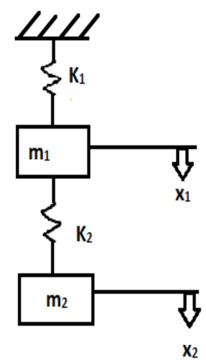
UNIT II MULTI DEGREE FREEDOM SYSTEMS:

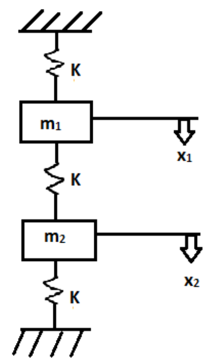
- Closed and coupled far system
- Determination of mass & stiffness matrix
- Eigen value problems
- Orthogonality of mode shapes
- Modal analysis
- The natural frequency is ωi; the components Xi =(Xi1,Xi2) are called ”normal modes”.

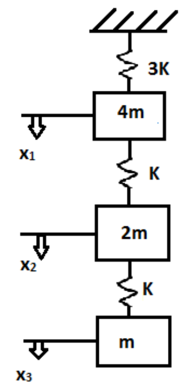
INTRODUCTION:

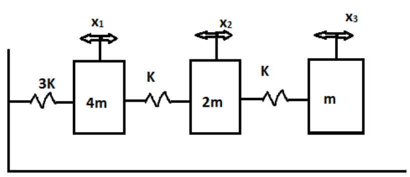
MULTIPLE DEGREES OF FREEDOM (MDOF):

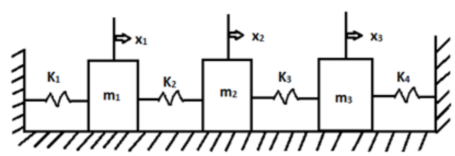
- A system having more than one degree of freedom system. Various methods are employed to determine the natural frequencies, mode shapes.











PRINCIPAL MODE OF VIBRATION OR NORMAL MODE OF VIBRATION:

- **When the** mass of a system are oscillating in such a manner that they reach maximum amplitude simultaneously and pass their equilibrium points simultaneously or all the moving parts of the system are oscillating in the same frequency and phase, such mode of vibration is called **principal mode of vibration**
- If at the principle mode of vibration, the amplitude of one of the masses is considered equal to unity, the mode of vibration is called **normal mode of vibration**
- Normal mode vibrations are free vibrations that depend only on the mass and stiffness of the system and how they are distributed
- In case of two degree of freedom system, mass will vibrate in two different modes
    - First principal mode and second principal mode

ORTHOGONALITY PRINCIPLE:

- The principal mode or normal modes of vibration for system having two or more degrees of freedom are orthogonal. This is known as **Orthogonality Principle**
- It is an important property while finding the natural frequency.
- It states that the principal nodes are orthogonal to each other. In other words, expansion theorem applied to vibration problems indicates that any general motion (x) for n masses

may be broke into components each of which corresponds to a principal node. This forms the basis of obtaining the response of vibration systems and is called modal analysis.

Types of coupling:

- n-degree of freedom system requires **n-**independent coordinates to specify the system completely at any instant.
- When these coordinates are independent of each other and equal in number of degree of freedom of the system, they are called as **generalized coordinates**
- To represent the motion of system one may use a number of generalized coordinate system. While using these coordinates the mass and stiffness matrices may be coupled or uncoupled.
- When the mass matrix is coupled, the system is said to be **dynamically coupled** and the stiffness matrix is coupled, the system is known as **statically coupled**.
- Static coupling due to static displacement. It contains terms as functions of coordinates like spring force coupling
- Dynamic coupling due to inertia force. It contains coupling terms as function of time derivatives of coordinates, like inertia coupling, damping.

PRINCIPAL COORDINATES:

An n degree freedom system requires ‘n’ independent coordinates and there will be ‘n’ number of differential equation of motion. It is always possible to fine a particular set of coordinates such that each equation of motion contains only one unknown quantity. Then the equation of motion can be solved independently and the unknown quantity can be found out. Such a particular set of coordinates is called **Principal Coordinates:**

CLOSED COUPLED SYSTEM:

In fig, an n degree translational and rotational close coupled system is given. For free vibration, the following equations can be written:

$$x=e$$
$$m_1\ddot{e}_1+K_1e_1+K_2(e_1-e_2)=0$$
$$m_1\ddot{e}_2+K_2(e_2-e_1)+K_3(e_2-e_3)=0$$

$$m_i\ddot{e}_i+K_i(e_i-e_{i-1})+K_{i+1}(e_i-e_{i+1})=0$$

We can ignore damping for free vibration, as the natural frequencies are not significantly affected by presence of damping. The above equations can also be used directly for torsion with appropriate alternation.

$$[M]\{\ddot{e}\}+[K]\{e\}=0$$

[M] is a square mass matrix
[K] is a stiffness matrix

In above equation, [K]{e} defines local force under static condition. The method of writing equation in this form called stiffness method using the displacements also called displacement method. In fact, for close coupled systems, the finite element method yields identical equations as above.

For free vibrations, the above solutions of can written as

$$\{x\}=\{X\} \cos \omega t$$

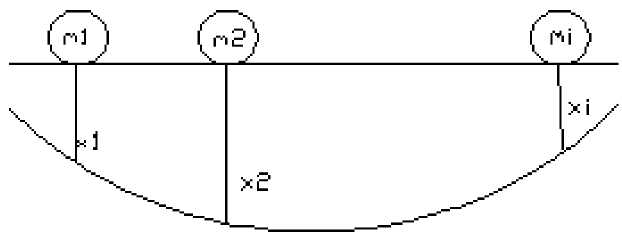
Where {X} represents the amplitudes of all the masses and ω is the natural frequency. These above equations reduce to

$$\{[K]- \omega 2[M]\}\{X\} = 0$$

The above equations known as Eigen value problem in matrix algebra and it can be solved with the aid of computer for a fairly large no of masses. ω2 is called as the characteristic value of equation.

There will be **n** such values of an **n** degree system. For each of the Eigen value, there exists a corresponding Eigen vector{x}, also called as a characteristic vector. This Eigen vector will represent the mode shape for a given frequency of the system

FAR COUPLED SYSTEM:



In fig an **n** degree far coupled system is shown with masses m1,m2,….,mi,at station 1,2,…..,i respectively. For a freely vibrating beam,the only external load is the inertia load due to masses m1,m2….. Following the influence coefficient procedure discussed two degree of freedom, we can write

$$x=e$$

$$\alpha_{11}m_1\ddot{e}_1+ \alpha_{12}m_2\ddot{e}_2+\ldots\ldots\ldots..+ \alpha_{1i}m_i\ddot{e}_i+e_1=0$$
$$\alpha_{21}m_1\ddot{e}_1+ \alpha_{22}m_2\ddot{e}_2+\ldots\ldots\ldots..+ \alpha_{2i}m_i\ddot{e}_i+e_2=0$$

$$\alpha_{i1}m_1\ddot{e}_1+ \alpha_{i2}m_2\ddot{e}_2+\ldots\ldots\ldots..+ \alpha_{ii}m_i\ddot{e}_i+e_i=0$$

In arriving at the above equations, the forces are treated as unknowns and expressed in terms of flexibility factor or influences coefficients, hence this approach is called force or flexibility method, contrary to the method adopted for close coupled systems where displacement are considered unknowns and the stiffness matrix setup.

Denoting [α] as the influences coefficient matrix, the above equation can be written as

$$[D] \{ \ddot{e}\} + [I] \{e\} = 0$$

Or

$$\{ \ddot{e}\}+ [\alpha]^{-1}\{e\} = 0$$

[D] is the dynamic matrix

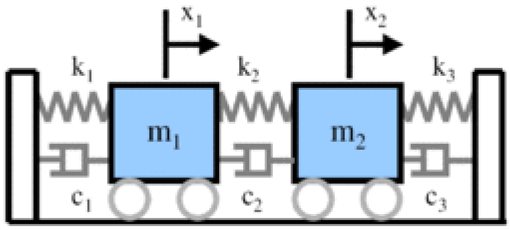
For a four degree of freedom, the natural frequencies are obtained by setting the determinant equation to zero.

$$\left[[D] - \frac{1}{p^2}[I]\right] \{X\} = 0$$

DETERMINATION OF MASS & STIFFNESS MATRIX- EIGEN VALUE PROBLEMS

Multiple degrees of freedom systems and mode shapes

- The simple mass–spring damper model is the foundation of vibration analysis, but what about more complex systems? The mass–spring–damper model described above is called a single degree of freedom (SDOF) model since the mass is assumed to only move up and down. In the case of more complex systems the system must be discretized into more masses which are allowed to move in more than one direction – adding degrees of freedom.
- The major concepts of multiple degrees of freedom (MDOF) can be understood by looking at just a 2 degree of freedom model as shown in the figure.



The equations of motion of the 2DOF system are found to be:

$$m_1\ddot{x_1} + (c_1 + c_2)\dot{x_1} - c_2\dot{x_2} + (k_1 + k_2)x_1 - k_2x_2 = f_1,$$
$$m_2\ddot{x_2} - c_2\dot{x_1} + (c_2 + c_3)\dot{x_2} - k_2x_1 + (k_2 + k_3)x_2 = f_2.$$

This can be rewritten in matrix format

:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x_1} \\ \ddot{x_2} \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{Bmatrix} \dot{x_1} \\ \dot{x_2} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}.$$

A more compact form of this matrix equation can be written as:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f\}$$

where $[M]$, $[C]$, and $[K]$ are symmetric matrices referred respectively as the mass, damping, and stiffness matrices. The matrices are NxN square matrices where N is the number of degrees of freedom of the system.

In the following analysis involves the case where there is no damping and no applied forces (i.e. free vibration). The solution of a viscously damped system is somewhat more complicated.[7]

$$[M]\{\ddot{x}\} + [K]\{x\} = 0.$$

This differential equation can be solved by assuming the following type of solution:

$$\{x\} = \{X\}e^{i\omega t}.$$

Note: Using the exponential solution of $\{X\}e^{i\omega t}$ is a mathematical trick used to solve linear differential equations. Using Euler's formula and taking only the real part of the solution it is the same cosine solution for the 1 DOF system. The exponential solution is only used because it is easier to manipulate mathematically.

The equation then becomes:

$$\left[-\omega^2[M]+[K]\right]\{X\}e^{i\omega t}=0.$$

Since $e^{i\omega t}$ cannot equal zero the equation reduces to the following.

$$\left[[K]-\omega^2[M]\right]\{X\}=0.$$

EIGENVALUE PROBLEM

This is referred to an eigenvalue problem in mathematics and can be put in the standard format by pre-multiplying the equation by $[M]^{-1}$

$$\left[[M]^{-1}[K]-\omega^2[M]^{-1}[M]\right]\{X\}=0$$

and if: $[M]^{-1}[K]=[A]$ and $\lambda=\omega^2$

$$\left[[A]-\lambda[I]\right]\{X\}=0.$$

- The solution to the problem results in N **eigenvalues** (i.e. $\omega_1^2,\omega_2^2,\cdots\omega_N^2$), where N corresponds to the number of degrees of freedom.
- The eigenvalues provide the natural frequencies of the system. When these eigenvalues are substituted back into the original set of equations, the values of $\{X\}$ that correspond to each eigenvalue are called the **Eigenvectors**.
- These eigenvectors represent the mode shapes of the system. The solution of an eigenvalue problem can be quite cumbersome (especially for problems with many degrees of freedom), but fortunately most math analysis programs have eigenvalue routines.

The eigenvalues and eigenvectors are often written in the following matrix format and describe the model of the system:

$$\left[\diagdown\omega_r^2\diagdown\right]=\begin{bmatrix}\omega_1^2 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \omega_N^2\end{bmatrix}$$ and $$[\Psi]=\left[\{\psi_1\}\{\psi_2\}\cdots\{\psi_N\}\right].$$

Eigen value problem:

ORTHOGONALITY OF MODE SHAPES:

The mode shapes of a dynamic system exhibit orthogonality property, which is very useful in simplifying the analysis for forced and transient vibrations.

In general, the differential equation of motion for any vibrating system can be written in matrix form as follow $[M]\{\ddot{x}\}+[K]\{x\}=0.$

Let [M] and [K] represent the mass and stiffness properties of a system,

Assuming harmonic motion i.e

$$x=X\sin\omega t$$

$\lambda = \omega^2$

The equation for the *i*th mode be

$$KX_i = \lambda_i MX_i$$

Premultiplying by the transpose of mode j,

$$X_j{}'KX_i = X_j{}'\lambda_i MX_i = \lambda_i (X_j MX_i) \tag{1}$$

Now start with the equation for the jth mode and premultiplying by $X_i{}'$ to obtain,

$$X_i{}'KX_j = \lambda_i (X_i{}'MX_j) \tag{2}$$

Since *K* and *M* are symmetric matrices

$$X_j{}'MX_i = X_i{}'MX_j \text{ and } X_j{}'KX_i = X_i{}'KX_j$$

Subtracting (2) from (1)

$$(\lambda_i - \lambda_j)X_2{}'MX = 0$$

As $\lambda_i \neq \lambda_j$

$$X_2{}'MX_j = 0 \text{ and } X_2{}'KX_j = 0$$

The above equation shows the orthogonal character of the normal modes.

If i=j,

$$X_2{}'MX_i = M_i$$

$$X_2KX_i = K_i$$

M*i* and Ki are known as the generalized mass and generalized stiffness of the ith mode

Modal analysis:

- An engineering system, when given an initial disturbance and allowed to execute free vibrations without a subsequent forcing excitation, will tend to do so at a particular "preferred" frequency and maintaining a particular "preferred" geometric shape.
- This frequency is termed a "natural frequency" of the system, and the corresponding shape (or motion ratio) of the moving parts of the system is termed a "mode shape." Any arbitrary motion of a vibrating system can be represented in terms of its natural frequencies and mode shapes.
- The subject of modal analysis primarily concerns determination of natural frequencies and mode shapes of a dynamic system.
- Once the modes are determined, they can be used in understanding the dynamic nature of the systems, and also in design and control.
- Modal analysis is extremely important in vibration engineering. Natural frequencies and mode shapes of a vibrating system can be determined experimentally through procedures of modal testing.
- The subject of modal testing, experimental modeling (or model identification), and associated analysis and design is known as *experimental modal analysis*