

$$-P_L W \Delta x \qquad \textbf{(13.10)}$$

Gravity force on CV is

$$\rho\, g\, \Delta x L W \qquad \textbf{(13.11)}$$

Substituting these terms into the momentum balance in z direction, we get

$$(\rho v_z w \Delta x v_z)|_{z=0} - (\rho v_z w \Delta x v_z)|_{z=L} + (LW\tau_{rz})|_{r=r} - (LW\tau_{rz})|_{r=r+\Delta r}$$
$$+P_0 w\Delta x - P_L w\Delta x + \rho g \Delta x LW = 0$$

Since, vz is not a function of z, the first two convective momentum terms represented by Equations (13.5) and (13.6) are equal and hence cancel out from the above equation and we get

$$(LW\tau_{xz})\Big|_{x=x} - (LW\tau_{xz})\Big|_{x=x+\Delta x} + P_0 w\Delta x - P_L w\Delta x + \rho g \Delta x LW = 0$$

Dividing Equation (13.13) by the volume of the control volume ΔxLW , we obtain

$$\frac{(\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x})}{\Delta x} = \frac{P_L - P_0}{L} - \rho g$$

Combining the pressure force with gravity, and taking the limit as $\Delta x \rightarrow 0$, we have

$$\frac{d}{dx}(\tau_{xz}) = \left(\frac{(P_0 - \rho g z(0)) - (P_L + \rho g z(L))}{L}\right)$$

or

$$\frac{d}{dx}(\tau_{xz}) = \left(\frac{P_{c0} - P_{cL}}{L}\right)$$

where, $P_c = P - \rho g z$

$$\frac{d}{dx}(\tau_{xz}) = \left(\frac{P_{c0} - P_{cL}}{L}\right)x + c_1$$

Substituting Newton’s law of viscosity, we have

$$\tau_{xz} = -\mu \frac{dv_z}{dx}$$

or

$$-\mu \frac{dv_z}{dx} = \left(\frac{P_{cL} - P_{c0}}{L} \right) x + c_1$$

and finally after integration, we get

$$v_z = -\left(\frac{P_{c0} - P_{cL}}{\mu L} \right) \frac{x^2}{2} - \frac{c_1}{\mu} x + c_2$$

Boundary conditions are

1. At x=0 , the velocity profile must be symmetric. Therefore, $\left. \frac{dv_z}{dx} \right|_{x=0} = 0$
or

$$c_1 = 0$$

2. At x=B , no slip boundary condition is applicable. Thus, $v_z = 0$

or

$$c_2 = \left(\frac{P_{c0} - P_{cL}}{\mu L} \right) \frac{B^2}{2}$$

Thus, velocity profile may be written as

$$v_z = \left(\frac{P_{c0} - P_{cL}}{\mu L} \right) \frac{B^2}{2} \left[1 - \left(\frac{x}{B} \right)^2 \right]$$

Equation (13.23) describes the velocity profile in the narrow slit.

Mass flow rate and average velocity

Mass flow rate = Volumetric flow rate × Density

$$= \rho \int_0^W \int_{-B}^{B} v_z \, dx dy$$

By substituting the value of velocity from Equation (13.23), we have

$$= \int_0^W \int_{-B}^{B} \left[\left(\frac{P_{c0} - P_{cL}}{\mu L} \right) \frac{B^2}{2} \left[1 - \left(\frac{x}{B} \right)^2 \right] dx dy \right] \rho$$

or

$$m = \frac{2}{3} \frac{\rho W \left(P_{c0} - P_{cL} \right) B^3}{\mu L}$$

Average velocity = Volumetric flow rate/ Area of cross section

$$= \frac{\frac{2}{3} \frac{W \left(P_{c0} - P_{cL} \right) B^3}{\mu L}}{2 B W}$$

or

$$\langle v_z \rangle = \frac{1}{3} \frac{\left(P_{c0} - P_{cL} \right) B^2}{\mu L}$$

Annular flow with inner cylinder moving axially

In a wire coating machine, a wire of radius kR is moving into a cylindrical hollow die. The radius of the die is R , and the wire is moving with a velocity v0 along the axis. The die is filled with a Newtonian fluid, a coating material. The pressure at both ends of the die is same. Find the velocity distribution in the narrow annular region. Obtain the viscous force acting on the wire of length L . Also, find the mass flow rate through the annular region.

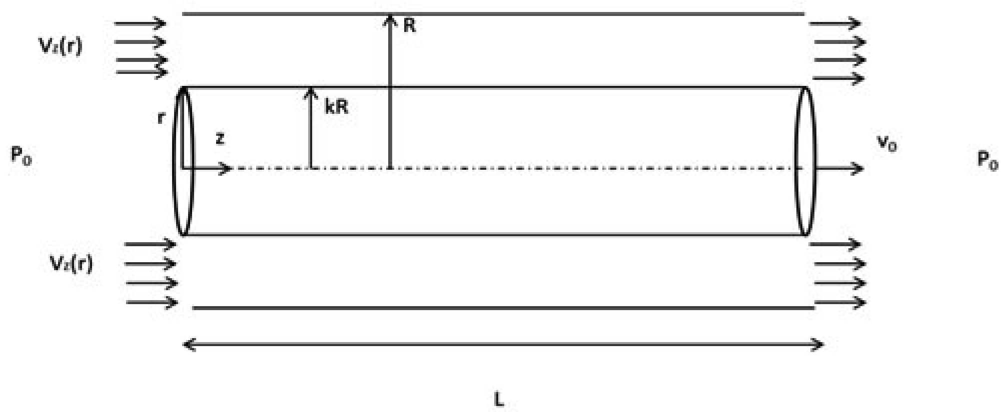


Fig 13.3 Annular flow with the inner cylinder moving axially

Assumptions

- density and viscosity are constant
- steady state.
- laminar (simple shear flow).
- Newton's law of viscosity is applicable.

Velocity components

The fluid is moving due to the motion of the wire in z direction so vz is the only important velocity component. There is no solid boundary in θ direction, and the flow is steady, therefore vz will not depend on θ and t. Hence,

$$v_z = v_z(r, z)$$

Now, applying the equation of continuity in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

or

$$\rho \frac{dv_z}{dz} = 0$$

Thus,

$$v_z = v_z(r)$$

This result indicates that τ_{rz} is the only significant shear stress among the 9 components for momentum balance in z direction. Now, consider a control volume of differential thickness dr and length L at a distance r away from the center. We may write the momentum balance in z direction.

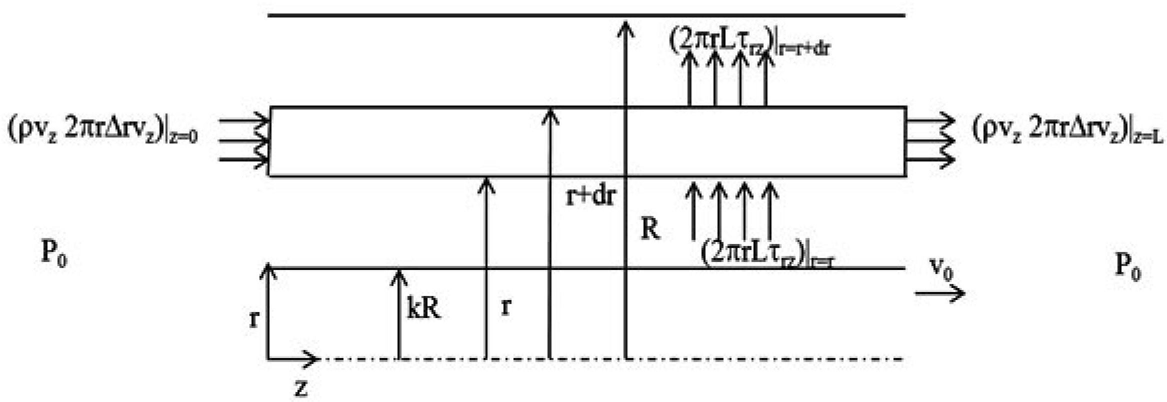


Fig 13.4 Control volume for annular flow with the inner cylinder moving axially

Convective momentum entering at z=0 is

$$(\rho v_z 2\pi r\Delta r v_z)|_{z=0}$$

Convective momentum leaving at z=L is

$$(\rho v_z 2\pi r\Delta r v_z)|_{z=L}$$

Momentum entering control volume by viscous transport at r = r is

$$(2\pi rL\tau_{rz})|_{r=r}$$

Momentum leaving control volume by viscous transport at at r = r +Δr is

$$(2\pi rL\tau_{rz})|_{r=r+\Delta r}$$

Now, the momentum balance over the control volume is below

$$(\rho v_z 2\pi r\Delta r v_z)|_{z=0} - (\rho v_z 2\pi r\Delta r v_z)|_{z=L} + (2\pi r L\tau_{rz})|_{r=r} - (2\pi r L\tau_{rz})|_{r=r+\Delta r} = 0$$

Since velocity vz is not dependent on z coordinate therefore the convective terms represented by equations (13.29) and (13.30) are equal and hence cancelled out. Leaving with the following equation,

$$(2\pi r L\tau_{rz})|_{r=r} - (2\pi r L\tau_{rz})|_{r=r+\Delta r} = 0$$

Dividing equation (13.34) by volume of the control volume, $2\pi r\Delta r L$

$$\frac{(r\tau_{rz}|_r - r\tau_{rz}|_{r+\Delta r})}{r\Delta r} = 0$$

Taking the limit as dr→0, we have

$$\frac{1}{r}\frac{d}{dr}(r\tau_{rz}) = 0$$

and after integration

$$\tau_{rz} = \frac{c_1}{r}$$

where c_1 is an integration constant. Now, using Newton’s law of viscosity, we get

$$-\mu\frac{dv_z}{dr} = \frac{c_1}{r}$$

or

$$v_z = -\frac{c_1}{\mu}\ln r + c_2$$

where c_2 is another integration constant.

Boundary conditions are

at r = kR , $v_z = v_0$ **(13.40)**

or

$$v_0 = -\frac{c_1}{\mu} \ln kR + c_2$$

and at r = R, $v_z = 0$ **(13.42)**

or

$$c_2 = \frac{c_1}{\mu} \ln R$$

From Equation (13.41)

$$v_0 = -\frac{c_1}{\mu} \ln kR + \frac{c_1}{\mu} \ln R$$

or

$$v_0 = \frac{c_1}{\mu} \ln \frac{R}{kR}$$

or

$$c_1 = \frac{v_0 \mu}{\ln(1/k)}$$

By substituting the value of c1 into Equation (13.39), the velocity profile may be obtained as

$$v_z = -\frac{v_0 \ln r}{\ln(1/k)} + \frac{v_0 \ln R}{\ln(1/k)}$$

or

$$\frac{v_z}{v_0} = \frac{ln(r / R)}{ln(1 / k)}$$

Mass flow rate in the annular region

$$w = \rho \int_0^{2\pi} \int_{kR}^{R} v_z r dr d\theta$$

or

$$w = \frac{2\pi\rho v_0}{ln\; k} \int_{kR}^{R} r\, ln\left(\frac{r}{R}\right) dr$$

or

$$w = \frac{\pi R^2 \rho v_0}{2}\left[\frac{(1-k^2)}{ln(1/k)} - 2k^2\right]$$

Drag force acting on the wire may be calculated as

$$F = 2\pi kRL + \left(-\tau_{rz}\right)\Big|_{r=kR}$$

or

$$F = 2\pi kRL\left(\mu \frac{dv_z}{dr}\bigg|_{r=kR}\right)$$

By substituting the value of velocity vz, we obtain

$$F = 2\pi kRL\mu \frac{d}{dr}\left[v_0 \frac{ln(r / R)}{ln(1 / k)}\right]\Bigg|_{r=kR}$$

Finally, we obtain the expression for drag force as

$$F = \frac{2\pi L \mu v_0}{ln(1/k)}$$

1.10 Flow of two immiscible fluids between two parallel plates

Two immiscible liquids are flowing in between two adjacent, parallel plates. Solve the problem for velocity profile and mass flow rate.

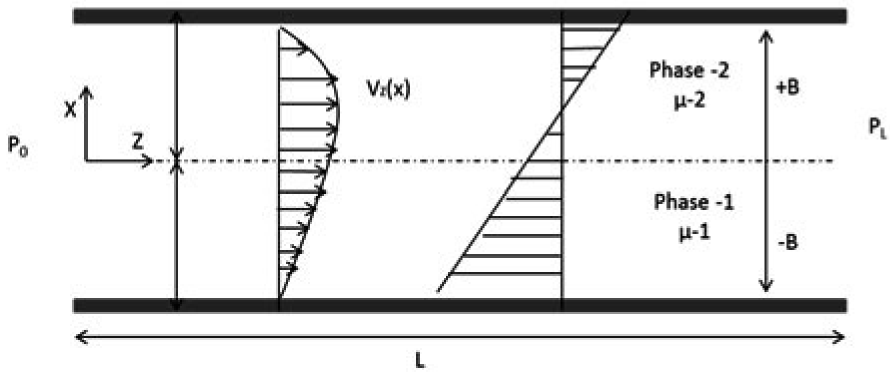


Fig 14.1 Flow of two immiscible fluids between a pair of horizontal plates

Assumptions

- Density and viscosity are constants.
- Steady state.
- Laminar (simple shear flow) fully developed.
- Newton’s law of viscosity is applicable.

Since fluid is flowing in z direction only, therefore vz is the only non-zero velocity component. We can assume that end effects are negligible in y direction and hence, vz is not a function of y. thus,

$$v_z = v_z(x, z) \quad (1$$

$$v_x = 0$$
$$v_y = 0 \quad (:$$

Now using equation of continuity for Cartesian coordinate system

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho v_x)+\frac{\partial}{\partial y}(\rho v_y)+\frac{\partial}{\partial z}(\rho v_z)=0$$

$$\rho \frac{dv_z}{dz} = 0$$

which implies that

$$v_z = v_z(x)$$

As before, we may concluded that τ_{xz} is only important shear stress component for momentum balance in z direction. Choosing a differential strip of thickness dx and length L as a control volume, we have

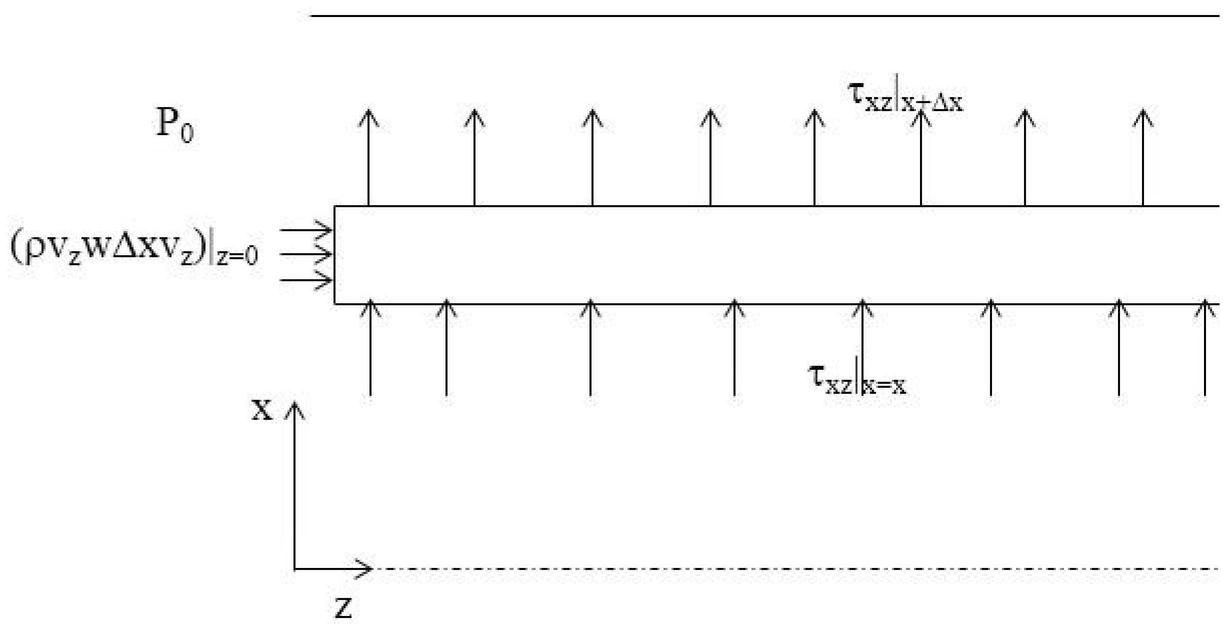


Fig 14.2 Control volume for Flow of two immiscible fluid between a pair of horizontal plates

Momentum balance in control volume

Convective momentum entering CV at z=0 is

$$(\rho v_z w \Delta x v_z)|_{z=0} \qquad \textbf{(14.5)}$$

Convective momentum leaving CV at z=0 is

$$(\rho v_z w \Delta x v_z)|_{z=L} \quad \textbf{(14.6)}$$

Momentum entering CV by viscous transport at x=x is $(LW\tau_{xz})|_{x=x}$ **(14.7)**

Momentum leaving CV by viscous transport at x=x+Δx is $(LW\tau_{xz})|_{x=x+\Delta x}$ **(14.8)**

Pressure force at z=0 is

$$P_0 \Delta x w \quad \textbf{(14.9)}$$

Pressure force at z=L is

$$-P_L \Delta x w \quad \textbf{(14.10)}$$

The equation for momentum balance can be written as

$$(\rho v_z w \Delta x v_z)|_{z=0} - (\rho v_z w \Delta x v_z)|_{z=L} + (LW\tau_{rz})|_{x=x} - (LW\tau_{rz})|_{x=x+\Delta x} + P_0 w \Delta x - P_L w$$

As before, convective terms cancel out and Equation (14.11) reduces to the following equation.

$$(LW\tau_{rz})|_{x=x} - (LW\tau_{rz})|_{x=x+\Delta x} + P_0 w \Delta x - P_L w \Delta x = 0$$

Dividing Equation (14.12) by volume of control volume ΔxLW, we obtain

$$\frac{(\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x})}{\Delta x} = \frac{P_L - P_0}{L}$$

Now, as Δx →0 Equation (14.13) becomes

$$\frac{d}{dx}(\tau_{xz}) = \left(\frac{P_0 - P_L}{L}\right)$$

After substituting Newton’s law of viscosity in Equation (14.14) and integrating it, we obtain

$$v_z = -\left(\frac{P_0 - P_L}{\mu L}\right)\frac{x^2}{2} + \frac{c_1}{\mu}x + c_2$$

This equation is valid for both regions. Therefore,

$$v_z^1 = -\left(\frac{P_0 - P_L}{\mu L}\right)\frac{x^2}{2} + \frac{c_1^1}{\mu}x + c_2^1$$

$$v_z^2 = -\left(\frac{P_0 - P_L}{\mu L}\right)\frac{x^2}{2} + \frac{c_1^2}{\mu}x + c_2^2$$

Here, superscript (1) represents the phase-1 and superscript (2) represents the phase-2.

Boundary conditions

There are four boundary conditions needed to solve the problem and given below

1. $x=0,\ v_z^1 = v_z^2$
2. $x=-b,\ v_z^1 = 0$
3. $x=+b,\ v_z^2 = 0$
4. $x=0,\ \frac{dv_z^1}{dx}\Big|_{x=0} = \frac{dv_z^2}{dx}\Big|_{x=0}$

This leads to the solution

$$c_1^{\,1} = c_1^{\,2}$$

$$c_2^{\,1} = c_2^{\,2}$$

$$c_1^{\,2} = \left(\frac{P_0 - P_L}{2\mu^1 L}\right) b^2 \left(\frac{2\mu^1}{\mu^1 + \mu^2}\right)$$

$$\tau_{xz} = \left(\frac{P_0 - P_L}{L}\right)\left[\frac{x}{b} - \frac{1}{2}\left(\frac{\mu^1 - \mu^2}{\mu^1 + \mu^2}\right)\right]$$

and

$$v_z^{\,1} = \left(\frac{P_0 - P_L}{2\mu^1 L}\right) b^2 \left[\left(\frac{2\mu^1}{\mu^1 + \mu^2}\right) + \left(\frac{\mu^1 - \mu^2}{\mu^1 + \mu^2}\right)\frac{x}{b} - \left(\frac{x}{b}\right)^2\right]$$

$$v_z^{\,2} = \left(\frac{P_0 - P_L}{2\mu^1 L}\right) b^2 \left[\left(\frac{2\mu^2}{\mu^1 + \mu^2}\right) + \left(\frac{\mu^1 - \mu^2}{\mu^1 + \mu^2}\right)\frac{x}{b} - \left(\frac{x}{b}\right)^2\right]$$

1.11 Derivation of equation of motion

In this section, we derive the equation of motion, which may be used for solving any fluid mechanics problem. This equation is based on axiom 2, i.e., the momentum is conserved. We consider a control volume having volume Δx,Δy,Δz fixed in space.

According to the momentum conservation equation,

$$\begin{pmatrix} Rate\ of\ accumulation\ of \\ momentum\ in\ control\ volume \end{pmatrix} = \begin{pmatrix} Net\ rate\ of\ inflow\ of \\ momentum\ by\ convectio \end{pmatrix}$$

$$+ \begin{pmatrix} Net\ rate\ of\ inflow\ of\ m \\ by\ viscous\ transpor \end{pmatrix}$$

$$+ (Pressure\ forces) + (For$$

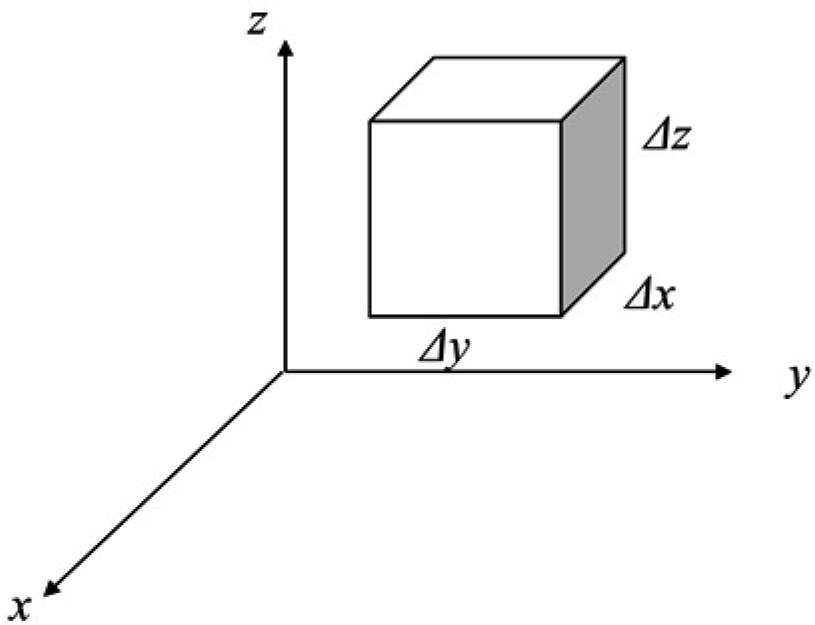


Fig 15.1 Cubical control volume fixed in space
Momentum balance in x direction

Rate of accumulation of x directed momentum in control volume

$$= \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z v_x)$$

Net rate of inflow of x directed momentum into CV by convection from x-phases

$$\left[(\rho v_x \Delta z \Delta y) v_x\right]_x - \left[(\rho v_x \Delta z \Delta y) v_x\right]_{x+\Delta x}$$

Net rate of inflow of x-momentum into CV from y-phases

$$= \left[(\rho v_y \Delta x \Delta z) v_x\right]_y - \left[(\rho v_y \Delta x \Delta z) v_x\right]_{y+\Delta y}$$

Net rate of inflow of x-momentum into CV from z-phases

$$= \left[(\rho v_z \Delta x \Delta y) v_x\right]_z - \left[(\rho v_z \Delta x \Delta y) v_x\right]_{z+\Delta z}$$

Net rate of inflow of momentum into CV due to viscous transport

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

In all shear stress component, the second index shows the direction of momentum flux and first index shows the direction in which the momentum is flowing. For example, τ_{yx} denotes the x directed momentum flowing in y direction. Therefore, the x directed momentum fluxes are τ_{xx}, τ_{yx} and τ_{zx}. Thus,

Net rate of inflow of x directed momentum by viscous transport from x phase are

$$= (\tau_{xx}\Delta z\Delta y)|_x - (\tau_{xx}\Delta z\Delta y)|_{x+\Delta x}$$

Net rate of inflow of x directed momentum by viscous transport from y phase are

$$= (\tau_{yx}\Delta x\Delta z)|_y - (\tau_{yx}\Delta x\Delta z)|_{y+\Delta y}$$

Net rate of inflow of x directed momentum by viscous transport from z phase are

$$= (\tau_{zx}\Delta x\Delta y)|_z - (\tau_{zx}\Delta x\Delta y)|_{z+\Delta z}$$

Net pressure force in x direction =

$$(P\Delta y\Delta z)|_x - (P\Delta y\Delta z)|_{x+\Delta x} \qquad \textbf{(15.9)}$$

Gravity force in x direction =

$$(\rho\Delta x\Delta y\Delta z)\, g_x \qquad \textbf{(15.10)}$$

Adding all the above terms and dividing by the volume of control volume Δx,Δy,Δz and finally taking the limits,
$\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$,and $\Delta z \rightarrow 0$, we obtain

$$\frac{\partial(\rho v_x)}{\partial t}=-\left[\frac{\partial(\rho v_x v_x)}{\partial x}+\frac{\partial(\rho v_y v_x)}{\partial y}+\frac{\partial(\rho v_z v_x)}{\partial z}\right]-\left[\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right]-\frac{\partial p}{\partial x}+\rho g$$

The above equation is the x component of equation of motion in cartesian coordinate system. Similarly, for y-direction

$$\frac{\partial(\rho v_y)}{\partial t}=-\left[\frac{\partial(\rho v_x v_y)}{\partial x}+\frac{\partial(\rho v_y v_y)}{\partial y}+\frac{\partial(\rho v_z v_y)}{\partial z}\right]-\left[\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right]-\frac{\partial p}{\partial y}$$

and for z-direction

$$\frac{\partial(\rho v_z)}{\partial t}=-\left[\frac{\partial(\rho v_x v_z)}{\partial x}+\frac{\partial(\rho v_y v_z)}{\partial y}+\frac{\partial(\rho v_z v_z)}{\partial z}\right]-\left[\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right]-\frac{\partial p}{\partial z}$$

The above three equation may be combined in vector tensor form as

$$\frac{\partial(\rho \underset{\sim}{v})}{\partial t}=-\underset{\sim}{\nabla}.(\rho\underset{\sim}{v}\underset{\sim}{v})-\underset{\sim}{\nabla}.\underset{\sim}{\tau}-\underset{\sim}{\nabla}p+\rho\underset{\sim}{g}$$

In above form, the equation of motion may be used in any coordinate system.

Equation (15.14) may be written in substantial derivative form as shown below

$$\frac{\partial(\rho \underset{\sim}{v})}{\partial t}+\underset{\sim}{\nabla}.(\rho\underset{\sim}{v}\underset{\sim}{v})=-\underset{\sim}{\nabla}.\underset{\sim}{\tau}-\underset{\sim}{\nabla}p+\rho\underset{\sim}{g}$$

if $\underset{\sim}{x}$ and $\underset{\sim}{z}$ are the two vectors. We may use the following vector identity.

$$\nabla.(\underset{\sim}{x}\underset{\sim}{z})=\underset{\sim}{x}.\underset{\sim}{\nabla}\underset{\sim}{z}+\underset{\sim}{z}(\underset{\sim}{\nabla}.\underset{\sim}{x})$$

Now, replace $\underset{\sim}{x}$ by $\rho\underset{\sim}{v}$ and $\underset{\sim}{z}$ by $\underset{\sim}{v}$, then we have

$$\nabla.(\underset{\sim}{x}\underset{\sim}{z})=\nabla.(\rho\underset{\sim}{v}\underset{\sim}{v})=\rho\underset{\sim}{v}.\underset{\sim}{\nabla}\underset{\sim}{v}+\underset{\sim}{v}(\underset{\sim}{\nabla}.\rho\underset{\sim}{v})$$

also,

$$\frac{\partial(\rho \underset{\sim}{v})}{\partial t} = \rho \frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v} \frac{\partial \rho}{\partial t} \qquad \textbf{(15.}$$

After substituting Equations (15.17) and (15.18) in Equation (15.14), the equation of motion reduces to

$$\underset{\sim}{v} \frac{\partial \rho}{\partial t} + \rho \frac{\partial \underset{\sim}{v}}{\partial t} + \rho \underset{\sim}{v}.\underset{\sim}{\nabla} \underset{\sim}{v} + \underset{\sim}{v}(\underset{\sim}{\nabla}.\rho \underset{\sim}{v}) = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla} p + \rho \underset{\sim}{g}$$

Rearranging the terms on the left hand side, we have

$$\rho\left[\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla} \underset{\sim}{v}\right] + \underset{\sim}{v}\left[\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}.\rho \underset{\sim}{v}\right] = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla} p + \rho \underset{\sim}{g}$$

But from the equation of continuity

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} . \rho \underset{\sim}{v} = 0$$

Hence,

$$\rho\left[\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla} \underset{\sim}{v}\right] = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla} p + \rho \underset{\sim}{g}$$

or

$$\rho \frac{D\underset{\sim}{v}}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{\tau} - \underset{\sim}{\nabla} p + \rho \underset{\sim}{g}$$

Equations (15.20) and (15.21) are the generalized form of equation of motion without any assumption and may be applied to any coordinate system. The detailed form of this equation in cartesian, cylindrical and spherical coordinate system is given in Appendix-3.
Navier Stokes Equation for incompressible Newtonian fluid

The equation of motion may be further simplified by substituting the Newton's law of viscosity for the momentum flux term appearing in the equation of motion.
For a one-dimensional system where

vy=0,vz=0.. (16.1)

we have seen that the Newton's law of viscosity may be written as,

$$\tau_{yx} = -\mu \frac{\partial v_x}{\partial y}$$

where, τ_{yx} represents x directed momentum flowing in the y direction However, in general, for a three dimensional flow, all 9 components of shear stress may be important. Thus,

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Here, τ_{xx} , τ_{yy} and τ_{zz} are the normal stresses and the remaining are shear stress.

Axiom 3: Moment of momentum is conserved

This axiom 3 leads to a very simple conclusion that the shear stress tensor is symmetric in nature. The derivative itself is lengthy and is not reproduced here. $\underset{\sim}{\tau}$ is symmetric implies that

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{xz} = \tau_{zx}$$

$$\tau_{yz} = \tau_{zy}$$

Newton's law of viscosity may now be generalized as given below. Again, the basis for this representation is not shown here, but it may be found in any standard books in fluid mechanics.

$$\underset{\sim}{\tau} = -\mu \underset{\sim}{\Delta} + \frac{2}{3}\mu(\underset{\sim}{\nabla} \cdot \underset{\sim}{v})\delta_{ij}\underset{\sim}{\delta}_i \underset{\sim}{\delta}_j$$

where,

$$\underset{\sim}{\Delta} = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \underset{\sim}{\delta}_i \underset{\sim}{\delta}_j$$

Hence, we have the nine components of shrear stress as

$$\tau_{xx} = -2\mu \frac{\partial v_x}{\partial x} + \frac{2}{3} \mu \underset{\sim}{\nabla} . \underset{\sim}{v}$$

$$\tau_{yy} = -2\mu \frac{\partial v_y}{\partial y} + \frac{2}{3} \mu \underset{\sim}{\nabla} . \underset{\sim}{v}$$

$$\tau_{zz} = -2\mu \frac{\partial v_z}{\partial z} + \frac{2}{3} \mu \underset{\sim}{\nabla} . \underset{\sim}{v}$$

$$\tau_{xy} = \tau_{yx} = -\mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$

$$\tau_{yz} = \tau_{zy} = -\mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right)$$

$$\tau_{xz} = \tau_{zx} = -\mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)$$

The detail form of Newtons law of viscosity in all coordinate system is given in Appendix- 01. Now, consider the situation when an incompressible fluid is flowing only in x direction and depends on y coordinate only. In such a case, we have $v_x = v_x(y)$, $v_y = 0$ and $v_z = 0$. We can easily see that for this case,

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = 0$$

$$\tau_{yz} = \tau_{zy} = 0$$

$$\tau_{xz} = \tau_{zx} = 0$$

and only significant components of stress are τ_{xy} and τ_{yx}. Also, the expression for τ_{yx} is the same as given earlier as Newton's law of viscosity. For rectangular coordinate system, substituting the value of $-\nabla.\tau$ in the x component of equation of motion, we obtain

$$-\left(\nabla.\tau\right)_x = -\left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right]$$

Assuming that ρ and μ are constant, we obtain

$$-\nabla.\tau = -\left[\frac{\partial}{\partial x}\left(-2\mu\frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(-\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right)\right]$$

or

$$= \mu\left[\frac{\partial}{\partial x}\frac{\partial v_x}{\partial x} + \frac{\partial}{\partial y}\frac{\partial v_x}{\partial y} + \frac{\partial}{\partial y}\frac{\partial v_y}{\partial x} + \frac{\partial}{\partial x}\frac{\partial v_x}{\partial x} + \frac{\partial}{\partial z}\frac{\partial v_z}{\partial x} + \frac{\partial}{\partial x}\frac{\partial v_x}{\partial x}\right]$$

or

$$= \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial}{\partial x}\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right]\right]$$

But from equation of continuity for an incompressible fluid, we have

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Therefore,

$$-(\underset{\sim}{\nabla}.\underset{\sim}{\tau})_x = \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right]$$

$$= \mu\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] v_x$$

or

$$-(\underset{\sim}{\nabla}.\underset{\sim}{\tau})_x = \mu \underset{\sim}{\nabla}^2 v_x$$

or

$$-(\underset{\sim}{\nabla}.\underset{\sim}{\tau})_y = \mu \underset{\sim}{\nabla}^2 v_y$$

similarly

$$-(\underset{\sim}{\nabla}.\underset{\sim}{\tau})_z = \mu \underset{\sim}{\nabla}^2 v_z$$

Therefore, in vector and tensor form

$$-(\underset{\sim}{\nabla}.\underset{\sim}{\tau}) = \mu \underset{\sim}{\nabla}^2 \underset{\sim}{v}$$

Thus, the equation of motion reduce to

$$\rho \frac{D\underset{\sim}{v}}{Dt} = -\underset{\sim}{\nabla} p + \mu \underset{\sim}{\nabla}^2 \underset{\sim}{v} + \rho \underset{\sim}{g}$$

Equation (16.25) is known as the Navier Stokes equation and is used for solving problems involving Newtonian fluids of constant density and viscosity. For non-Newtonian and compressible fluids, the generalized form of equation of motion given earlier must be used. The detailed forms of the equations of motion along with Navier Stokes equations in cartesian, cylindrical and spherical coordinates are given in the Appendix-03.
Solution of momentum transport problems using Navier Stokes equation

In this section, transport problems involving Newtonian fluids are solved by making use of the equation of motion or Navier Stokes equation. We will firstly solve the falling film problem and flow through a circular tube for comparing the solutions obtained earlier by using the shell momentum balance method. We will then proceed to solve some more fluid mechanics problems.

1.12 Falling film on an inclined surface

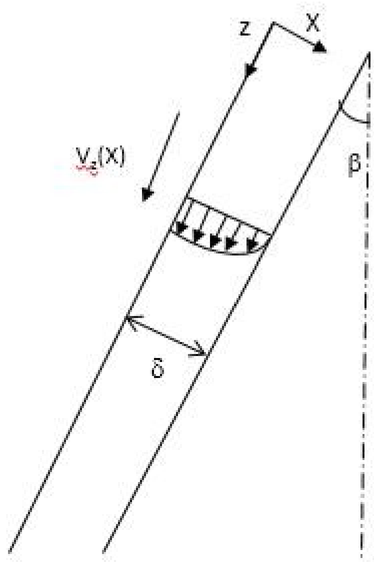


Fig 17.1 Falling film on inclined surface

This problem was solved earlier by the shell momentum balance technique. We will now try to solve this problem by using the Navier Stokes equations.
We are again required to make the same necessary assumptions as done earlier using the shell momentum balance technique. We postulate the non- zero components of the velocity and from there, determine the non-zero components of the shear stress tensor. These steps are the same as earlier and lead us to conclude that $v_z = v_z(r)$ and τ_{rz} is the only important component of shear stress. We now use the Navier Stokes equation in cartesian coordinates as given in Appendix-03.

x component is

$$\rho g_x = 0 \qquad (17.1)$$

y component is

$$\rho g_y = 0 \qquad (17.2)$$

z component is

$$\mu \frac{d^2 v_z}{dx^2} + \rho g_z = 0$$

where

$$g_z = g\cos\beta$$

Integrating Equation (17.3), we have

$$\frac{dv_z}{dx} = -(g\cos\beta)x + c_1$$

and

$$v_z = -(g\cos\beta)x + c_1 x + c_2$$

The boundary conditions are also the same as used earlier,

at

$$x = 0,\ \left.\frac{dv_z}{dx}\right|_{x=0} = 0$$

and

at

$$x = \delta,\ v_z = 0$$

This leads to the solution for velocity profile, as

$$v_z = -\frac{g\delta^2\cos\beta}{2\mu}\left[1-\left(\frac{x}{\delta}\right)^2\right]$$

which is same as obtain earlier using shell momentum balance approach.

Fluid flow through a vertical tube

A Newtonian fluid is flowing inside a vertical tube having circular cross section due to pressure difference and gravity. Solve the problem using the Navier Stokes equations.

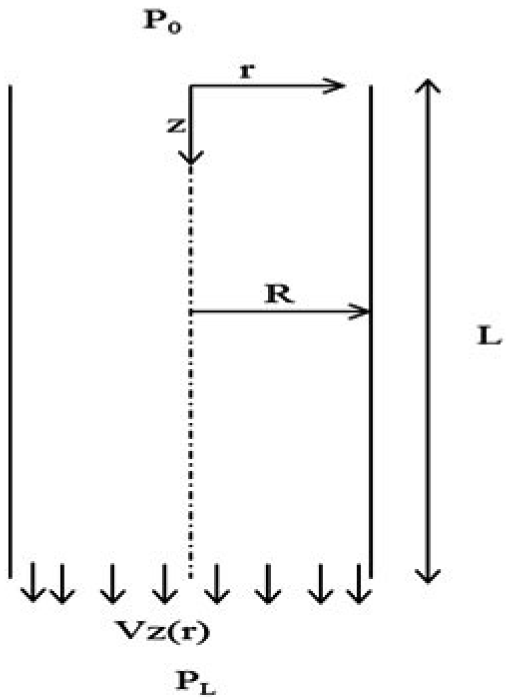


Fig 17.2 Flow through a vertical circular tube

A similar type of problem (for a horizontal pipe) was solved earlier using the shell momentum balance technique. Therefore, the initial steps are the same and include making appropriate assumptions and postulating the non- zero velocity components. As shown earlier, it leads to the conclusion that $v_z = v_z(r)$

Now using the Navier Stokes equation for cylindrical co-ordinates, after eliminating all zero terms, we have r- component of Navier Stokes equation

$$\frac{\partial P}{\partial r} = 0$$

θ- component

$$\frac{\partial P}{\partial \theta} = 0$$

z - component

$$-\frac{\partial P}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \rho g_z = 0$$

We can combine gravity and pressure forces as to rewrite Equation (17.11) as,

$$-\frac{\partial P_c}{\partial z}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=0$$

where, Pc is the effective pressure including the gravity, and is defined as,

$$P_c = P - \rho g_z z$$

Note that since pressure changes in only z direction and vz is a function of r only the partial derivative may be converted to total derivative. Furthermore, in Equation (17.12), the first term is only a function of z and the second term is only a function of r, i.e.,

$$F_1(z) + F_2(r) = 0$$

This leads to result that F1 and F2 both are constants as Equation (17.13) is true for all values of z and r.

$$F_1(z) = c_1;\ F_2(r) = -c_1$$

Therefore,

$$\frac{dP_c}{dz} = c_1$$

By integrating the Equation (17.15)

$$P_c = c_1 z + c_2$$

Boundary conditions are

at

$$z=0,\ P_c = P_{c0}$$

and

at

$$z=L,\ P_c = P_{cL}$$

This leads to the following solution

$$\frac{P_{c0} - P_{cL}}{L} = c_1$$

By substituting in Equation (17.12)

$$-\left(\frac{P_{c0} - P_{cL}}{L}\right) + \mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = 0$$

or

$$v_z = -\left(\frac{P_{c0} - P_{cL}}{4\mu L}\right) r^2 + c_3 \ln r + c_4$$

Boundary conditions are

at r= 0, v_z is finite

and

at r = R,
$v_z = 0$

This leads to

$$v_z = \left(\frac{P_{c0} - P_{cL}}{4\mu L}\right) R^2 \left[1 - \left(\frac{r}{R}\right)^2\right]$$

which is again similar to what we have seen for a horizontal tube except for pressure difference term. In fact, it can be shown that the velocity profile given in Equation (17.22) is valid for any configuration, horizontal, vertical, or inclined, with effective pressure is defined as

$$P_c = P - \rho g_z z$$

$$T_z = \int_0^{2\pi}\int_0^R (-\tau_{\theta\phi})\big|_{\theta=\pi/2} r\,dr\,r\,d\phi = 2\pi\left(\frac{\mu\Omega}{\psi}\right)\int_0^R r^2\,dr$$

Radial flow between two parallel discs

A part of a lubrication system consists of two circular discs and the lubricant flows in the radial direction. The flow takes place because of modified pressure *(p_1- p_2)* between the inner and outer radii r_1 and r_2 respectively. Formulate the problem for velocity profile and mass flow rate through the system.

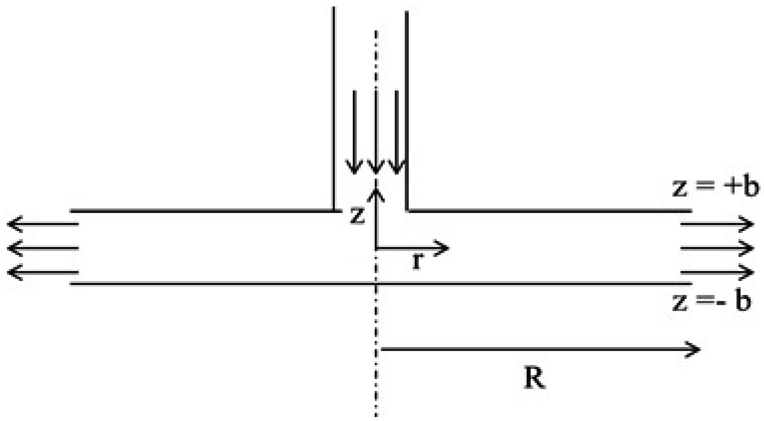


Fig 19.1 Radial flow in space between two parallel circular discs

Assumptions

- Density and viscosity are constant
- Steady state.
- Laminar flow (simple shear flow).
- Newton's law of viscosity is applicable.

Velocity profile

The fluid is flowing in the *r* direction. Hence, the only non-zero component of velocity is v_r and it depends on the both *r* and *z*. It will not depend on the *θ* coordinate due to cylindrical symmetry. i.e.,

$$v_r = v_r(r, z) \qquad \textbf{(19.1)}$$

Applying the equation of continuity in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \qquad \textbf{(19.2)}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) = 0 \quad \textbf{(19.3)}$$

or

$$\frac{\rho}{r}\frac{\partial}{\partial r}(r v_r) = 0 \quad \textbf{(19.4)}$$

Thus, $r v_r$ is a constant and which may be a function of the z,

$$r v_r = F(z) \quad \textbf{(19.5)}$$

Using the *r*-component of the Navier–Stokes equation in cylindrical co-ordinate systems, we have

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial P}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{\partial^2 v_r}{\partial z^2}\right] \quad \textbf{(19.6)}$$

By substituting Equation (19.5), we get

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial P}{\partial r} + \mu\frac{\partial^2 v_r}{\partial z^2} \quad \textbf{(19.7)}$$

Equation (19.7) is a second order partial differential equation and may not solve analytically. However, we may obtain an analytical solution for the limiting case when the flow is very slow (also called a creeping flow). In such a scenario, we may neglect the convective term (on the left hand side) in Equation (19.7) and thus, we have

$$\frac{\partial P}{\partial r} = \mu\frac{\partial^2 v_r}{\partial z^2} \quad \textbf{(19.8]}$$

Multiplying *r* on both sides, we have

$$r\frac{\partial P}{\partial r} = r\mu\frac{\partial^2 v_r}{\partial z^2}$$

or

$$r\frac{\partial P}{\partial r} = \mu\frac{\partial^2 F}{\partial z^2} \quad \textbf{(19.9)}$$

In Equation (19.9), the left hand side is a function of *r* only, while the right hand side is a function of *z* only. Since this equation is valid for all possible values of *r* and *z*, both the terms should be equal to each other, and in turn equal to a constant, c_2, independent of *r* and *z*. Therefore,

$$r\frac{dP}{dr} = c_2 \quad \textbf{(19.10)}$$

or

$$\mu\frac{d^2 F}{dz^2} = c_2 \quad \textbf{(19.11)}$$

From Equation (19.10), we get

$$\int_{P_1}^{P_2} dP = c_2\int_{r_1}^{r_2}\frac{dr}{r} \quad \textbf{(19.12)}$$

or

$$c_2 = \frac{P_2 - P_1}{ln\left(\frac{r_2}{r_1}\right)} \quad \textbf{(19.13)}$$

Substituting Equation (19.13) into Equation (19.11), we find

$$\mu \frac{d^2F}{dz^2} = -\frac{P_1 - P_2}{\mu \ln\left(\frac{r_2}{r_1}\right)} \qquad (19.14)$$

or

$$F = \frac{P_1 - P_2}{2\mu \ln\left(\frac{r_2}{r_1}\right)} z^2 + c'_1 z + c'_2 \qquad (19.15)$$

Boundary conditions

No-slip is valid at both the plates. Thus,

at

$$z = \pm b, v_r = 0 \qquad (1$$

By substituting these boundary condition*s* in Equation (19.15), we have

$$c'_2 = \frac{P_1 - P_2}{2\mu \ln\left(\frac{r_2}{r_1}\right)} b^2 \qquad (19.17)$$

At *z=0*, the velocity profile is symmetric. Therefore, this is the second required boundary condition for the problem

$$\left.\frac{\partial F}{\partial z}\right|_{z=0} = 0 \qquad (19.$$

This leads to the solution

$$c'_1 = 0$$

and

$$F = \frac{P_1 - P_2}{2\mu \ln\left(\frac{r_2}{r_1}\right)} b^2 \left[1 - \left(\frac{z}{b}\right)^2\right] \qquad \text{(19.}$$

or

$$r v_r = \frac{P_1 - P_2}{2\mu \ln\left(\frac{r_2}{r_1}\right)} b^2 \left[1 - \left(\frac{z}{b}\right)^2\right] \qquad \text{(19.}$$

Finally, we obtain the velocity profile

$$v_r = \frac{P_1 - P_2}{2r\mu \ln\left(\frac{r_2}{r_1}\right)} b^2 \left[1 - \left(\frac{z}{b}\right)^2\right] \qquad \text{(19.}$$

The mass flow rate of at any *r* in the system must be the same (in fact that was the reason, why we got $r v_r$ constant for a given in the first place). Select the surface at $r = r_2$ to obtain mass flow rate

$$w = \int_0^{2\pi} \int_{-b}^{+b} \rho v_r \big|_{r=r_2} \, r\,dz\,d\theta$$

or

$$w = \frac{4\pi(P_1 - P_2) b^2 \rho}{3\mu \ln\left(\frac{r_2}{r_1}\right)} \qquad \text{(19.2}$$

Parallel – disc viscometer

A fluid is placed in a gap (of thickness B) between two parallel discs of radius *R*. The lower disc

is kept stationary while the upper disc is made to rotate at a constant angular velocity Ω .
Formulate the problem for determining the viscosity at low shear rates.

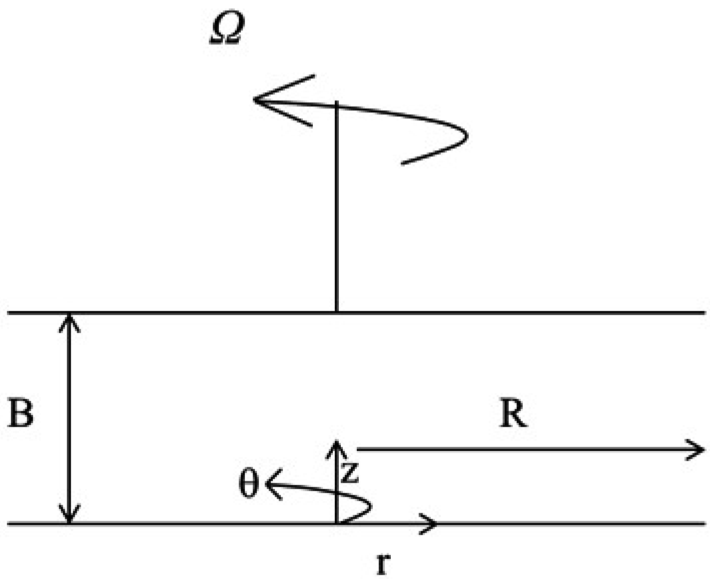


Fig 19.2 Front view of two-plate viscometer

Assumptions

- Density and viscosity are constant.
- Steady state.
- Laminar flow (simple shear flow).
- Newton's law of viscosity is applicable.

Velocity profile

The fluid is sheared in the θ direction; hence, v_θ is the non-zero component of velocity. Applying the equation of continuity in cylindrical coordinate, we obtain

$$\frac{1}{r}\frac{\partial}{\partial\theta}(\rho r v_\theta) = 0 \qquad \textbf{(19.}$$

Thus, v_θ, does not depend on the θ coordinate, or

$$v_\theta = v_\theta(r, z)$$ **(19**

For simplifying the problem further, we may assume that for low shear rates

$$v_\theta = r f(z)$$ **(19**

Using the θ component of the Navier – Stokes equation for cylindrical co-ordinate systems

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{\partial^2 v_\theta}{\partial z^2} = 0$$ **(19**

By substituting Equation (19.25), we get

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 f(z)\right)\right) + \frac{r\partial^2 f(z)}{\partial z^2} = 0$$ **(19.**

or

$$\frac{\partial^2 f(z)}{\partial z^2} = 0$$

After integration, we finally obtain

$$f(z) = c_1 z + c_2$$ **(1**

where c_1 and c_2 are the integral constant.

Boundary conditions are

at $z = 0$, $v_\theta = 0$ or $f(z) = 0$

thus,

$$c_2 = 0$$

and at $z = B, v_\theta = v$ or

$$f(z) = v/r = \Omega \qquad \textbf{(19}$$

This boundary condition leads to the solution

$$c_1 = \frac{\Omega}{B} \qquad \textbf{(1}$$

and

$$f(z) = \frac{\Omega z}{B} \qquad \textbf{(1}$$

Finally, we obtain the velocity profile

$$v_\theta = \frac{\Omega zr}{B} \qquad \textbf{(1}$$

Now, the *z*-component of the torque exerted on the fluid by the upper rotating disc, may be calculated as

$$T_z = \int_0^{2\pi}\int_0^R (-\tau_{z\theta})\, r\big|_{z=B}\, r dr d\theta \qquad \textbf{(19}$$

or

$$= 2\pi\mu \frac{\Omega}{B}\int_0^R r^3 dr$$

Finally, we obtain the value of torque.

$$T_z = \pi\mu\frac{\Omega}{2B}R^2 \qquad \textbf{(}$$

Thus, by plotting the angular velocity Ω vs torque T_z, the viscosity may be determined.

Non-Newtonian fluids

Non-Newtonian fluids are the fluids which do not obey Newton's law of viscosity. For describing Non-Newtonian fluids, let's recall the Newton's law of viscosity experiment. There are two long parallel plate situated at distance h to each other. Top plate is stationary and bottom plate is moving with velocity $\underset{\sim}{v}$ as shown in Fig. (20.1).

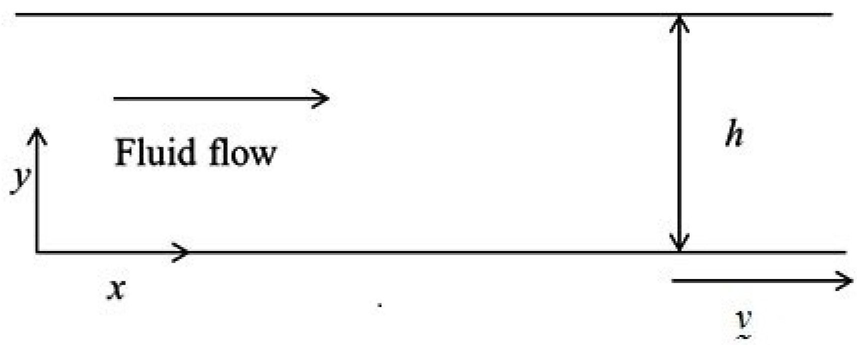


Fig 20.1 Non-Newtonian flow between two parallel plates

If a force, F, is applied to move plate, then (τ_{xy})

$$\tau_{xy} = \frac{F}{A}$$

and under steady state conditions when h is small and when

$$\frac{d v_x}{d y} = \frac{\underset{\sim}{v}}{h}$$

Now, we calculate τ_{xy} by repeating experiments for different applied forces and velocity achieved by the bottom plate and plotting a graph as shown in Fig. (20.2). Depending on the nature of fluid, different types of curves may be obtained.

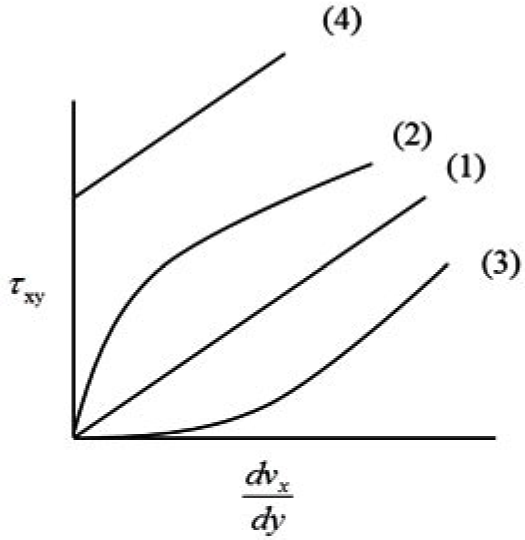


Fig 20.2 Shear stress vs. shear strain diagram for Newtonian and non-Newtonian fluids

1.13 Rheological behaviour of fluids

If fluid shows the behaviour like curve (1) then it is a Newtonian fluid. Other fluids are non-Newtonian fluids. Curve (2) represents a Pseudo-plastic fluid, curve (3) represents a Dilatant fluid, and curve (4) represents a Bingham plastic fluid. There are several Theoretical and empirical models available to describe the rheological behaviour of non-Newtonian fluids. Here, we discuss some of them, which come under the group of generalized Newtonian models. Basic equation for a generalized non-Newtonian fluid is given below

$$\tau_{yx} = -\eta \frac{dv_x}{dy}$$

Here, η is the apparent viscosity, which is clearly a function of shear rate as may be seen from Fig. (20.2). Therefore,

$$\eta = f\left(\frac{dv_x}{dy}\right) \qquad (20.3)$$

If the apparent viscosity increases with increase in shear rate, $\frac{dv_x}{dy}$, then the fluid is called Dilatant fluid and if it decreases with increase in shear rate, $\frac{dv_x}{dy}$ then fluid is called Pseudo-plastic fluid. Some fluids require a critical shear stress to initiate the flow. These fluids are called Bingham fluids. Some important rheological models for non-Newtonian fluids are given below.

1 Power Law or Ostwald De Waele model

Power law or Ostwald De Waele model is the most generalized model for non-Newtonian fluids. The expression of this model is given in Equation (20.3)

$$\tau_{yx} = -m\left|\left(\frac{dv_x}{dy}\right)\right|^{n-1}\frac{dv_x}{dy}$$

Here, apparent viscosity η is defined as,

$$\eta = m\left|\left(\frac{dv_x}{dy}\right)\right|^{n-1}$$

This is a two-parameter model where m and n are the two parameters.

If n = l then η = m

where m is similar to the viscosity of the fluid and model shows the Newtonian behaviour .

If n>1, then η increases with increasing shear rate and the model shows the Dilatant behaviour.

If n<1, then η decreases with increasing shear rate and the model shows the Pseudo-plastic behaviour.

Modulus sign

In power law model, modulus sign can be removed according to the value of shear rate.

1. If $\frac{dv_x}{dy}$ is positive, then

$$\eta = m\left(\frac{dv_x}{dy}\right)^{n-1}$$

2. If $\frac{dv_x}{dy}$ is negative, then

$$\eta = m\left(-\frac{dv_x}{dy}\right)^{n-1}$$

Several fluids do not show single type of rheological behaviour. They show Newtonian behaviour for a range of shear stress and Non-Newtonian behaviour for some other ranges of shear stresses. Several models have been suggested for these types of fluids. Some popular models like Eyring model, Ellis model, Reiner Philipp off model and Bingham Fluid model are discussed here.

2. Eyring model

Eyring model is a two-parameter model. The equation of Eyring model is as follow

$$\sinh\left(\frac{\tau_{yx}}{A}\right) = -\frac{1}{B}\frac{dv_x}{dy}$$

where A, B are the two parameters.

In Eyring model, if $\tau_{xy} \to 0$ which means very low shear forces, we have

$$\sinh\left\{\frac{\tau_{yx}}{A}\right\} \to \frac{\tau_{yx}}{A}$$

Therefore, as $\tau_{yx} \to 0$, the model shows Newtonian behaviour

$$\tau_{yx} = -\frac{A}{B}\frac{dv_x}{dy}$$

Here, viscosity = $\left(\frac{A}{B}\right)$

If τ_{yx} is very large, the model shows Non-Newtonian behaviour as shown Fig. 20.3

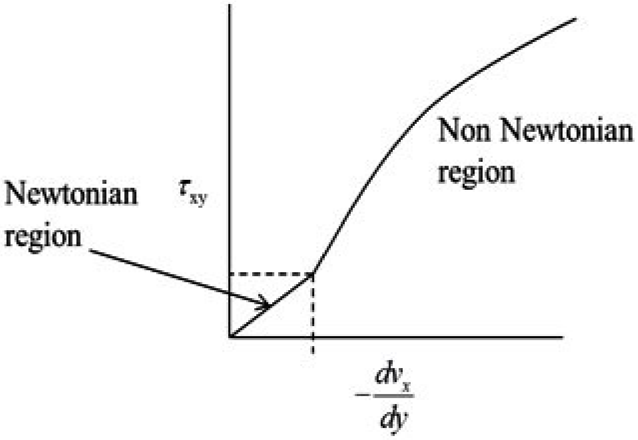


Fig 20.3 Shear stress vs. shear strain diagram for Eyring model

Therefore, Eyring model may be used for a fluid which shows Newtonian behaviour at low shear rates and non- Newtonian behaviour at high shear rates.

3. Ellis model

Ellis model is a three-parameter model. The equation of this model is as follows

$$-\frac{dv_x}{dy} = \left\{\varphi_o + \varphi_1 \left|\tau_{yx}\right|^{\alpha-1}\right\}\tau_{yx}$$

Here, φ_0, φ_1 and α are the three parameters .

Here, we consider some special cases,

1. If $\varphi_1 = 0$ then Equation (20.11) reduce to

$$\frac{dv_x}{dy} = -\varphi_0 \tau_{yx}$$

or

$$\tau_{yx} = -\frac{1}{\varphi_0}\frac{dv_x}{dy}$$

which is same as Newton’s law of viscosity with $\left(\frac{1}{\varphi_0}\right)$ as the viscosity of the fluid.

2. If $\varphi_0 = 0$, then

$$-\frac{dv_x}{dy} = -\varphi_1 |\tau_{yx}|^{\alpha-1} \tau_{yx}$$

which is similar to a Power law model

3. If α >1 and τ_{yx} is small then the second term is approximately zero and equation reduces to

$$\tau_{yx} = -\frac{1}{\varphi_0}\frac{dv_x}{dy}$$

which is similar to Newton's law of viscosity.

4. If α <1 and τ_{yx} is very large, then again, second term is negligible and we have

$$\tau_{yx} = -\frac{1}{\varphi_0}\frac{dv_x}{dy}$$

Which again shows Newtonian behaviour. Therefore, Ellis model may be used for fluids which show Newtonian behaviour at very low and very high shear stresses, but non-Newtonian behaviour at intermediate value of shear stresses.

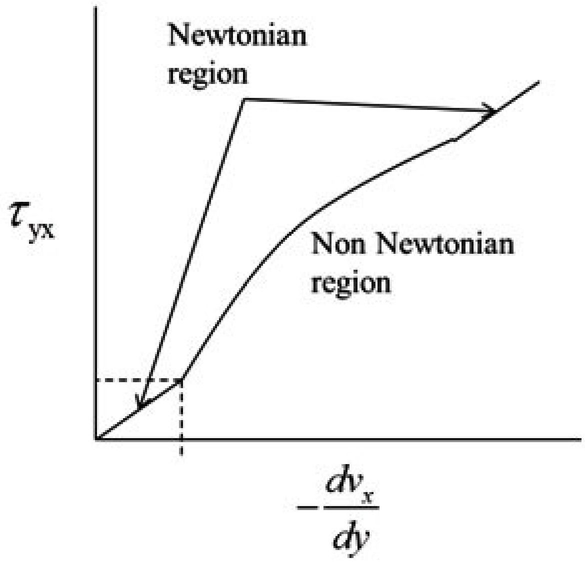


Fig 20.4 Shear stress vs. shear strain diagram for Ellis model

This type of behaviour may be shown by some polymer melts

4. Reiner Philipp off model

This is also a three-parameter model. The equation of Reiner Philipp off model is as follows,

$$-\frac{dv_x}{dy} = \left[\frac{1}{\mu_\infty + \dfrac{\mu_0 - \mu_\infty}{1 + \left(\dfrac{\tau_{yx}}{\tau_s} \right)^2}} \right] \tau_{yx}$$

where, μ_0 , μ_∞ and τ_s are the three parameters.

In Reiner Philipp off model, if τ_{yx} is very large, the equation reduces to,

$$-\frac{dv_x}{dy} = \frac{1}{\mu_\infty} \tau_{yx}$$

or

$$\tau_{yx} = -\mu_{\infty} \frac{dv_x}{dy}$$

which is same as the Newton's law of viscosity,

If τ_{yx} is very small then equation reduces to

$$-\frac{dv_x}{dy} = \frac{1}{\mu_0}\tau_{yx}$$

or

$$\tau_{yx} = -\mu_0 \frac{dv_x}{dy}$$

which is also same as the Newton's law of viscosity. Therefore, Reiner Philipp off model may be used for a fluid which shows Newtonian behaviour at very low and very high shear stresses but non-Newtonian behaviour for intermediate values of shear stress. Here, μ_0 and μ_{∞} represent the viscosity of fluid at very low and very high shear stress conditions respectively.

5. Bingham Fluid model

Bingham fluid is special type of fluid which require a critical shear stress to start the flow. The equation of Bingham fluid model are given below

$$\tau_{yx} = -\left[\mu + \frac{\tau_0}{\left|\frac{dv_x}{dy}\right|}\right]\frac{dv_x}{dy}$$

if

$$|\tau_{yx}| > \tau_0 \qquad \textbf{(20.19)}$$

$\frac{dv_x}{dy} = 0$ if $|\tau_{yx}| \leq \tau_0$ or

$\eta = 0$ **(20.2**

A typical shear stress vs. shear rate diagram for a Binghum model is shown below

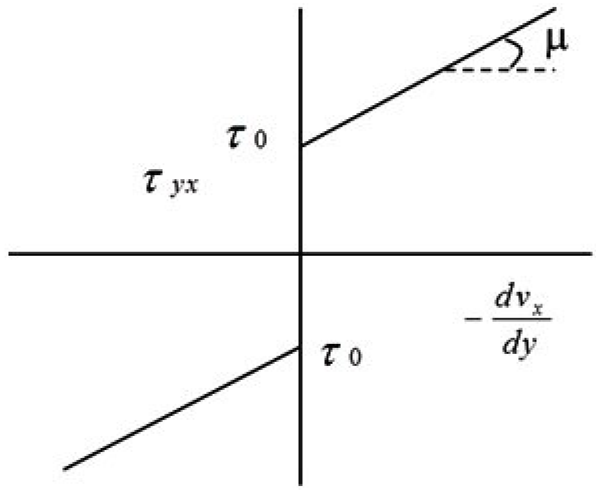


Fig 20.5 Shear stress vs. shear strain diagram for Bingham model

Momentum transport problems involving Power law and Bingham fluids:

In this section, we will solve fluid mechanics problem for Power law and Bingham plastic fluids. These problems have been earlier solved for Newtonian fluids. We have chosen the same problems here for better understanding.

Falling film on inclined plane

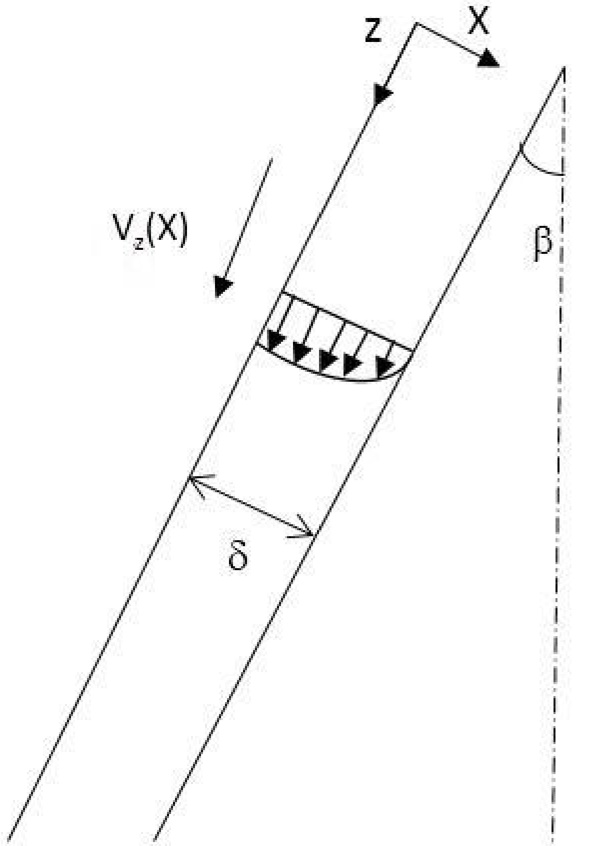


Fig 21.1 Falling film problem for non-Newtonian fluid

Initial steps, such as making appropriate assumptions, finding important velocity components, applying equation of continuity, and determining important shear stress components are similar as steps seen for Newtonian fluid in lecture 11 and 17. As before $v_z = v_z(x)$ and is the only non-zero velocity component and τ_{xz} is the only important shear stress component.

(Note: Since the forms of shear stress $\underset{\sim}{\tau}$ for Newtonian and non-Newtonian fluids are same, the only difference is the viscosity μ for Newtonian fluids and apparent viscosity η for non-Newtonian fluids and furthermore as non-zero components of velocities are also same, the same components of shear stress $\underset{\sim}{\tau}$ are significant for both Newtonian and non- Newtonian fluids.)

To solve the problem, we start with the generalized equation of motion in terms of $\underset{\sim}{\tau}$. Since the fluid is moving in z direction, discarding all terms which are zero, z-component of the equation of motion reduces to

$$-\frac{d\tau_{xz}}{dx}+\rho g_z=0$$

where

$$g_z = g\ cos\beta$$

therefore,

$$\tau_{xz}=\rho g\ cos\ \beta x+c_1$$

For Power law fluids

$$\tau_{xz} = -\eta\ \frac{dv_z}{dx}$$

$$\eta = m\left|\frac{dv_z}{dx}\right|^{n-1}$$

Since vz is decreasing with increasing value of x , the negative sign should be used for removing the modulus sign, i.e. ,

$$\tau_{xz}=-m\left(-\frac{dv_z}{dx}\right)^{n-1}\left(\frac{dv_z}{dx}\right)$$

or

$$\tau_{xz} = m\left(-\frac{dv_z}{dx}\right)^{n}$$

By substituting Equation (21.7) in Equation. (21.1), we obtain

$$\left(-\frac{dv_z}{dx}\right)^n = \frac{\rho g \cos\beta}{m} x + c_1$$

By applying the boundary condition, at $x = 0, \quad \tau_{xz}\big|_{air} = \tau_{xz}\big|_{fluid}$
which simplifies to

$$\left.\frac{d v_z}{d x}\right|_{x=0} = 0$$

as disused in lecture 11

By substituting this boundary condition in Equation (21.8), we get $c_1 = 0$. Therefore,

$$\left(-\frac{dv_z}{dx}\right)^n = \frac{\rho g \cos\beta}{m} x$$

or

$$v_z = -\sqrt[n]{\frac{\rho g \cos\beta}{m}} \frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1} + c_2$$

Here, c_2 is another integral constant.

Now, using the second boundary condition, at $x = \delta$, $v_z = 0$, we finally obtain

$$v_z = \frac{n}{n+1}\sqrt[n]{\frac{\rho g \cos\beta}{m}}\,\delta^{\frac{n+1}{n}}\left[1-\left(\frac{x}{\delta}\right)^{\frac{n+1}{n}}\right]$$

Tube flow problem for Power law fluid

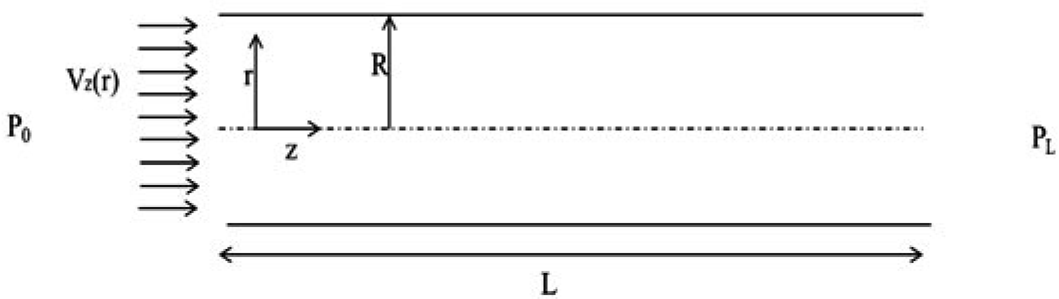


Fig 21.1 Flow through pipe for non-Newtonian fluid

As we discussed in lecture 10, the only non-zero component of velocity is vz, which depends on r only. The important component of shear stress is τ_{rz} .

By applying general equation of motion in cylindrical co-ordinate, we get

$$\frac{\partial P}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = 0$$

Equation (21.11) may be further simplified as before

$$\frac{P_0 - P_L}{L} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = 0$$

or

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r + \frac{c_1}{r}$$

By applying the boundary condition, at r=0, velocity is finite, we obtain

$$c_1 = 0$$

and for power law fluids

$$\tau_{rz} = m\left(-\frac{dv_z}{dr}\right)^n$$

(Note: Since vz is decreasing with increasing value of r, the negative sign should be used for removing the modulus sign.)

By substituting Equation (21.13) to Equation (21.12), we get

$$m\left(-\frac{dv_z}{dr}\right)^n = \frac{(P_0 - P_L)}{2L} r$$

Integrating above equation, we obtain

$$v_z = -\sqrt[n]{\frac{(P_0 - P_L)}{2Lm}}\, \frac{r^{\frac{n+1}{n}}}{\frac{n+1}{n}} + c_2$$

Now, by applying the no-slip boundary condition at, $r = R,\ v_z = 0$, we obtain

$$c_2 = -\sqrt[n]{\frac{(P_0 - P_L)}{2Lm}}\, \frac{R^{\frac{n+1}{n}}}{\frac{n+1}{n}}$$

Thus,

$$v_z = \sqrt[n]{\frac{(P_0 - P_L)}{2Lm}}\, \frac{R^{\frac{n+1}{n}}}{\frac{n+1}{n}} \left[1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]$$

Equation (21.15) represents the velocity profile of freely falling film on an inclined surface for a Power law non-Newtonian fluid. If we substitute the n=1 and m=μ in this expression, we get Equation (10.25) which was derived earlier for a Newtonian fluid.

Tube Flow Problem for a Bingham Fluid

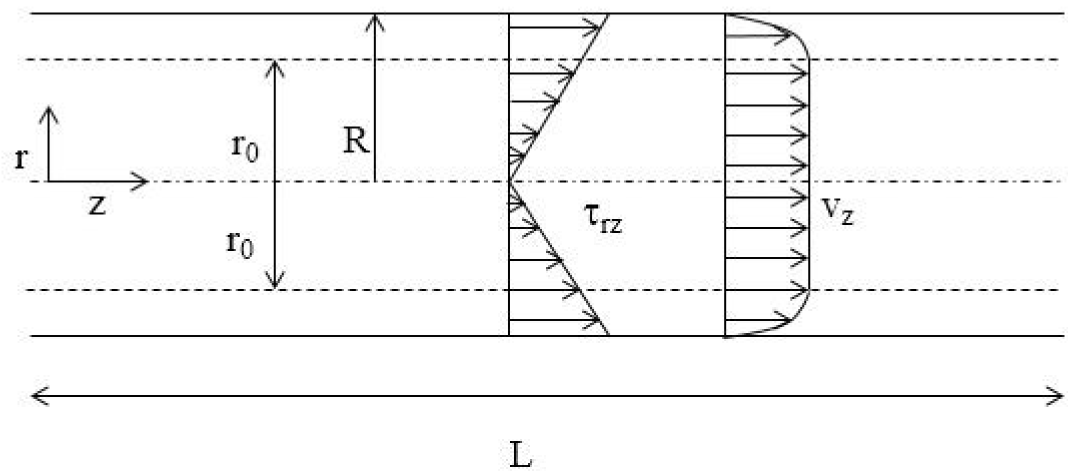


Fig 22.1 Flow through pipe for Bingham fluid

As mentioned in the previous lecture, the forms of shear stress $\underset{\sim}{\tau}$ for Newtonian and non-Newtonian fluids are the same. Therefore, Equation (21.12) is applicable for a Bingham fluids also, i.e.,

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

Equations (20.19) and (20.20) may be written for this system

1. For $\tau_{rz} < \tau_0$ ($r < r_0$), where r_0 is to be determine latter,

$$\frac{dv_z}{dr} = 0$$, or

$$v_z = v_0 \text{ (constant)}$$

2. For $\tau_{rz} \geq \tau_0$ ($r \geq r_0$)

$$\eta = \mu_0 + \frac{\tau_0}{\left|\frac{dv_z}{dr}\right|}$$

In Equation (22.2), $\frac{dv_z}{dr}$ is negative. Therefore, after removing the modulus sign, we obtain

$$\eta = \mu_0 - \frac{\tau_0}{\left(\frac{dv_z}{dr}\right)}$$

Thus,

$$\tau_{rz} = -\left\{\mu - \frac{\tau_0}{\frac{dv_z}{dr}}\right\}\frac{dv_z}{dr}$$

or

$$\tau_{rz} = -\mu\frac{dv_z}{dr} + \tau_0$$

Condition for movement of fluid

As we start to pressurize the fluid by imposing pressure difference , fluid does not move initially. As we continue to increase the pressure difference the fluid may start to move at some critical pressure difference ($P_{c0} - P_{cL}$). This critical value may be determined by setting $\tau_{rz}\big|_{r=R} = \tau_0$. Thus,

$$\tau_0 = \frac{(P_{c0} - P_{cL})}{2L}R$$

Thus, the fluid will flow if

$$\tau_0 \le \frac{(P_{c0} - P_{cL})}{2L}R$$

Suppose the pressure difference across the tube exceeds this critical value of pressure ($P_{c0} - P_{cL}$) then the fluid will start to flow. Now, under this condition we may calculate the value of (r0) where the value of $\tau_{rz} = \tau_0$. For r<r0, the velocity gradient is zero and the

fluid flows with a constant velocity. The detail calculation for two different regions r<r0 and r>r0 are given below.

$$\tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

At $r = r_0, \tau_{rz} = \tau_0$. Thus,

$$\tau_0 = \frac{(P_0 - P_L)}{2L} r_0$$

or

$$r_0 = \frac{2\tau_0 L}{(P_0 - P_L)}$$

For r<r0, we equate Equations (21.12) and (22.4), that is

$$-\mu \frac{dv_z}{dr} + \tau_0 = \tau_{rz} = \frac{(P_0 - P_L)}{2L} r$$

Finally, we obtain,

$$v_z = \frac{\tau_0}{\mu} r - \left(\frac{(P_0 - P_L)}{4\mu L} r^2 \right) + c_1$$

No slip Boundary condition at r=R , $v_z = 0$ may be used to calculate c1 as shown below

Substituting this value in Equation (22.11), we get

$$c_1 = \left(\frac{(P_0 - P_L)}{4\mu L} R^2 \right) - \frac{\tau_0}{\mu} R$$

Finally, the velocity profile is given by

$$v_z = \frac{\tau_0}{\mu}(r-R) + \left(\frac{(P_0-P_L)}{4\mu L}R^2\right)\left(1-\left(\frac{r}{R}\right)^2\right)$$

Equation (22.12) gives the velocity profile is region $r_0 \le r \le R$. as shown in Fig. 22.2. Equation (22.9) shows that as we keep increasing the pressure difference (P_0-P_L), the value of r0 keep on decreasing and the velocity profile changes as shown in Fig. 22.2.

The value of r0 also depends on τ_0 , and reduces with it. If we substitute $\tau_0 = 0$ in Equation (22.12), we obtain the same expression for velocity profile as we had earlier obtain for Newtonian fluids. This result implies that if the value of pressure difference (P_0-P_L) is significantly high then the Bingham fluid may show behaviour similar to Newtonian fluids.

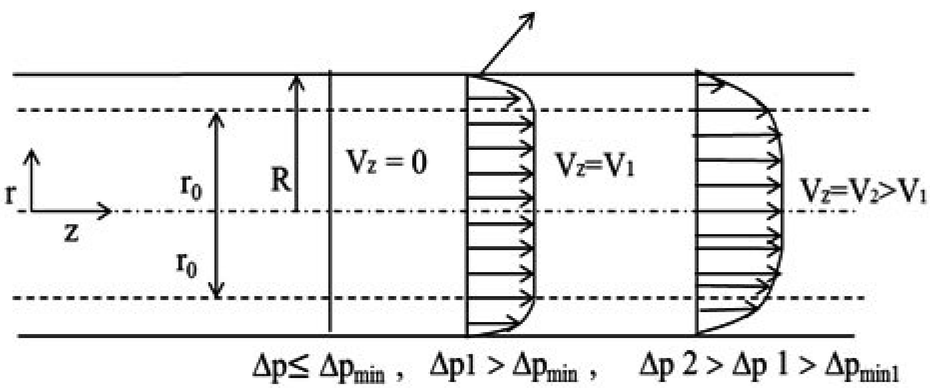


Fig 22.2 Effect of differential pressure flow through pipe for Bingham fluid

Now, we may determine the velocity profile in the plug flow region (r>r0) by substituting r= r0 in Equation (22.12)

$$v_0 = \frac{\tau_0}{\mu}(r_0-R) + \left(\frac{(P_0-P_L)}{4\mu L}R^2\right)\left(1-\left(\frac{r_0}{R}\right)^2\right)$$

Falling film problem for Bingham fluid

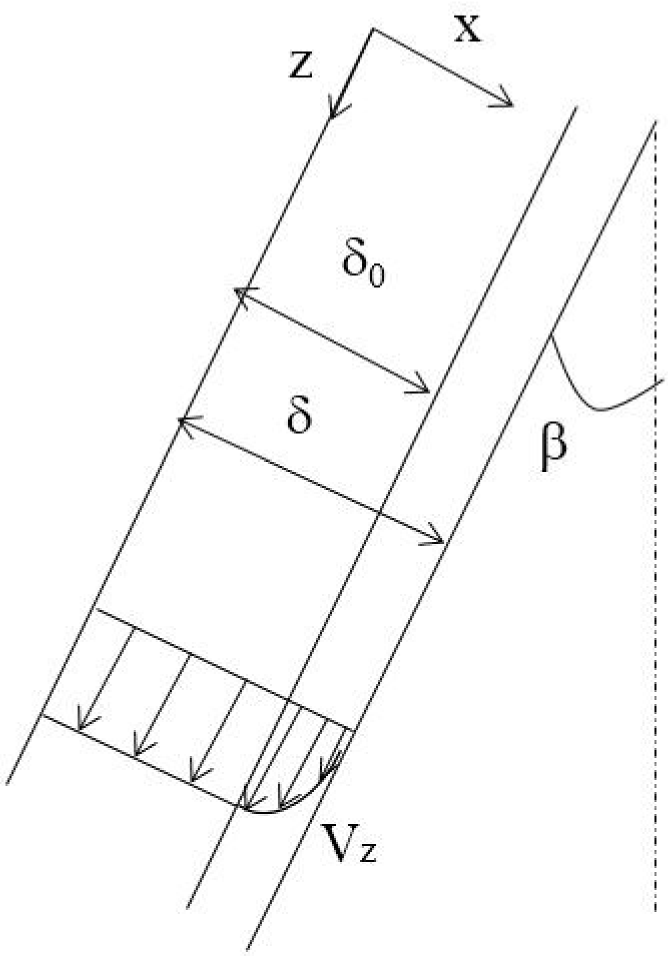


Fig 22.3 Flow on inclined surface for Bingham fluid

As we discussed earlier, the expression of shear stress is same, as we had derived for Newtonian fluids and Power law fluids in lecture 11 and lecture 21. Therefore, from Equation (21.3)

$$\tau_{xz} = \rho g x \cos\beta$$

For this system, Bingham fluid model may be written as,

1. For $\tau < \tau_0$, $x < \delta_0$

$$\frac{dv_z}{dx} = 0$$

2. For $\tau \geq \tau_0$, $x \geq \delta_0$

$$\eta = \mu_0 \pm \frac{\tau_0}{\left|\frac{dv_z}{dx}\right|}$$

As the critical thickness of film δ_0 is unknown, (the fluid flows only when $\delta \geq \delta_0$) we may calculated from Equation (22.17), i.e.,

$\tau_{xz} = \tau_0$ at $x = \delta_0$ or,

$$\tau_0 = \rho g \delta_0 \cos \beta$$

or

$$\delta_0 = \frac{\tau_0}{\rho g \cos \beta}$$

From region (1) where $\tau < \tau_0$ and $x < \delta_0$, we have

$$\frac{dv_z}{dx} = 0$$ or $$v_z = v_0$$

For region (2) where $\tau \geq \tau_0$ and $\delta \geq \delta_0$, we have

$$\eta = \left\{ \mu \pm \frac{\tau_0}{\left|\frac{dv_z}{dx}\right|} \right\}$$

Here $\frac{dv_z}{dx}$ is negative. Therefore, after removing the modulus sign and substituting the value of η in Generalized Newton's law of viscosity. we obtain,

$$\tau_{xz} = -\left\{ \mu \frac{dv_z}{dx} - \tau_0 \right\} = \rho g \cos \beta x$$

or

$$-\mu \frac{dv_z}{dx} = \rho g \cos \beta x - \tau_0$$

or

$$\frac{dv_z}{dx} = \frac{-\rho g \cos \beta x}{\mu} + \frac{\tau_0}{\mu}$$

Finally, we obtain the velocity profile, as given below

$$v_z = \frac{-\rho g \cos \beta}{\mu} \frac{x^2}{2} + \frac{\tau_0}{\mu} x + c_2$$

where c2 is an integral constant. By using no slip boundary condition at $x = \delta$, $v_z = 0$, we obtain

$$c_2 = \frac{\rho g \cos \beta}{\mu} \frac{\delta^2}{2} - \frac{\tau_0}{\mu} \delta$$

Therefore,

$$v_z = \frac{\rho g \cos \beta \delta^2}{2\mu} \left[1 - \frac{x^2}{\delta^2} \right] - \frac{\tau_0 \delta}{\mu} \left[1 - \frac{x}{\delta} \right]$$

Equation (22.22) shows the velocity profile in region $\delta_0 \leq x \leq \delta$. From Equation(22.22), we may also calculate the velocity of plug flow region by substituting the value $x = \delta_0$. Thus,

$$v_0 = \frac{\rho g \cos\beta \delta^2}{2\mu}\left[1 - \frac{\delta_0^{\,2}}{\delta^2}\right] - \frac{\tau_0 \delta}{\mu}\left[1 - \frac{\delta_0}{\delta}\right]$$

$$\tau_{r\theta} = \frac{c_1}{r^2}$$