ENGINEERING MATHEMATICS - III (Common to ALL Branches except BIO Groups, CSE and IT)

SUBJECT CODE: SMT1201

COURSE OBJECTIVE:

The ability to identify, reflect upon, evaluate and apply different types of information and knowledge to form independent judgments. Analytical, logical thinking and conclusions based on quantitative information will be the main objective of learning this subject.

UNIT 1 COMPLEX VARIABLES

. Analytic functions - Cauchy- Riemann equations in cartesian and polar form - Harmonic functions - properties of analytic functions - Construction of analytic functions using Milne - Thompson method - Bilinear transformation.

UNIT – I

COMPLEX VARIABLES

Complex Variable:

$z = x+iy$ is a complex variable where x & y are real variables.

Function of a complex variable:

$w = f(z) = u(x,y) + i\,v(x,y)$ is a function of the complex variable $z = x+iy$. Where $u(x,y)$ is the real part and $v(x,y)$ is the imaginary part of the complex function.

Example: $f(z) = x^2 - y^2 + 2ixy$ is a function of a complex variable z.

Derivative of a complex function:

A function $f(z)$ is said to be differentiable at a fixed point z if the $\lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$ exists.

ANALYTIC FUNCTION:

A function defined at a point z_0 is said to be analytic at z_0, if it has a derivative at z_0 and at every point in some neighbourhood of z_0.

A function $f(z)$ is said to be analytic in a region R, if it is analytic at every point of R.

The necessary condition for $f(z)$ to be analytic.

Cauchy-Riemann Equations:

The necessary conditions for a complex function $f(z) = u(x,y) + iv(x,y)$ to be analytic are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Sufficient condition for $f(z)$ to be analytic:

The function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if

(i) $u(x,y)$ and $v(x,y)$ are differentiable in D and $u_x = v_y$ and $u_y = -v_x$

(ii) The partial derivatives u_x, u_y, v_x, v_y are all continuous in D.

Polar form of Cauchy-Riemann Equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

PROBLEMS:

1) Test the analyticity of the function

(i) $f(z) = e^x(\cos y + i\sin y)$

$u_x = e^x\cos y$ | $v_x = e^x\sin y$

$u_y = -e^x\sin y$ | $v_y = e^x\cos y$

Here $u_x = v_y$ & $u_y = -v_x$

$\Rightarrow$ $f(z)$ satisfies C-R equations

$\therefore$ The given function is analytic.

(ii) $f(z) = \frac{1}{z} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

$u_x = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\qquad v_x = \frac{2xy}{(x^2+y^2)^2}$

$u_y = \frac{-2xy}{(x^2+y^2)^2}$ $\qquad v_y = \frac{y^2-x^2}{(x^2+y^2)^2}$

$u_x = v_y$, $u_y = -v_x$

Hence $f(z)$ is analytic.

(iii) $f(z) = e^{\bar{z}} = e^{x-iy} = e^x e^{-iy} = e^x(\cos y - i\sin y)$

$u_x = e^x\cos y$ $\qquad v_x = -e^x\sin y$

$u_y = -e^x\sin y$ $\qquad v_y = -e^x\cos y$

$u_x \neq v_y$, $u_y \neq -v_x$, $f(z)$ is not satisfied by C-R equations. $\therefore$ $f(z) = e^{\bar{z}}$ is not analytic.

2) Show that the function $f(z) = |z|^2$ is differentiable only at the origin.

$f(z) = |z|^2$

$|z|^2 = x^2 + y^2 \Rightarrow f(z) = x^2 + y^2$

$u = x^2 + y^2, \quad v = 0$

$u_x = 2x \qquad v_x = 0$

$u_y = 2y \qquad v_y = 0$

If $f(z)$ is differentiable, then

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x = 0 \Rightarrow x = 0$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow 2y = 0 \Rightarrow y = 0$

$\therefore$ C-R equations are satisfied only when $x = 0$, $y = 0$.

Hence the given function $f(z)$ is differentiable only at the origin $(0,0)$.

3. Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Given $f(z) = \bar{z} = x - iy$

$u = x, \quad v = -iy$

$u_x = 1, \quad v_x = 0$

$u_y = 0, \quad v_y = -1$

At all point (x,y), $u_x = 1$ and $v_y = -1$. Hence CR equations are not satisfied anywhere. Hence $f(z) = \bar{z}$ is nowhere differentiable.

PROPERTIES OF ANALYTIC FUNCTION:

Property 1: Both the real and imaginary parts of any analytic function satisfies Laplace's equation.

Proof: let $f(z) = u + iv$ be an analytic function

Then we know that $u_x = v_y$, $u_y = -v_x$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Similarly, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Property: 2

If $f(z)$ and $\overline{f(z)}$ are analytic in a region R in that region.

then $f(z)$ is constant

proof: Given $f(z)$ and $\overline{f(z)}$ are analytic.

Since $f(z)$ is analytic both u & v satisfies C-R equations. $u_x = v_y$, $u_y = -v_x$

Since $\overline{f(z)}$ is analytic both u & $-v$ satisfies C-R equations. $u_x = -v_y$, $u_y = +v_x$

From the above equations

$$u_x = u_y = v_x = v_y = 0$$

Hence both u & v are constants.

i.e., $f(z) = u + iv$ is constant.

Property; 3

An analytic function with constant real part is a constant and an analytic function with constant imaginary part is a constant.

Proof:

(i) Given real part is constant.

$$u(x,y) = c_1$$

$$u_x = 0, \quad u_y = 0$$

Since $f(z)$ is analytic, it satisfies CR equations.

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

$$\Rightarrow v_y = 0 \qquad\qquad v_x = 0$$

Since v_x & v_y are zero it is clear that v is independent of x & y.

i.e., $v(x,y) = c_2$

$f(z) = c_1 + ic_2 = c$ (complex constant)

Hence an analytic function with constant real part is constant.

(ii) Given $v(x,y) = c_2$

$$v_x = 0$$

$$v_y = 0$$

Since $f(z)$ is analytic, it satisfies CR equations

$$v_x = -u_y \qquad \& \qquad u_x = v_y$$

$$u_y = 0 \qquad\qquad u_x = 0$$

$\therefore$ $u(x,y)=c_1$ is a constant.

$\therefore$ $w=f(z)=c_1+ic_2=c$ (complex constant)

Hence an analytic function with constant imaginary part is a constant.

Property: 4

An analytic function with constant modulus is a constant.

Proof:

Given $|f(z)|=\sqrt{u^2+v^2}=$ constant.

i.e., $u^2+v^2=c$

Differentiating partially with respect to x & y we get,

$$2uu_x+2vv_x=0$$

$$uu_x+vv_x=0 \quad \text{--- (2)}$$

$$uu_y+vv_y=0 \quad \text{--- (3)}$$

i.e., $vu_x-uv_x=0$ ($\because u_y=-v_x, u_x=v_y$)

Equation (2) & (3) have only trivial solutions

$u_x=0$ & $v_x=0$

$\Rightarrow u_y=0, \; v_y=0$

Since $u_x=u_y=v_x=v_y=0$, we have both u & v are independent of x and y.

i.e., $u(x,y)=$ a constant $=k_1$

$v(x,y)=$ a constant $=k_2$

$f(z)=u+iv=k_1+ik_2=k$ (a constant)

Hence an analytic function with constant modulus is a constant.

Property: 6

If $w = u + iv$ is an analytic function, then the curves of the family $u(x,y) = c_1$ cut orthogonally the curves of the family $v(x,y) = c_2$ where c_1 and c_2 are constants.

Proof:

Consider the first curve $u(x,y) = c_1$

we know that $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y}$$

Then the slope of the curve

$$m_1 = -\frac{u_x}{u_y}$$

Consider the curve $v(x,y) = c_2$

$$\therefore \frac{dv}{dx} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\partial v/\partial x}{\partial v/\partial y}$$

Then the slope of the curve

$$\therefore m_2 = -\frac{v_x}{v_y}$$

Since $f(z)$ is analytic, $u_x = v_y$, $u_y = -v_x$

$$\therefore m_2 = \frac{u_y}{u_x}$$

$$\therefore m_1 m_2 = \frac{-u_x}{u_y}\,\frac{u_y}{u_x} = -1$$

i.e., product of the slopes $= -1$

Hence the curves cut each other orthogonally.

CONSTRUCTION OF ANALYTIC FUNCTION:

MILNE-THOMSON METHOD:

To find f(z) when u is given (real part)

$$f(z) = \int [u_x(z,0) - i\, u_y(z,0)]\, dz$$

To find f(z) when v is given (imaginary part)

$$f(z) = \int [v_y(z,0) + i\, v_x(z,0)]\, dz$$

HARMONIC FUNCTION:

Any function which has continuous second order partial derivatives and which satisfies Laplace equation is called Harmonic function.

Two harmonic functions u and v which are such that u+iv is an analytic function are called conjugate harmonic functions.

PROBLEMS:

1) Prove that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and determine its conjugate. Also find f(z).

$$\frac{\partial u}{\partial x} = \frac{x}{(x^2+y^2)}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$ Therefore u is harmonic.

To find conjugate of u

We know that $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

$$= \frac{x\,dy - y\,dx}{(x^2+y^2)}$$

$$dv = \frac{x\,dy - y\,dx}{(x^2)} \frac{1}{1+\left(\frac{y}{x}\right)^2}$$

$$\int dv = \int \frac{1}{1+\left(\frac{y}{x}\right)^2} d(y/x)$$

$$v = \tan^{-1}(y/x)$$

$\therefore$ The required analytic function

$$f(z) = u + iv$$
$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$
$$= \log(x+iy) \quad \text{(Using logarithm of a complex number)}$$
$$= \log z$$

2) Find an analytic function whose imaginary part is $3x^2y - y^3$.

Given $v = 3x^2y - y^3$

$$\frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$v_x(z,0) = 0, \quad v_y(z,0) = 3z^2$$

$$f(z) = \int 3z^2\,dz = z^3 + c.$$

3) If $f(z) = u + iv$ is an analytic function and $u - v = e^x(\cos y - \sin y)$. find $f(z)$ in terms of z.

Solution:

Given $f(z) = u + iv$ —— ①

$if(z) = iu - v$ —— ②

① + ② $\Rightarrow (1+i) f(z) = (u-v) + i(u-v)$

$F(z) = U + iV$

Given $U = u - v = e^x(\cos y - \sin y)$

$U_x = e^x(\cos y - \sin y)$

$U_y = e^x(-\sin y - \cos y)$

$F(z) = \int [U_x(z,0) - i\, U_y(z,0)]\, dz$

$= \int e^z - i(-e^z)\, dz$

$= (1+i) \int e^z dz$

$F(z) = (1+i) e^z + c$

$f(z) = \dfrac{F(z)}{1+i} = e^z + c$

$\boxed{f(z) = e^z + c}$

4) If $u + v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u + iv$, find $f(z)$ in terms of z.

Given $V = (x-y)(x^2+4xy+y^2)$

$\dfrac{\partial V}{\partial x} = 3x^2 + 6xy - 3y^2$

$\dfrac{\partial V}{\partial y} = (-1)(x^2+4xy+y^2) + (x-y)(4x+2y)$

$V_x(z,0) = 3z^2$

$V_y(z,0) = -z^2+4z^2 = 3z^2$

$F(z) = \int (1+i)\,3z^2\,dz$

$F(z) = (1+i)z^3$

$$\boxed{f(z) = z^3 + c}$$

BILINEAR TRANSFORMATION

A transformation of the form $w = \dfrac{az+b}{cz+d}$ where a, b, c and d are complex constants and $ad-bc \neq 0$ is known as bilinear transformation.

To find bilinear transformation which transforms three distinct points into three specified three distinct points.

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

1. Find the bilinear transformation which maps the points $z_1=2$, $z_2=i$ and $z_3=-2$ into the points $w_1=1$, $w_2=i$ and $w_3=-1$.

Soln: Substituting z_1, z_2, z_3, w_1, w_2 & w_3 in the formula

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

$$\frac{(w-1)}{-(w+1)} \frac{(1+i)^2}{(1-i)(1+i)} = \frac{(z-2)}{-(z+2)} \frac{(i+2)^2}{(i+2)(2-i)}$$

$$\frac{2i(w-1)}{2(w+1)} = \frac{(z-2)(3+4i)}{(z+2)\,5}$$

$$\frac{w-1}{w+1} = \frac{(z-2)(3+4i)}{(z+2)\,5i} = \frac{(z-2)(4-3i)}{5(z+2)}$$

Using componendo and dividendo

$$\frac{(w-1)+(w+1)}{(w-1)-(w+1)} = \frac{(z-2)(4-3i)+5(z+2)}{(z-2)(4-3i)\cancel{+}-5(z+2)}$$

$$\frac{2w}{-2} = \frac{4z-3zi-8+6i+5z+10}{4z-8-3iz+6i-5z-10}$$

$$+w = \frac{3(3-i)z+2(1+3i)}{+(1+3i)z+6(3-i)}$$

$$w = \frac{3z+\frac{2(1+3i)}{3-i}}{\frac{1+3i}{(1-3i)}z+6} = \frac{3z+\frac{2(1+3i)(3+i)}{(3-i)(3+i)}}{\frac{(1+3i)^2 z}{(1-3i)(1+3i)}+6}$$

$$\boxed{w = \frac{3z+2i}{iz+6}}$$

2. Determine the bilinear transformation which maps $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ into $w_1 = i$, $w_2 = -1$, $w_3 = -i$ respectively.

Let the bilinear transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Since $z_3 = \infty$, we take z_3 as a common term in both numerator & denominator.

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)\left(\frac{z_2}{z_3}-1\right)}{(z_1-z_2)\left(1-\frac{z}{z_3}\right)}$$

Substituting the given values we get

$$\frac{(w-i)[-1-(-i)]}{(w+i)(-1-i)} = \frac{z-0}{1-0}$$

$$\frac{(w-i)(1-i)}{(w+i)(1+i)} = z$$

$$\frac{w-i}{w+i}(-i) = z \qquad \left[\because \frac{1-i}{1+i} = -i\right]$$

$$\Rightarrow \frac{w-i}{w+i} = \frac{z}{-i}$$

Using componendo & dividendo rule,

$$\frac{w-i+w+i}{w-i-w-i} = -\left[\frac{z+i}{z-i}\right]$$

$$\frac{2w}{2i} = -\left[\frac{z+i}{i-z}\right]$$

$$w = -i\frac{(z+i)}{i-z}$$

$$\boxed{w = i\left(\frac{z+i}{z-i}\right)}$$

3) Show that, under the mapping $w = \frac{i-z}{i+z}$, the image of the circle $x^2+y^2<1$, is the entire half of the w-plane to the right of the imaginary axis.

Solution:

Given $w = \frac{i-z}{i+z}$

$$(i+z)w = i-z$$

$$iw + zw = i - z$$

$$z(w+1) = i(1-w)$$

$$z = \frac{i(1-w)}{w+1}$$

Also given $x^2+y^2<1$,

i.e., $|z| < 1$

$$\left|\frac{i(1-w)}{w+1}\right| < 1 \Rightarrow |i(1-w)| < |w+1|$$

$$\Rightarrow |1-u-iv| < |u+1+iv|$$

$$\Rightarrow (1-u)^2 + v^2 < (1+u)^2 + v^2$$

$$\Rightarrow 1+u^2-2u+v^2 < 1+2u+u^2+v^2$$

$$\Rightarrow -4u < 0 \Rightarrow 4u > 0$$

$$u > 0$$

Hence the circle $x^2+y^2<1$ is mapped into the entire half of the w-plane to the right of the imaginary axis.

i.e., $x^2+y^2=1$ is mapped into $u=0$ which is the imaginary axis of w-plane.

INVARIANT POINTS OR FIXED POINTS:

The fixed points of the transformation are such that the image of z is itself.

The invariant points of the transformation $w=f(z)$ is given by the solution of $z=f(z)$.

PROBLEMS:

1. Find the invariant point of the transformation

$$w=\frac{1}{z-2i}$$

Solution:

The invariant points are given by

$$z=\frac{1}{z-2i}$$

$$z^2-2iz-1=0 \Rightarrow z=\frac{2i\pm\sqrt{-4+4}}{2}$$

$\boxed{z=i}$ is the invariant point.