SUBJECT NAME: ENGINEERING MATHEMATICS III

(Common to ALL branches except BIO GROUPS, CSE & IT)

SUBJECT CODE: SMT1201

COURSE MATERIAL

UNIT IV PARTIAL DIFFERENTIAL EQUATIONS

Formation of equations by elimination of arbitrary constants and arbitrary functions - Solutions of PDE - general, particular and complete integrals - Solutions of First order Linear PDE (Lagrange's linear equation) - Solution of Linear Homogeneous PDE of higher order with constant coefficients.

INTRODUCTION

A partial differential equation is an equation involving a function of two or more variables and some of its partial derivatives. Therefore a partial differential equation contains one dependent variable and more than one independent variable

Notations in PDE

$p = \partial z/\partial x$ $q = \partial z/\partial y$ $r = \partial^2 z/\partial x^2$ $s = \partial^2 z/\partial x\partial y$ $t = \partial^2 z/\partial y^2$

Formation of partial differential equations:

There are two methods to form a partial differential equation.

(i) By elimination of arbitrary constants.

(ii) By elimination of arbitrary functions.

Formation of partial differential equations by elimination of arbitrary constants:

1. Form a p.d.e by eliminating the arbitrary constants a and b from $Z=(x+a)^2+(y-b)^2$

Solution:

Given $Z= (x+a)^2+(y-b)^2$

$P = \frac{\partial z}{\partial x} = 2(x+a)$, ie) $x+a = \frac{p}{2}$

$q = \frac{\partial z}{\partial y} = 2(y-b)$, ie) $y-b = \frac{q}{2}$

$$\therefore (1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$z = \frac{p^2}{4} + \frac{q^2}{4}$$

$4z = p^2+q^2$

which is the required p.d.e.

2. Find the p.d.e of all planes having equal intercepts on the X and Y axis.

Solution:

Intercept form of the plane equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Given : a=b. [Equal intercepts on the x and y axis]

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad .. \qquad (1)$$

Here a and c are the two arbitrary constants.

Differentiating (1) p.w.r.to 'x' we get

$$\frac{1}{a} + 0 + \frac{1}{c}\frac{\partial z}{\partial x} = 0$$

$$\frac{1}{a} + \frac{1}{c}p = 0.$$

$$\frac{1}{a} = -\frac{1}{c}p. \qquad (2)$$

Diff (1) p.w.r.to. 'y' we get

$$0+\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0.$$

$$\frac{1}{a}+\frac{1}{c}q=0$$

$$\frac{1}{a}=-\frac{1}{c}q \qquad (3)$$

From (2) and (3) $\Rightarrow -\frac{1}{c}p=-\frac{1}{c}q$

p = q ,which is the required p.d e.

3. Form the p.d.e by eliminating the constants a and b from $z = ax^n+by^n$.

Solution:

Given: $z = ax^n+by^n$. (1)

$P = \frac{\partial z}{\partial x} = anx^{n-1}$

$\frac{p}{n} = ax^{n-1}$

Multiply ' x' we get, $\frac{px}{n} = ax^n$ (2)

$q = \frac{\partial z}{\partial y} = bny^{n-1}$

$\frac{q}{n} = by^{n-1}$

Multiply ' y' we get , $\frac{qy}{n} = by^n$ (3)

Substitute (2) and (3) in (1) we get the required p.d.e $z = \frac{px}{n}+\frac{qy}{n}$

$zn = px+qy$.

Formation of partial differential equations by elimination of arbitrary functions:

1. Eliminate the arbitrary function f from z= $f\left(\frac{y}{x}\right)$ and form a partial differential equation.

   Solution:

   Given z = $f\left(\frac{y}{x}\right)$ (1)

   Differentiating (1) p.w.r.to 'x' we get

   $$P = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \quad (2)$$

   Differentiating (1) p.w.r.to y we get

   $$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad (3)$$

   $$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{-y}{x}$$

   $\therefore$ px = -qy

   ie) px+qy = 0 is the required p.d.e.

2. Eliminate the arbitrary functions f and g from z = f(x+iy)+g(x-iy) to obtain a partial differential equation involving z,x,y.

   Solution:

   Given : z = f(x+iy)+g(x-iy) (1)

   $$P = \frac{\partial z}{\partial x} = f'(x+iy)+g'(x-iy) \quad (2)$$

   $$q = \frac{\partial z}{\partial y} = if'(x+iy)-ig'(x-iy) \quad (3)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+iy)+g''(x-iy) \qquad (4)$$

$$t = \frac{\partial^2 z}{\partial y^2} = -f''(x+iy)-g''(x-iy) \qquad (5)$$

r + t = 0 is the required p.d.e.

3. Form the p.d.e by eliminating arbitrary function ϕ from the relation $\phi(xyz, x^2 + y^2 + z^2) = 0$

Solution:

The pde is obtained from $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$

$$\begin{vmatrix} yz + xyp & 2x + 2zp \\ xz + xyq & 2y + 2zq \end{vmatrix} = 0$$

(yz+xyp)(2y+2zq)-(xz+xyq)(2x+2zp)=0

SOLUTION OF PDE

Complete solution: A solution which contains as many arbitrary constants as there are independent variables is called a complete integral (or)complete solution.(number of arbitrary constants=number of independent variables)

Particular solution: A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral (or) particular solution.

General solution: A solution of a p.d.e which contains the maximum possible number of arbitrary functions is called a general integral (or) general solution.

1. Find the general solution of $\frac{\partial^2 z}{\partial y^2} = 0$

Solution:

Given $\frac{\partial^2 z}{\partial y^2} = 0$

ie) $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=0$

Integrating w.r.to ‘y’ on both sides

$\frac{\partial z}{\partial y}$ = a (constants)

ie) $\frac{\partial z}{\partial y}$ =f(x)

Again integrating w.r.to ‘y’ on both sides.

z = f(x) y + b which is the required solution.

Lagrange’s linear equations:

The equation of the form Pp + Qq = R is known as Lagrange’s equation, where P, Q and R are functions of x, y and z. To solve this equation it is enough to solve the subsidiary equations.

dx/P = dy/Q = dz/R

If the solution of the subsidiary equation is of the form u(x, y, z) = c_1 and v(x, y, z) = c_2 then the solution of the given Lagrange’s equation is Φ(u, v) = 0.

To solve the subsidiary equations we have two methods:

1 Method of Grouping:

Consider the subsidiary equation dx/P = dy/Q = dz/R..Take any two members say first two or last two or first and last members. Now consider the first two members dx/P = dy/Q. If P and Q contain z (other than x and y) try to eliminate it. Now direct integration gives u(x, y) = $c_{1.}$ Similarly take another two members dy/Q = dz/R. If Q and R contain x(other than y and z) try to eliminate it. Now direct integration gives v(y, z) = c_2 . Therefore solution of the given Lagrange’s equation is Φ(u, v) = 0.

1. Solve px + qy = z

Solution:

The Lagrange's eqn is Pp + Qq = R

and the auxilliary eqn. is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

ie $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$ (1)

Taking the first two ratios,

$$\frac{dx}{x}=\frac{dy}{y}$$

Integrating, logx = logy + loga

$$\frac{x}{y}=a \qquad (2)$$

Similarly, taking last two ratios of eqn (1),

$$\frac{dy}{y}=\frac{dz}{z}$$

Integrating, logy = logz + logb

$$\frac{y}{z}=b \qquad (3)$$

Eqns (2) and (3) are independent solns of (1).

Hence the complete soln of the given eqn. is φ(u,v)=0

ie; $\phi\left(\frac{x}{y},\frac{y}{z}\right)=0$

Method of multiplier's

Choose any three multipliers l, m, n may be constants or function of x, y and z such that in $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}=\frac{ldx+mdy+ndz}{lP+mQ+nR}$

the expression lP + mQ + nR = 0. Hence ldx + mdy + ndz = 0

[since each of the above ratios equal to a constant $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}=\frac{ldx+mdy+ndz}{lP+mQ+nR}=k(say)$

$$ldx+mdy+ndz=k(lP+mQ+nR)$$

If $lP + mQ + nR = 0$ then $ldx + mdy + ndz = 0$]

Now direct integration gives u(x, y, z) = c_1.

similarly choose another set of multipliers l′, m′, n′

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$$

the expression $l'P + m'Q + n'R = 0$

therefore $l'dx + m'dy + n'dz = 0$ (as explained earlier)

Now direct integration gives v(x, y, z) = c_2.

Therefore solution of the given Lagrange's equation is Φ(u, v) = 0.

1. Solve $x(y^2 - z^2)p - y(z^2 + x^2)q = z(x^2 + y^2)$

Solution:

The Lagrange's eqn is Pp + Qq = R

and the auxilliary eqn. is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}$$

Taking multpliers as x,y,z;

$$\frac{dx}{x\left(y^2 - z^2\right)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)} = \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = k(say)$$

$$xdx + ydy + zdz = k(x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2))$$

$$xdx + ydy + zdz = 0$$

Integrating , $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c}{2}$

ie; $x^2 + y^2 + z^2 = c$

$u = x^2 + y^2 + z^2$ (1)

Again taking the multipliers as 1/x,-1/y,-1/z,

$$\frac{dx}{x\left(y^2 - z^2\right)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)} = \frac{\frac{1}{x}dx + \frac{-1}{y}dy + \frac{-1}{z}dz}{(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)} = k(say)$$

$$\frac{1}{x}dx + \frac{-1}{y}dy + \frac{-1}{z}dz = k(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)$$

$$\frac{1}{x}dx + \frac{-1}{y}dy + \frac{-1}{z}dz = 0$$

Integrating, log x- log y- log z= log C'

$$\frac{x}{yz} = c'$$

$v = \frac{x}{yz}$ (2)

solution is $\phi(x^2 + y^2 + z^2, \frac{x}{yz}) = 0$

Homogeneous Linear partial differential equations:

Equation of the form $a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^{n-1} z}{\partial x^{n-1} \partial y} + ...a_n \frac{\partial^n z}{\partial y^n} = F(x, y)$

$F(x, y) = [a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots\ldots + a_n D'^n]z$

where $D = \partial/\partial x$ and $D' = \partial/\partial y$

Solution of Homogeneous Linear partial differential equations:

The Complete solution consists of two parts namely complementary function and particular integral.

i.e) Z = C.F + P.I

To find the Complementary function (C.F.):

The complementary function is the solution of the equation

$$a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots\ldots + a_n D'^n = 0.$$

In this equation, put D = m and D′ = 1 then we get an equation, which is called auxiliary equation.Hence the auxiliary equation is

$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_n = 0.$

Let the root of this equation be $m_1, m_2, m_3, \ldots\ldots m_n$.

Case 1: If the roots are real or imaginary and different say $m_1 \neq m_2 \neq m_3 \neq \ldots\ldots \neq m_n$. then the

$$\text{C.F. is } Z = f_1 (y + m_1x) + f_2 (y + m_2x) + \ldots\ldots + f_n (y + m_nx)$$

Case 2: If any two roots are equal, say $m_1 = m_2 = m$, and others are different then the C.F. is

$$Z = f_1 (y + mx) + xf_2 (y + mx) + f_3 (y + m_3x) + \ldots\ldots + f_n (y + m_nx)$$

Case 3: If three roots are equal, say $m_1 = m_2 = m_3 = m$, then the C.F. is

$$Z = f_1 (y + mx) + xf_2 (y + mx) + x^2f_3 (y + mx) + \ldots\ldots + f_n (y + m_nx) .$$

To find the Particular Integral:

Rule1: If $F(x, y) = e^{ax+by}$ then

$$\text{P.I.} = \frac{1}{\phi(D, D')} e^{ax+by}$$

$= 1 / \Phi (a, b).\ e^{ax+by}$ provided $\Phi (a, b) \neq 0$ [Replace D by a and D′ by b]

If $\Phi (a, b) = 0$ refer rule 4.

Rule2: If F(x, y) = sin (mx + ny) or cos (mx + ny) then

$$\text{P.I.} = \frac{1}{\phi(D, D')} \sin(mx + ny) \quad or \quad \cos(mx + ny)$$

Replace D^2 by $-m^2$, D'^2 by $-n^2$ and DD′ by –mn in provided the denominator is not equal to zero. If the denominator is zero refer rule 4.

Rule3: If $F(x, y) = x^m y^n$

$$\text{P.I.} = \frac{1}{\phi(D, D')} x^m y^n$$

$$= [\Phi(D, D')]^{-1} x^m y^n$$

Expand $[\Phi(D, D')]^{-1}$ by using binomial theorem and then operate on $x^m y^n$

Note: 1/ D denotes integration w.r.t x, 1/ D′ denotes integration w.r.t y.

Rule4: If F(x, y) is any other function, resolve Φ (D, D′) in to linear factor say $(D - m_1 D')$

$(D - m_2 D')$ etc. then the P.I. = $\frac{1}{(D - m_1 D')(D - m_2 D')} F(x, y)$

Note:1

$$\frac{1}{(D - mD)} F(x, y) = \int F(x, c-mx)\, dx, \text{ where } y = c-mx.$$

Note:2

If the denominator is zero in rule (1) and (2) then apply Rule (4)

1. Solve $(D^2-2DD'+ D'^2)z = 0$

   Solution:

   Given $(D^2-2DD'+ D'^2) z = 0$

   The auxiliary eqn is $m^2-2m+1=0$

   ie) $(m-1)^2 =0$

   m =1,1

   The roots are equal.

   ∴ C.F = $f_1(y+x)+xf_2(y+x)$

   Hence z = C.F

   z = $f_1(y+x)+xf_2(y+x)$.

2. Solve $(D^4 - D'^4)z = 0$

Solution:

Given $(D^4 - D'^4)\, z = 0$

The auxiliary equation is $m^4 - 1 = 0$

[Replace D by m and D′ by 1]

Solving $(m^2-1)(m^2+1) = 0$

$m^2-1=0$, $m^2+1=0$

$m^2=1$, $m^2=-1$

$m=\pm 1$, $m = \pm\sqrt{-1} = \pm i$

ie) $m = 1, -1, i, -i$

The solution is $z = f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix)$.

3. Find the P.I of $[D^2 + 4DD']y = e^x$

Solution:

$$\text{P.I} = \frac{1}{D^2 + 4DD'} e^x$$

$$= \frac{1}{D^2 + 4DD'} e^{x+0y}$$

$$= e^x \left[\frac{1}{1+4(1)(0)} \right] \quad \text{Replace D by 1 and } D' \text{ by 0}$$

$$= e^x .$$

Solution is $y = e^x$.

4. Solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

Solution:

The symbolic form is $(D^3 - 3D^2 D' + 4D'^3)z = e^{x+2y}$

A.E is $m^3 - 3m^2 + 4 = 0$

m = -1 , 2 ,2

C.F is z = $f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$

$$\text{P.I} = \frac{1}{D^3 - 3D^2D^1 + 4D'^3} e^{x+2y}$$

$$= \frac{1}{1-(3)(1)(2)+(4)(8)} e^{x+2y}$$

$$= \frac{1}{27} e^{x+2y}$$

The complete solution is

$$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

5. Solve $[D^2 - 2DD' + D'^2]$ z = cos(x-3y).

Solution:

Given $[D^2 - 2DD' + D'^2]z$ = cos(x-3y).

The auxiliary equation is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

m =1,1

C.F = $f_1(y+x) + xf_2(y+x)$.

$$\text{P.I} = \frac{1}{D^2 - 2DD' + D'^2} \cos(x-3y)$$

$$= \frac{\cos(x-3y)}{-1-2(3)-9}$$

$$= \frac{-1}{16} \cos(x-3y)$$

$\therefore$ The complete solution is $Z = f_1(y+x) + xf_2(y+x) - \frac{1}{16}\cos(x-3y)$.

6. Solve $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$

Solution:

The symbolic form is $[D^2 + 3DD' + 2D'^2]z = x + y$

A.E is $m^2 + 3m + 2 = 0$

$m = -1 , -2$

C.F is $z = f_1(y-x) + f_2(y-2x)$

$$\text{P.I} = \frac{1}{D^2 + 3DD' + 2D'^2} x + y$$

$$= \frac{1}{D^2\left[1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right]} x + y$$

$$= \frac{1}{D^2}\left[1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right]^{-1} x + y$$

$$= \frac{1}{D^2}\left[1 - (\frac{3D'}{D} + \frac{2D'^2}{D^2}) + ...\right] x + y$$

$$= \frac{1}{D^2}\left[1 - \frac{3D'}{D}\right] x + y$$

$$= \frac{1}{D^2}\left[(x + y) - \frac{3D'}{D}(x + y)\right]$$

$$= \frac{1}{D^2}\left[(x+y)-3x\right]$$

$$= \frac{1}{D^2}\left[y-2x\right]$$

$$= \frac{1}{D^2}\left[(x+y)-3x\right]$$

$$= \frac{1}{D^2}\left[y-2x\right]$$

$$= \frac{yx^2}{2}-\frac{x^3}{3}$$

The complete solution is

$$z = f_1(y-x) + f_2(y-2x) + \frac{yx^2}{2}-\frac{x^3}{3}$$

7. Solve $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-6\frac{\partial^2 z}{\partial y^2}=y\cos x$

Solution:

The symbolic form is $[D^2+DD'-6D'^2]z = y\cos x$

A.E is $m^2 + m - 6 = 0$

m = -3 , 2

C.F is z = $f_1(y-3x) + f_2(y+2x)$

P.I $= \frac{1}{D^2+DD'-6D'^2}\,y\cos x$

$$= \frac{1}{(D+3D')(D-2D')}\,y\cos x$$

$$= \frac{1}{(D+3D')}\int (c-2x)\cos x\ dx$$

$$= \frac{1}{(D+3D')}\int[((c-2x)\sin x - \int -2\sin x]\ dx$$

$$= \frac{1}{(D+3D')}[(y+2x-2x)\sin x - 2\cos x]$$

$$= \frac{1}{(D+3D')}[y\sin x - 2\cos x]$$

$$= \int[(c+3x)\sin x - 2\cos x]\ dx$$

$$= (y - 3x + 3x)\cos x + 3\sin x - 2\sin x$$

$$= -y\cos x + \sin x$$

The complete solution is

$$z = f_1(y-3x) + f_2(y+2x) - y\cos x + \sin x$$