UNIT 4: Friction Factor and Fluid Flow

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4.1 Derivation of equation of energy

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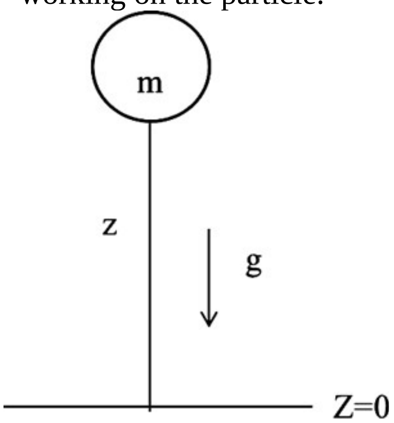
Equation of mechanical energy

For understanding the nature of mechanical energy, consider a simple case of a single particle moving in one direction as shown in Fig. 30.1. Assume the particle has mass m and is located at height h from a reference plane and moving upward with velocity $\underset{\sim}{v}$. Gravity is the only force working on the particle.



Starting with Newton's second law of motion, we have

Force = mass x acceleration

where

$$\underset{\sim}{F} = m\underset{\sim}{a}$$

or

$$\underset{\sim}{F} = m\frac{d\underset{\sim}{v}}{dt}$$

By taking dot product of equation (30.2) with velocity , we find that

$$\underset{\sim}{v}.\left[\underset{\sim}{F} = m\frac{d\underset{\sim}{v}}{dt}\right]$$

or

$$\underset{\sim}{v}.\underset{\sim}{F} = m\underset{\sim}{v}.\frac{d\underset{\sim}{v}}{dt}$$

Using vector identity, we have

$$\frac{d\left(\underset{\sim}{v}.\underset{\sim}{v}\right)}{dt} = \underset{\sim}{v}.\frac{d\underset{\sim}{v}}{dt} + \frac{d\underset{\sim}{v}}{dt}.\underset{\sim}{v} = 2\underset{\sim}{v}.\frac{d\underset{\sim}{v}}{dt}$$

or

$$\underset{\sim}{v}.\frac{d\underset{\sim}{v}}{dt} = \frac{1}{2}\frac{d\left(v^2\right)}{dt}$$

where, ν is the magnitude of the velocity vector $\underset{\sim}{v}$.

By substituting

$$\underset{\sim}{v}.\underset{\sim}{F} = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

$$v_1F_1 + v_2F_2 + v_3F_3 = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

For the example given above, we have ,

F1 = 0, F2 = 0, F3=-mg

and

ν1 = 0, ν2 = 0, ν3 = ν

$$-vmg = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

Substitute ν = (dz/dt). Thus we obtain,

$$-mg\frac{dz}{dt} = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

Since, m and g are constants. We may rewrite above equation as,

$$-\frac{d(mgz)}{dt} = \frac{d\left(m\frac{v^2}{2}\right)}{dt}$$

or

$$\frac{d}{dt}\left(mgz + m\frac{v^2}{2}\right) = 0$$

Thus $$mgz + m\frac{v^2}{2} = constant$$

$$\underset{\sim}{v}\cdot\left[\rho\frac{D\underset{\sim}{v}}{Dt} = \rho\underset{\sim}{g} - \underset{\sim}{\nabla}P - \underset{\sim}{\nabla}\cdot\underset{\sim}{\tau}\right]$$

As before,

$$\underset{\sim}{v}\cdot\frac{D\underset{\sim}{v}}{Dt} = \frac{D\left(\frac{v^2}{2}\right)}{Dt}$$

(Note: substantial derivatives behave like normal derivatives.). Thus,

$$\rho\frac{D}{Dt}\left(\frac{v^2}{2}\right) = \rho\underset{\sim}{g}\cdot\underset{\sim}{v} - \underset{\sim}{v}\cdot\left(\underset{\sim}{\nabla}P\right) - \underset{\sim}{v}\cdot\left(\underset{\sim}{\nabla}\cdot\underset{\sim}{\tau}\right)$$

The following vector and tensor identities may be used for simplifying Equation (30.14)

$$\nabla \cdot (P\, v) = P(\nabla \cdot v) + v \cdot (\nabla P)$$

and if τ is a second order symmetric tensor then we also have

$$\nabla \cdot (\tau \cdot v) = v \cdot \nabla \cdot \tau + \tau : \nabla v$$

Thus, we obtain

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right) = v \cdot (\rho g) - \left[\nabla \cdot (P v) + \left(-P(\nabla \cdot v)\right)\right] - \left[\nabla \cdot (\tau \cdot v) + \left(-\tau : \nabla v\right)\right]$$

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right) = v \cdot (\rho g) - \nabla \cdot (P v) + P(\nabla \cdot v) - \nabla \cdot (\tau \cdot v) + \tau : \nabla v$$

Equation is called the equation of mechanical energy for fluids.

Significance of each term is given below.

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right)\begin{Bmatrix} \text{Rate of change of} \\ \text{kinetic energy} \\ \text{per unit volume} \end{Bmatrix} = v \cdot (\rho g)\begin{Bmatrix} \text{work done by gravity} \\ \text{force on the system} \end{Bmatrix}$$

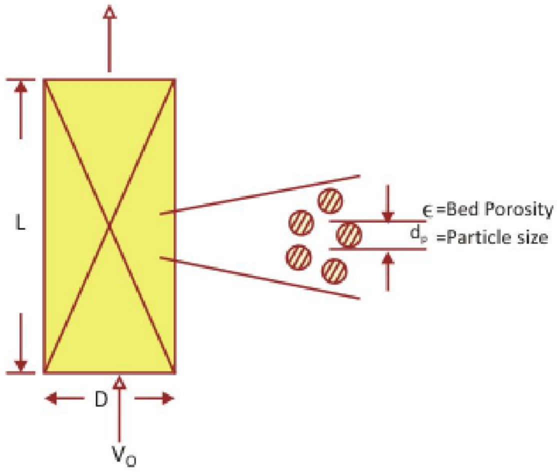
$$-\nabla \cdot (P v)\begin{Bmatrix} \text{work done by pressure} \\ \text{force on the system} \end{Bmatrix} + P(\nabla \cdot v)\begin{Bmatrix} \text{reversible conversion of} \\ \text{kinetic energy into} \\ \text{the internal energy} \end{Bmatrix}$$

$$-\nabla \cdot (\tau \cdot v)\begin{Bmatrix} \text{work done by viscous} \\ \text{forces on system} \end{Bmatrix} + \tau : \nabla v\begin{Bmatrix} \text{irreversible conversion of} \\ \text{kinetic energy in to the heat} \end{Bmatrix}$$

$$\rho\frac{D(\hat{U})}{Dt}\begin{Bmatrix}\text{Rate of change of}\\ \text{internal energy}\\ \text{per unit volume}\end{Bmatrix} = -\nabla.\underset{\sim}{q}\begin{Bmatrix}\text{Heat transferred}\\ \text{by conduction}\end{Bmatrix} + S_c\begin{Bmatrix}\text{Heat generated / removed}\\ \text{by source or sink}\end{Bmatrix}$$

$$-P(\nabla.\underset{\sim}{v})\begin{Bmatrix}\text{reversible conversion of}\\ \text{kinetic energy into}\\ \text{the internal energy}\end{Bmatrix} - (\underset{\sim}{\tau} : \nabla\underset{\sim}{v})\begin{Bmatrix}\text{irreversible conversion of}\\ \text{kinetic energy in to the heat}\end{Bmatrix}$$

4.2 FLOW THROUGH PACKED BED



For the theoretical analysis to calculate pressure–drop, actual flow channels are replaced with parallel cylindrical conduits of constant cross–section. Particles are assumed to be of the same size and shape having constant sphericity, Φ_S.

Pressure–drop occurs due to inertial and viscous effects. At high Reynolds number, inertial effects prevail, whereas the viscous effects are important at low Reynolds number. In general,

$$(\Delta p)_{total} = (\Delta p)_{viscous} + (\Delta p)_{inertial}$$

$$\frac{F_D}{As} = \frac{\text{Drag over the channel} - \text{walls consiting of packed bed particles}}{\text{Total surface area of particles}}$$

$$\boxed{= K_1 \left(\frac{\mu_f V}{r_h}\right) + K_2 \left(\rho_f V^2\right)}$$

$$\tau = \frac{4\mu_f V}{r} \text{ or } \tau \, \alpha \, \frac{\mu_f V}{r_h}$$

$$r_h = \text{hydraulic radius} = \frac{\text{Total cross} - \text{section of conduits}}{\text{Wetted parameter}}$$

$$As = \underset{\textit{Total \# of particles}}{\underset{\downarrow}{N_p}} \times \underset{\textit{Surface area of one particle}}{\underset{\downarrow}{S_p}}$$

$$= \frac{S_o L(1-\epsilon)}{v_p} \times s_p, \text{ where } v_p = \text{volume of one partide}$$

$$\frac{v_p}{s_p} = \frac{6}{\phi_p \, d_p}; v = \frac{v_o}{\epsilon}, \text{ where } v_o = \text{superficial velocity}$$

Similarly, pressure–drop at high Reynolds number, $\Delta p \; \alpha \; \rho_f V^2$. Therefore, Pressure-drop in packed beds is related to pressure–drop due to viscous and inertial effects via two empirical constants, K_1 and K_2.

$= \frac{\text{Total volume of voids}}{\text{Total surface area of particles}}$ **(multiply both numerator and denominator by L)**

$= \frac{(S_o L)\epsilon}{As}$ **, S_0 = cross sectional area of packed–bed**

$$\frac{F_D}{As} = \frac{F_D \phi_p \, d_p}{S_o L(1-\epsilon) \times 6} = \left[K_1 \, \frac{\mu_f \, v_o}{\epsilon^2 \, S_o L} \times \frac{S_o L(1-\epsilon)6}{\phi_p \, d_p} + K_2 \rho_f \frac{V_o^2}{\epsilon^2}\right]$$
$$= \frac{\rho_f v_o^2}{\epsilon^2}\left[6K_1 \frac{\mu_f \,(1-\epsilon)}{v_o \phi_s \, d_p \;\; \rho_f} + K_2\right]$$
$$F_D = \text{drag} - \text{force} = (\Delta p)_{total} \times S_o \epsilon$$

$$\frac{(\Delta p)_{total} \; (S_o \epsilon) \phi_s \, d_p}{S_o L(1-\epsilon) \times 6} = \rho_f \frac{V_o^2}{\epsilon^2}\left[\frac{6K_1 \mu_f \,(1-\epsilon)}{\phi_s \; d_p v_o \; \rho_f} + K_2\right]$$

$$\frac{\Delta p}{L}\left(\frac{\epsilon^3}{1-\epsilon}\right)\left(\frac{\phi_s \; d_p}{\rho_f \; V_o^2}\right) = \frac{36 \; K_1 \; \mu_f \,(1-\epsilon)}{\phi_s \; d_p v_o \; \rho_f} + 6K_2$$

$$\boxed{\frac{\Delta p}{L}\left(\frac{\epsilon^3}{1-\epsilon}\right)\left(\frac{\phi_s \; d_p}{\rho_f \; V_o^2}\right) = \frac{150(1-\epsilon)\mu_f}{\phi_s \; d_p v_o \; \rho_f} + \; 1.75}$$

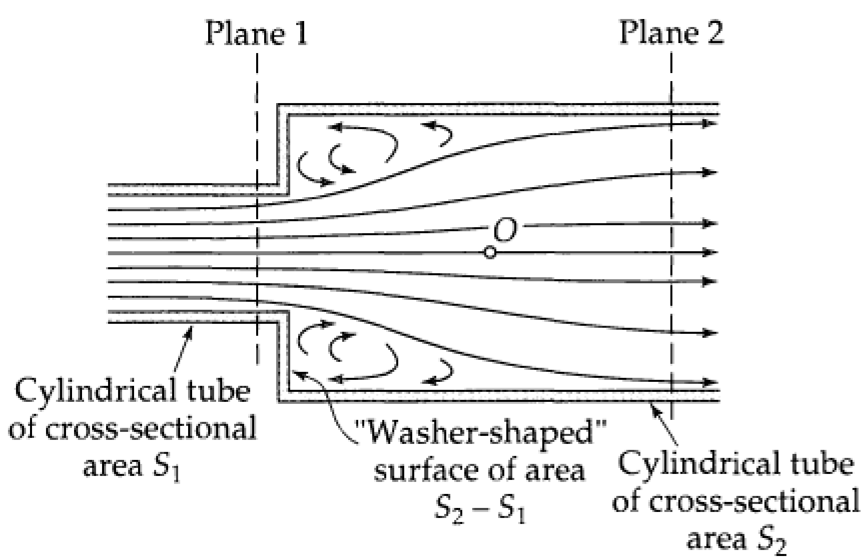
$$f_\rho = \frac{\Delta p}{L \; \rho_f \, V_o^2}\left(\frac{\phi_s \; d_p \epsilon^3}{(1-\epsilon)}\right)$$

$$= \left(\frac{\Delta p}{\rho}\right)_{Friction}$$

$$= \frac{150(1-\epsilon)^2\, \mu_f\, v_o L}{\epsilon^3\, d_p{}^2\, \phi_s{}^2\, \rho_f} + \frac{1.75\,(1-\epsilon)\, V_o^2 L}{\epsilon^3\, \phi_s\, d_p}$$

4.3 SUDDEN ENLARGEMENT

An incompressible fluid flows from a small circular tube into a large tube in turbulent flow, as shown in Fig. 7.6-1. The cross-sectional areas of the tubes are S_1 and S_2. Obtain an expression for the pressure change between planes 1 and 2 and for the friction loss associated with the sudden enlargement in cross section. Let $\beta = S_1/S_2$, which is less than unity.



(a) ***Mass balance.*** For steady flow the mass balance gives

$$w_1 = w_2 \quad \text{or} \quad \rho_1 v_1 S_1 = \rho_2 v_2 S_2$$

For a fluid of constant density, this gives

$$\frac{v_1}{v_2} = \frac{1}{\beta}$$

(b) ***Momentum balance.*** The downstream component of the momentum balance is

$$\mathbf{F}_{f\to s} = (v_1 w_1 - v_2 w_2) + (p_1 S_1 - p_2 S_2)$$

The force $\mathbf{F}_{f\to s}$ is composed of two parts: the viscous force on the cylindrical surfaces parallel to the direction of flow, and the pressure force on the washer-shaped surface just to the right of plane 1 and perpendicular to the flow axis. The former contribution we neglect (by intuition) and the latter we take to be $p_1(S_2 - S_1)$ by assuming that the pressure on the washer-shaped surface is the same as that at plane 1.

$$-p_1(S_2 - S_1) = \rho v_2 S_2 (v_1 - v_2) + (p_1 S_1 - p_2 S_2)$$

Solving for the pressure difference gives

$$p_2 - p_1 = \rho v_2 (v_1 - v_2)$$

or, in terms of the downstream velocity,

$$p_2 - p_1 = \rho v_2^2 \left(\frac{1}{\beta} - 1\right)$$

(c) ***Angular momentum balance.*** This balance is not needed. If we take the origin of coordinates on the axis of the system at the center of gravity of the fluid located between planes 1 and 2, then $[\mathbf{r}_1 \times \mathbf{u}_1]$ and $[\mathbf{r}_2 \times \mathbf{u}_2]$ are both zero, and there are no torques on the fluid system.

(d) ***Mechanical energy balance.*** There is no compressive loss, no work done via moving parts, and no elevation change, so that

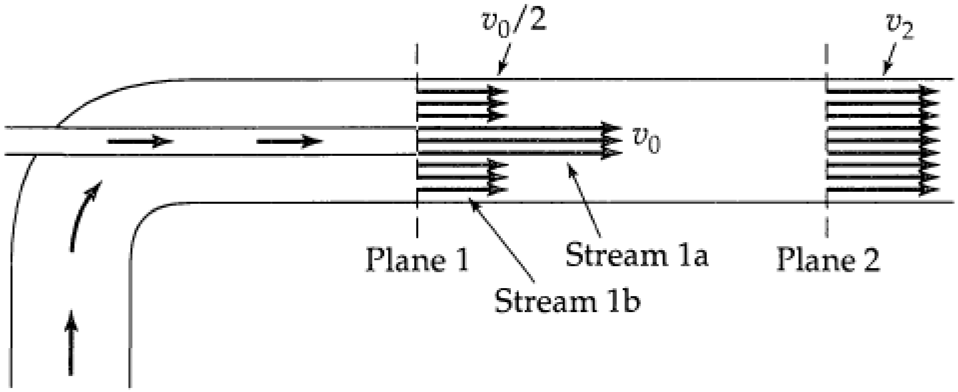
$$\hat{E}_v = \tfrac{1}{2}(v_1^2 - v_2^2) + \frac{1}{\rho}(p_1 - p_2)$$

Insertion of Eq. 7.6-6 for the pressure rise then gives, after some rearrangement,

$$\hat{E}_v = \tfrac{1}{2} v_2^2 \left(\frac{1}{\beta} - 1\right)^2$$

4.4 LIQUID - LIQUID EJECTOR

A diagram of a liquid–liquid ejector is shown in Fig. It is desired to analyze the mixing of the two streams, both of the same fluid, by means of the macroscopic balances. At plane 1 the two fluid streams merge. Stream 1a has a velocity v_0 and a cross-sectional area $\frac{1}{3}S_1$, and stream 1b has a velocity $\frac{1}{2}v_0$ and a cross-sectional area $\frac{2}{3}S_1$. Plane 2 is chosen far enough downstream that the two streams have mixed and the velocity is almost uniform at v_2. The flow is



(a) ***Mass balance.*** At steady state, Eq. (A) of Table 7.6-1 gives

$$w_{1a} + w_{1b} = w_2$$

or

$$\rho v_0(\tfrac{1}{3}S_1) + \rho(\tfrac{1}{2}v_0)(\tfrac{2}{3}S_1) = \rho v_2 S_2$$

Hence, since $S_1 = S_2$, this equation gives

$$v_2 = \tfrac{2}{3}v_0$$

for the velocity of the exit stream. We also note, for later use, that $w_{1a} = w_{1b} = \frac{1}{2}w_2$.

(b) ***Momentum balance.*** From Eq. (B) of Table 7.6-1 the component of the momentum balance in the flow direction is

$$(v_{1a}w_{1a} + v_{1b}w_{1b} + p_1S_1) - (v_2w_2 + p_2S_2) = 0$$

or using the relation at the end of (a)

$$\begin{aligned}(p_2 - p_1)S_2 &= (\tfrac{1}{2}(v_{1a} + v_{1b}) - v_2)w_2 \\ &= (\tfrac{1}{2}(v_0 + \tfrac{1}{2}v_0) - \tfrac{2}{3}v_0)(\rho(\tfrac{2}{3}v_0)S_2)\end{aligned}$$

from which

$$p_2 - p_1 = \tfrac{1}{18}\rho v_0^2$$

This is the expression for the pressure rise resulting from the mixing of the two streams.

(c) ***Angular momentum balance.*** This balance is not needed.

(d) ***Mechanical energy balance.*** Equation (D) of Table 7.6-1 gives

$$(\tfrac{1}{2}v_{1a}^2 w_{1a} + \tfrac{1}{2}v_{1b}^2 w_{1b}) - \left(\tfrac{1}{2}v_2^2 + \frac{p_2 - p_1}{\rho}\right)w_2 = E_v$$

or, using the relation at the end of (a), we get

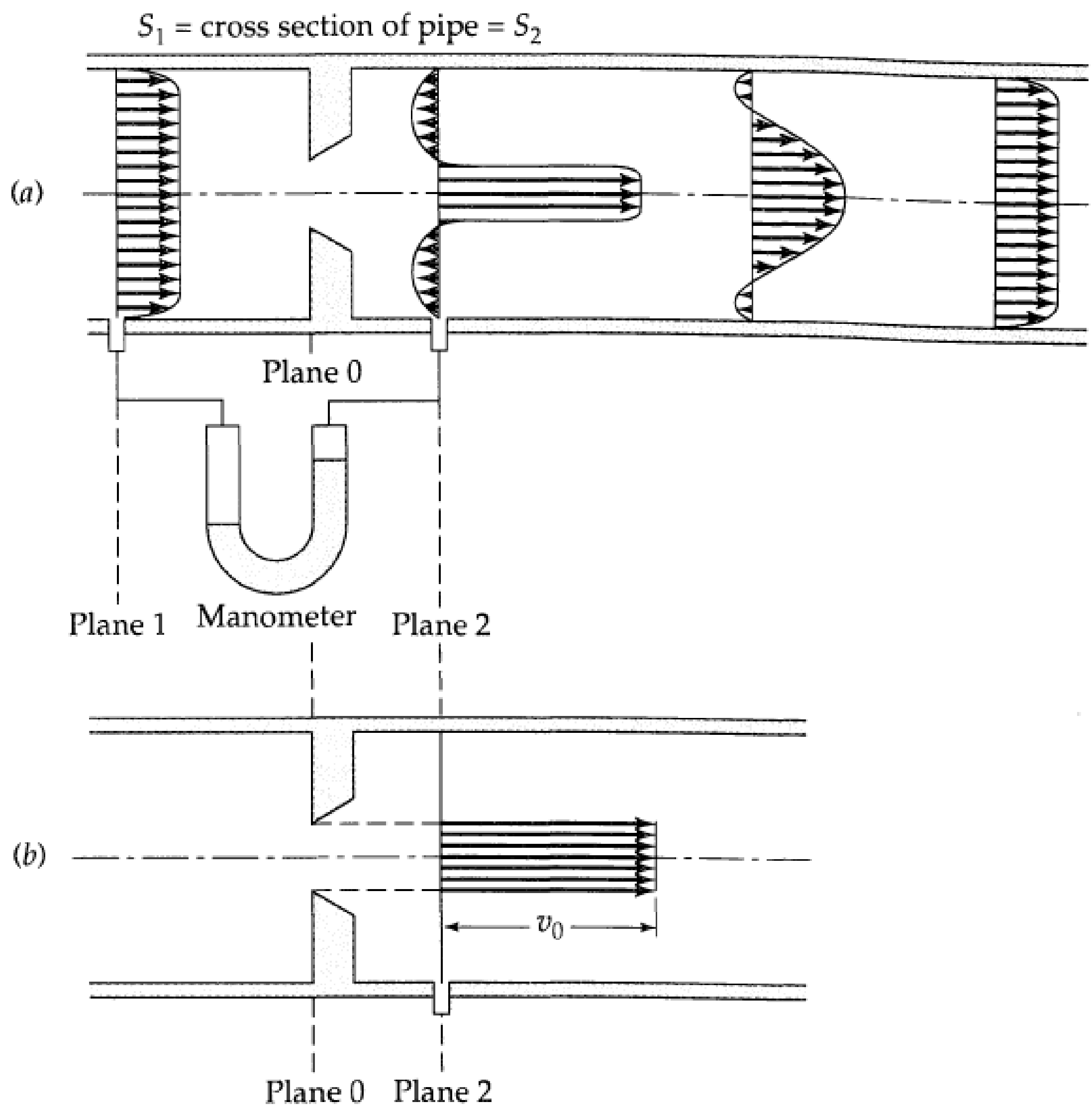
$$(\tfrac{1}{2}v_{1a}^2(\tfrac{1}{2}w_2) + \tfrac{1}{2}(\tfrac{1}{2}v_0)^2(\tfrac{1}{2}w_2)) - (\tfrac{1}{2}(\tfrac{2}{3}v_0)^2 + \tfrac{1}{18}v_0^2)w_2 = E_v$$

Hence

$$\hat{E}_v = \frac{E_v}{w_2} = \frac{5}{144}v_0^2$$

4.5 ISOTHERMAL FLOW OF AN LIQUID THROUGH AN ORIFICE

A common method for determining the mass rate of flow through a pipe is to measure the pressure drop across some "obstacle" in the pipe. An example of this is the orifice, which is a thin plate with a hole in the middle. There are pressure taps at planes 1 and 2, upstream and downstream of the orifice plate.


S_1 = cross section of pipe = S_2
(a)
Plane 0
Plane 1
Manometer
Plane 2
(b)
v_0
Plane 0
Plane 2

(a) ***Mass balance.*** For a fluid of constant density with a system for which $S_1 = S_2 = S$, the mass balance in Eq. 7.1-1 gives

$$\langle v_1 \rangle = \langle v_2 \rangle$$

With the assumed velocity profiles this becomes

$$v_1 = \frac{S_0}{S} v_0$$

and the volume rate of flow is $w = \rho v_1 S$.

(b) ***Mechanical energy balance.*** For a constant-density fluid in a flow system with no elevation change and no moving parts, Eq. 7.4-5 gives

$$\frac{1}{2}\frac{\langle v_2^3 \rangle}{\langle v_2 \rangle} - \frac{1}{2}\frac{\langle v_1^3 \rangle}{\langle v_1 \rangle} + \frac{p_2 - p_1}{\rho} + \hat{E}_v = 0$$

The viscous loss $\hat{E}_v$ is neglected, even though it is certainly not equal to zero. With the assumed velocity profiles, Eq.

$$\tfrac{1}{2}(v_0^2 - v_1^2) + \frac{p_2 - p_1}{\rho} = 0$$

$$v_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho}\frac{1}{(S/S_0)^2 - 1}}$$