UNIT 2: HEAT TRANSFER

CONTENTS

2.1 Introduction to Heat Transfer

In heat transfer, we deal with transfer of thermal energy or heat which takes place between different bodies/fluids. Here, we start with Axiom-4 of transport phenomena.

This axiom is similar to the first law of thermodynamics. It states that "energy is conserved" which means energy can not be created or destroyed. Energy may be transferred from one form to another or one place to another. Transfer of energy in a system depends how it interacts with the surrounding. Here, the system is defined as the region of an equipment / unit which is under investigation. The remainder of everything else is called the surrounding which is outside the boundaries of the system. The system may be classified in three types, based on how the system is interacting with the surrounding in terms of heat, work, and mass exchange.

(1) Isolated system

Here, the system can not exchange either heat, work or mass with the surrounding. Therefore, the total energy of an isolated system does not be change or $\Delta E = E_1 - E_2 = 0$ where ΔE is the change in total energy of the system at two different states 1 and 2.

(2) Closed system

Here, the system can not exchange mass with the surrounding but heat and work may be exchanged. Therefore, the change in total energy of a closed system within two different states can be calculated as $\Delta E = \Delta Q + \Delta W$ where, ΔE is the change in energy of the system, ΔQ is the heat added to the system, and ΔW is the work done on the system by the surrounding. The change in total energy of a system, ΔE equals to the summation of changes in potential, kinetic, and internal energies of the system. However, the change in potential and kinetic energies of the system are usually negligible and thus, the total energy E changes only due to the change in internal energy, U. Therefore, for a closed system, we may write,
$\Delta U = \Delta Q + \Delta W$

(3) Open system

In an open system all three mass, heat, and work may be exchanged with the surrounding.

Therefore, the change in total energy of an open system may be calculated as,

$$\Delta E = \Delta Q + \Delta W + \begin{pmatrix} \textit{Addition or removal of} \\ \textit{energy due to net inflow} \\ \textit{of mass in the system} \end{pmatrix}$$

the following manner.

$$\begin{pmatrix}\text{Rate of accumulation}\\ \text{of energy in the system}\end{pmatrix} = \begin{pmatrix}\text{Net rate of inflow}\\ \text{of energy by convection}\end{pmatrix} + \begin{pmatrix}\text{Net rate of heat}\\ \text{addtion by conduction}\end{pmatrix}$$

$$\begin{pmatrix}\text{Net rate of work}\\ \text{done on the system}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation / consumption}\\ \text{by the heat source or sink}\end{pmatrix}$$

$$+\begin{pmatrix}\text{Heat loss or gain}\\ \text{by radiation}\end{pmatrix}$$

The above equation takes into account the heat transfer by three different modes as shown below.

- **Conduction:** In heat transfer, heat conduction is the transfer of heat from higher temperature region to lower temperature region due to temperature gradients.
- **Convection:** The energy transfer may also occur due to the transport of material from the boundaries of the system.
- **Radiation:** This term implies transfer of heat energy due to electromagnetic waves under certain range of wavelength. Radiation does not require a material medium for energy transport like in conduction and convection. Unless the temperature is high, the heat addition by radiation may be neglected.

While studying the subject of heat transfer, the main objective is to find the rate of heat transfer from a body or a system. Fourier's law of heat conduction provides the relation between the rate of heat transfer and temperature gradients.

Fourier's law of heat conduction

When a temperature gradient exists in a body, experience has shown that the heat is transferred from higher temperatures to lower temperatures. Consider a solid block of surface area *A*, which is located between two parallel planes, set a distance *H* apart as shown in Fig. (24.1). Initially, for $t<0$, the solid block is maintained at a homogeneous temperature T_1 throughout. After some time $t=0$, lower plane is suddenly brought to a higher temperature T_2 and maintained at that temperature for $t>0$. Once the steady state is achieved, it is found that a constant heat flux in *x* direction is required to maintain the temperature difference, Q/A is required to maintain the constant temperature difference (T_2-T_1) across the solid block.

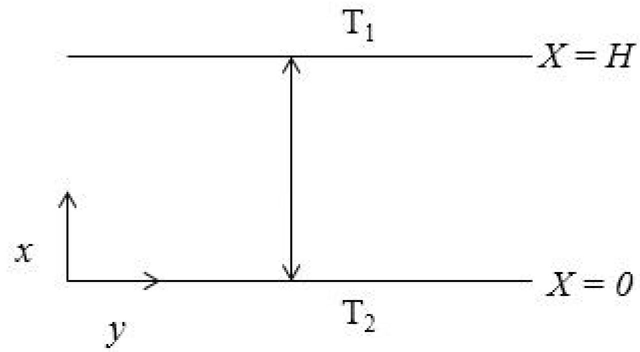

Fig 24.1 Flow of heat between two parallel plates

Repetition of the above experiment with different temperature differences $\Delta T=(T_2-T_1)$ shows that the heat flux is proportional to ΔT/H as shown in Fig. (24.2).

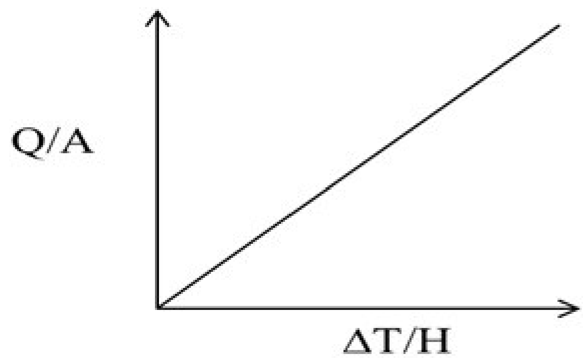

Fig 24.2 Heat flus vs. temperature gradient

This implies that

$$q_x = \frac{Q}{A} \propto \frac{\Delta T}{H}$$

or

$$q_x = -k\frac{dT}{dx}$$

where *k* is called thermal conductivity. Negative sign indicates that the heat flows from higher temperatures to lower temperatures.

Unit of thermal conductivity is

$$k = \frac{[q_x]}{\left[\frac{dT}{dx}\right]} = \frac{cal/cm^2\,sec}{{}^\circ c/cm} = \frac{cal}{cm - sec\,{}^\circ c}$$

The Equation (24.2) is called the Fourier's law of heat conduction. By extending this equation in three dimensions, we obtain

$$q_x = -k\frac{dT}{dx}$$

ector form as

$$q_y = -k\frac{dT}{dy}$$

the molecular heat transport or conduction for an isotropic body and ystem. The detail forms of Fourier's law in all coordinate systems are

$$q_z = -k\frac{dT}{dz}$$

ome simple heat transfer problems due to conduction by using shell energy balance.

2.2 Heat conduction through a composite wall

Consider a composite wall of height L , width W and thickness $\delta_1 + \delta_2$. The wall contains two layers of different materials which have the thermal conductivity K0 and K1, and different thickness δ_1 and δ_2 respectively. At x=0, the composite wall is maintained at a constant temperature T0, while at $x = \delta_1 + \delta_2$, it has a constant temperature T2 as shown in Fig. (25.1).

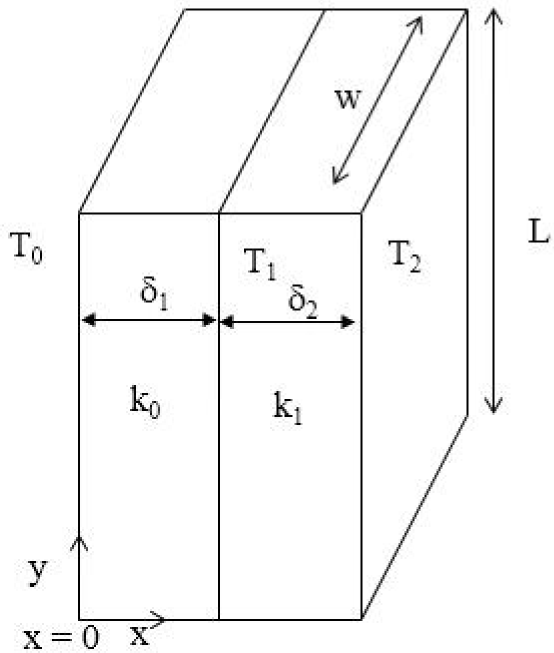

Fig 25.1 Heat conduction through a composite wall

Assumptions

- System is in steady state.
- Thermal conductivity for both walls, K0 , and K1 are constants.
- System follows Fourier's law of heat conduction.
- Heat loss from side walls in direction of y and z are negligible.

Non-zero components of heat flux and the control volume

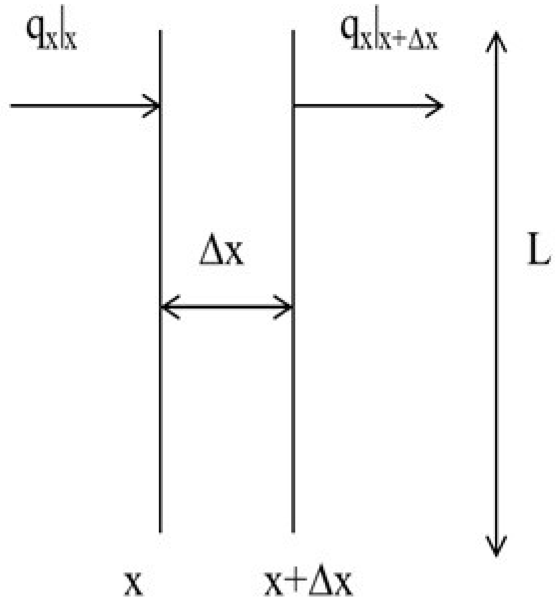

Fig 2 Control volume for heat flow through a composite wall

Since temperature is changing in x direction only, the control volume is chosen such that it has differential thickness in x direction as shown in Fig. (25.2) and qx is the only component of heat flux.

Energy balance

Heat flux entering into the control volume at x = x is

$$q_x HW \big|_x \qquad ($$

Heat flux leaving from the control volume at x = x + dx is p

$$q_x HW \big|_{x+\Delta x} \qquad ($$

Any source or sink of heat is not present in the control volume and work done on the system is zero. The thermal energy balance for this control volume may be written as

$$0 = \left(q_x HW\right)\Big|_{x=x} - \left(q_x HW\right)\Big|_{x+\Delta x}$$

After dividing by LWΔx and taking the limit Δx to zero, we get

$$\frac{dq_x}{dx} = 0$$

Integration of this equation gives

$$q_x = c_1$$

Here C1 is the integration constant. Equation (25.5) implies that heat flux is constant throughout the composite wall.
By applying the Fourier's law of heat conduction, we get

$$q_x = \frac{-kdT}{dx} = c_1$$

Equation (25.6) may also be written as

$$\frac{dT}{dx} = \frac{-c_1}{k}$$

Now, the problem may be solved for both layers of composite wall separately.
Layer 1: $0 < x < \delta_1$
Here, thermal conductivity is k0 Therefore, Equation (25.7) may be changed to

$$\frac{dT}{dx} = \frac{-c_1}{k_0}$$

At x=0, the temperature of the composite wall is given as T=T0. Also the temperature of the first at can be assumed as T=T11. By integration of Equation (25.8) and substituting the boundary conditions, we obtain

$$T_{11} - T_0 = -\frac{c_1\delta_1}{k_0}$$

Layer 2: $\delta_1 < x < \delta_2$

Similar to solution for layer one, solution for layer 2 may also be found and given below

$$T_2 - T_{12} = \frac{-c_1(\delta_2 - \delta_1)}{k_1}$$

Here, T12 is the assumed temperature of second layer of

$$x = \delta_1$$ **(25.1**

It may be noted that at the interface of the two layers at X=δ1, the heat fluxes are same for both layers. Thus, equating the heat fluxes we find the integration constant c1 is same for both layers. Also, thermal equilibrium may assume at interface and therefore,

$$T_{11} = T_{12}$$

Using above boundary condition in Equation (25.9) and (25.10) and adding, we obtain

$$T_2 - T_0 = -c_1 \left\{ \frac{\delta_1}{k_0} + \frac{\delta_2 - \delta_1}{k_1} \right\}$$

or

$$c_1 = q_x = \frac{T_0 - T_2}{\left\{ \frac{\delta_1}{k_0} + \frac{\delta_2 - \delta_1}{k_1} \right\}}$$

The above equation (25.14) provides resulting heat flow per unit area of the composite wall. In the last example, we had solved a heat transfer problem which involved cartesian coordinates. To understand the formulation of problem in other coordinate systems two more example are considered here. One in cylindrical coordinate system and second in spherical coordinate system.

2.3 Heat transfer in a cylindrical shell

Consider a long cylindrical shell of inner radius R1, outer radius R2, and length L shown in Fig. 26.1. The inner wall of cylindrical shell is maintained at constant temperature T1 and outer wall is maintained at constant temperature T2. Calculate the heat transfer rate in radial direction from the cylindrical shell.

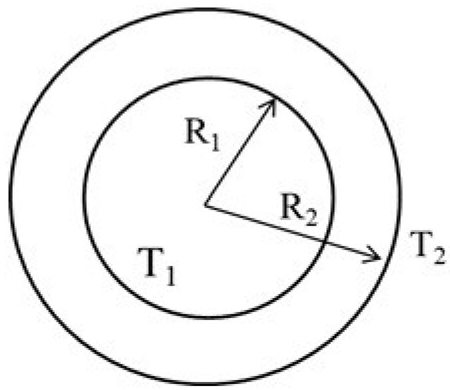

Fig 26.1 Heat transfer in cylindrical shell

Assumptions

- System is in steady state.
- Thermal conductivity, k, is constant.
- System follows Fourier's law of heat conduction.
- Heat loss in axial direction is negligible.

Non-zero heat flux component

Since temperature is changing in r direction only, qr is present. Now, consider a control volume of differential thickness Δr as shown below

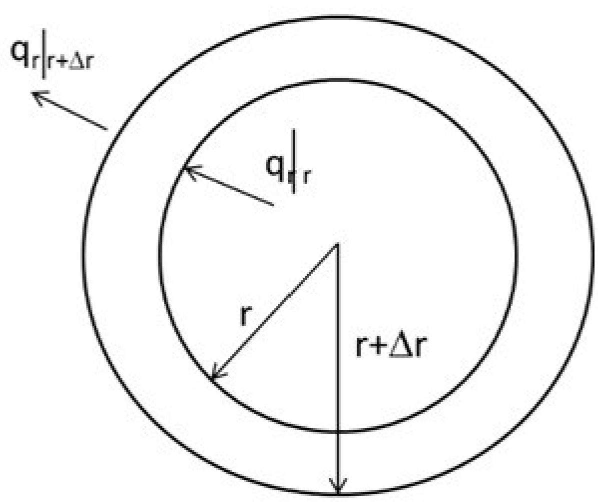

Fig 26.2 Control volume for heat transfer in cylindrical shell

Shell energy balance across the control volume

Heat flux entering the control volume at r=r is

$$2\pi r L q_r \big|_r \qquad (26.1)$$

Heat flux leaving from the control volume at r= r+ Δr is

$$2\pi r L q_r \big|_{r+\Delta r} \qquad (26.2)$$

No source or sink of heat is present in the control volume and work done on the system is zero. Thus, the thermal energy balance is reduced to

$$0 = q_r 2\pi r L\big|_r - q_r 2\pi r L\big|_{r+\Delta r}$$

By dividing Equation (26.3) to the volume of control volume 2πrLΔr and taking the limit Δr going to zero, we obtain

$$\frac{d(rq_r)}{dr} = 0$$

By integrating Equation (26.4), we get

$$q_r = \frac{c_1}{r}$$

where c1 is a integration constant.
Substituting Fourier's law of heat conduction in Equation (26.5) , we obtain

$$\frac{dT}{dr} = -\frac{c_1}{-kr}$$

or

$$T = -c_1 \ln r + c_2$$

Here, c2 is the constant of integrations.
Boundary conditions are

at $r = R_1$, $T = T_1$

(26.7)

and

at $r = R_2$, $T = T_2$ (26.8)

This leads to the solution

$$T_2 - T_1 = -c_1 \frac{ln(R_2 / R_1)}{k}$$

$$c_1 = \frac{k\left(T_1 - T_2\right)}{ln\left(R_2 / R_1\right)}$$

Substituting the value of c1 in Equation (26.5), we finally obtain

$$q_r = \frac{k\left(T_1 - T_2\right)}{r\,ln\left(R_2 / R_1\right)}$$

The rate of heat transfer through cylindrical shell may be calculated as shown below,

$$Q_0 = \left(2\pi Lr \times q_r\right)\Big|_{r=R_1 \; or \; r=R_2} = \frac{2\pi Lk\left(T_1 - T_2\right)}{ln\left(R_2 / R_1\right)}$$

2.4 Heat transfer in a spherical shell

Consider a spherical shell of inner radius R1 and outer radius R2, whose inside and outside surfaces are maintained at the constant temperatures T1 and T2 respectively as shown in Fig. 26.3. Calculate the heat flux from the spherical shell

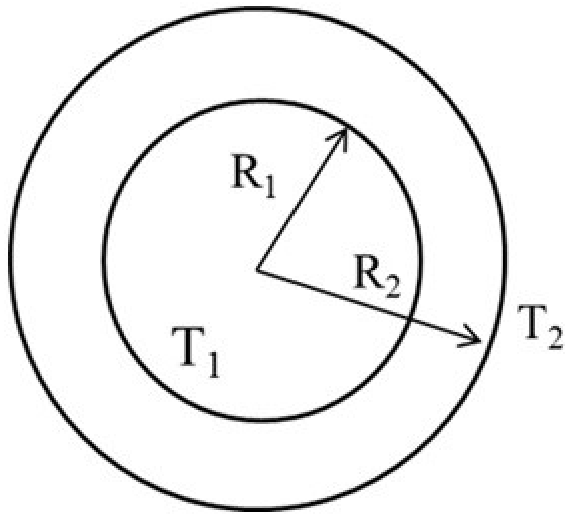

Fig Heat transfer in hollow sphere

Assumptions

- System is in steady state.
- Thermal conductivity, k, is constant.
- System follows Fourier's law of heat conduction.

Non-zero heat flux component

Since temperature changing in r direction, only qr is present. The control volume may be drawn of differential thickness δr as shown in Fig (26.4).

Shell energy balance across the control volume

Heat flux entering control volume at r= r is $4\pi r^2 q_r \big|_r$ (26.12)

Heat flux leaving control volume at r= r+ Δr is $4\pi r^2 q_r \big|_{r+\Delta r}$ (26.13)

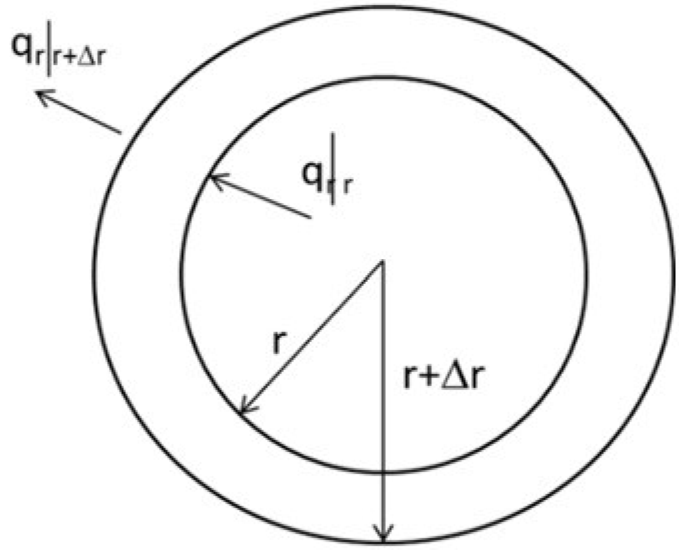

Fig 26.4 Control volume for heat transfer in hollow sphere

Any heat source or sink is not present in the control volume and work done on the system is zero. Thus, the thermal energy balance is reduced to

$$0=\left(q_r 4\pi r^2\right)\Big|_r-\left(q_r 4\pi r^2\right)\Big|_{r+dr}$$

Dividing Equation (26.14) by volume of control volume $4\pi r^2 \Delta r$ and taking the limit $\Delta r \rightarrow 0$, we get

$$\frac{d}{dr}(r^2 q_r) = 0$$

and integrating Equation (26.15), we find

$$q_r = \frac{c_1}{r^2}$$

where, c1 is an integration constant.

By substituting Fourier’s law of heat conduction in Equation (26.16) and integrating, we obtain

$$T = \frac{c_1}{kr} + c_2$$

where c2 is another integration constant.

The Equation (26.17) is subjected to the boundary conditions,

at $r = R_1$, $T = T_1$ (26.18)

and

at $r = R_2$, $T = T_2$ (26.19)

Using above boundary conditions and evaluating the constants of integration c1 and c2, we finally obtain heat flux through spherical shell as given below,

$$q_r = \frac{1}{r^2}\frac{k\left(T_1 - T_2\right)}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

2.5 Heat transfer from a cylindrical composite wall : Use of heat transfer coefficients

Consider a cylindrical composite wall whose inner surface is exposed to a fluid at constant temperature Tb and the outer surface is exposed to atmosphere at a temperature Ta. Ro is the inner radius of cylinder while outer radius is R1. The cylinder is insulated and radius of insulation changes from R1 to R2 as shown in Fig. (27.1). The inside and out side heat transfer coefficients are hb and ha respectively. K01 and K12 are the thermal conductivity of cylinder material and insulation respectively. Calculate the overall heat loss through the cylindrical wall.

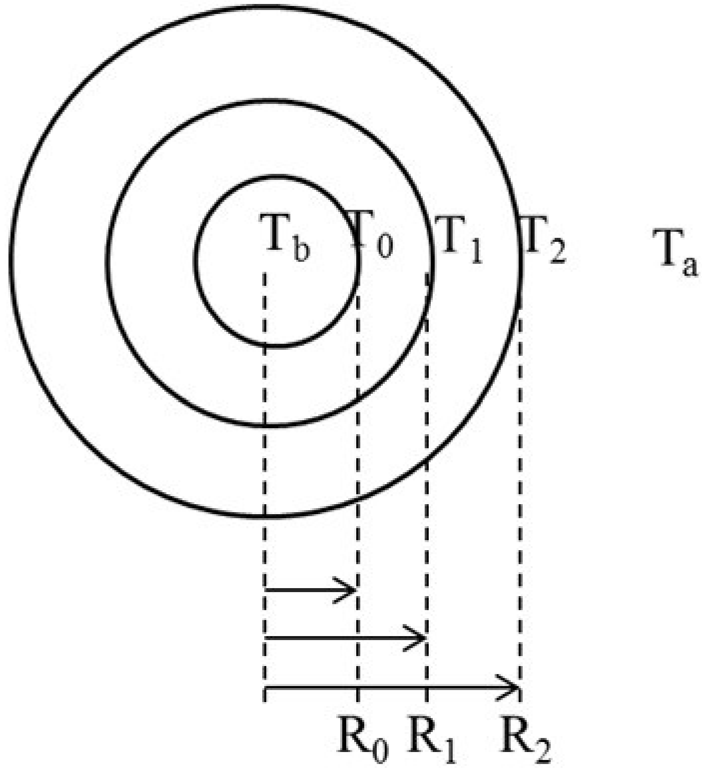

Fig 1 Heat transfer in cylindrical composite wall

Assumptions

- System is in steady state.
- Thermal conductivities K01and K12 are constants.
- System follows Fourier's law of heat conduction.
- Heat loss in axial directions are negligible.

Heat flux component

Here, the temperature is changing in the radial direction, and therefore, T=T(r) and qr is the only non-zero heat flux. As we have seen in the previous problems, the energy balance for cylinder material and insulation may be written as

$$rq_r = c_1 = constant$$

where, c1= R0q0 = R1 q1= R2q2 where q0, q1, and q2 are the heat fluxes at R0, R1 and R2 respectively. Substituting Fourier's law of heat conduction, we obtain,

$$-kr\frac{dT}{dr} = c_1$$

Assume the unknown temperatures are T0, T1 and T2 respectively as shown in Fig. (27.1). To solve Equation (27.2) for metal wall as well as for insulation material, it may be noted that the metal wall subjected to the following boundary conditions.

at $r = R_0$, $T = T_0$ (27.3)

and

at $r = R_1$, $T = T_1$ (27.4)

Similarly for insulation, we have the boundary condition (27.4) and second boundary condition is given below

$$r = R_2, T = T_2$$

The final solution in for Equation (27.2) is given below

$$q_0 R_0 = \frac{k_{01}(T_0 - T_1)}{ln(R_1/R_0)}$$

or

$$(T_0 - T_1) = q_0 R_0 \frac{ln(R_1/R_0)}{k_{01}}$$

or

and similarly

$$(T_1 - T_2) = q_0 R_0 \frac{ln(R_2/R_1)}{k_{12}}$$ (

Heat transfer coefficient

For complete solution of this problem, we need to solve problems of heat transfer in fluid inside the cylindrical tube as well as in the atmosphere outside the insulation. However, if we know the heat transfer coefficients, we may avoid finding these solutions. Recall the Newton's law of cooling which states the rate of heat transfer from a body is proportional to the difference in temperature between the body and its surrounding where q= AδT where A is the constant of proportionality. The heat transfer coefficients are similar to the coefficient A in the Newton's law of cooling, using these coefficients

$$q_0 = h_a\left(T_a - T_0\right)$$

or

$$\left(T_a - T_0\right) = \frac{q_0}{h_a} = \frac{R_0 q_0}{h_a R_0}$$

and for insulation, we have

$$q_2 = h_b\left(T_2 - T_b\right)$$

or

$$\left(T_2 - T_b\right) = \frac{q_2}{h_b} = \frac{R_2 q_2}{R_2 h_b}$$

By adding Equations (27.7), (27.8),(27.10) and (27.11) and, noting Equation (27.1), we get

$$\left(T_a - T_b\right) = q_0 R_0 \left\{ \frac{1}{h_b R_2} + \frac{ln(R_1/R_0)}{k_{01}} + \frac{ln(R_2/R_1)}{k_{12}} + \frac{1}{R_0 h_a} \right\}$$

or

$$q_0 = \frac{\left(T_a - T_b\right)}{R_0 \left\{ \frac{1}{h_b R_2} + \frac{ln(R_1/R_0)}{k_{01}} + \frac{ln(R_2/R_1)}{k_{12}} + \frac{1}{h_a R_0} \right\}}$$

Heat conduction with a heat source

A cylindrical rod of radius R0 and length L is producing heat which is equal to Sc per unit time per unit volume, may be due to conversion of electrical energy into heat. The surface of the cylinder is maintained at temperature T0. Determine the temperature profile.

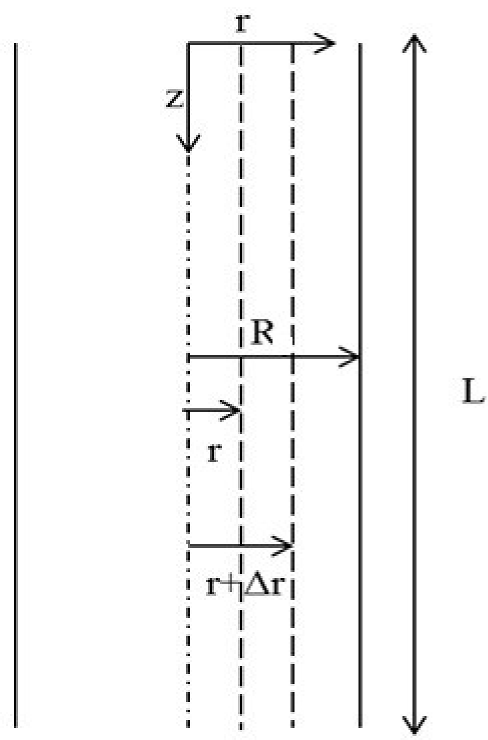

Fig 27.2 Heat transfer in cylindrical shell

Assumptions

- System is in steady state.
- Thermal conductivity of metal is constant.
- System follow Fourier’s law of thermal heat conduction.
- Heat loss in axial directions are negligible.

Once again, temperature is changing in r direction only and qr is the only non-zero component of heat flux. Writing the following terms required for energy balance for a control volume shown in Fig. 27.2, we obtain the following terms

Heat entering the control volume is $2\pi rLq_r\big|_r$

(27.13)

Heat leaving control volume is $2\pi rLq_r\big|_{r+\Delta r}$

(27.14)

the heat produced in metal rod is given by

$$Sc\times 2\pi r\Delta r\, L$$

Writing the energy balance, we find

$$2\pi rLq_r\big|_r - 2\pi rLq_r\big|_{r+\Delta r} + Sc2\pi r\Delta r\, L = 0$$

which leads to the following differential equation

$$\frac{1}{r}\frac{d}{dr}\left(rq_r\right) = Sc$$

Integrating Equation (27.17), we finally obtain

$$q_r = \frac{rSc}{2} + \frac{c_1}{r}$$

To evaluate the constant of integration c1, we may apply the boundary condition that at r=0, qr is finite; Therefore,

c1 =0 (27.19)

or

$$q_r = \frac{rSc}{2}$$

Applying Fourier’s law of heat conduction, we have

$$-k\frac{dT}{dr} = \frac{rSc}{2}$$

Thus, after integration, we obtain

$$T = -\frac{Scr^2}{4k} + c_2$$

The second boundary condition for this problem is that

at

$$r = R, T = T_0$$

or

$$c_2 = T_0 + \frac{ScR^2}{4k}$$

Thus, the temperature profile may be determined as below

$$(T - T_0) = \frac{ScR^2}{4k}\left\{1 - \left(\frac{r}{R}\right)^2\right\}$$

2.6 Critical radius of insulation

Consider a cylindrical rod which is insulated by an insulation material as shown in Fig. (27.3). The radius of the rod is R0 and rod is maintained at temperature T0. The insulated rod is surrounded by a medium at temperature Ta. The out-side heat transfer coefficient is ha. Determine the critical radius of insulation at which the heat loss is maximum.

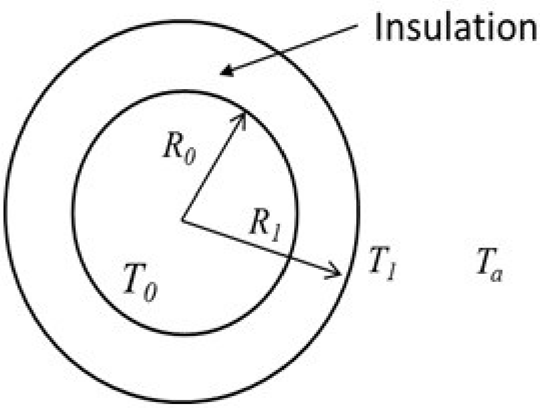

Fig 27.3Insulated cylindrical pipe

Once again, as shown earlier, the energy balance for cylindrical shell in insulation material leads to the following differential equation.

$$\frac{1}{r}\frac{d}{dr}\left(rq_r\right)=0$$

Integrating Equation (27.25) and applying Fourier's law of heat conduction, we obtain

$$q_r=\frac{c_1}{r}=-k_1\frac{dT}{dr}$$

The above equation may be integrated subject to the following boundary conditions

at $r=R_0$, $T=T_0$ (27.27)

and

at $r=R_1$, $T=T_1$ (27.28)

Also, since the out side heat transfer coefficient is given, we have

$$q_r\big|_{r=R_1}=h_a\left(T_1-T_a\right) \quad \textbf{(27.}$$

The solution of this problem is similar to the problem solved earlier for composite cylindrical shell. Thus, we finally obtain

$$q_0 = \frac{(T_0 - T_a)}{R_0\left\{\frac{1}{h_a R_1} + \frac{ln(R_1/R_0)}{k_I}\right\}}$$

In Equation (27.28), we may define the overall heat transfer resistance as

$$U = R_0\left\{\frac{1}{h_a R_1} + \frac{ln(R_1/R_0)}{k_I}\right\}$$

In Equation (27.30), the first term in right hand side represents the convective heat transfer resistance and the second term represents the conductive heat transfer resistance. It can be easily observed that convective resistance decreases and conductive resistance increases as we increase the thickness of the insulation, i.e., R1 thus, over all heat loss may initially increase and then decreases as shown in Fig. 27.8. The value of R1 where the heat loss is maximum or the overall heat transfer resistance is minimum, is called the critical radius of insulation.

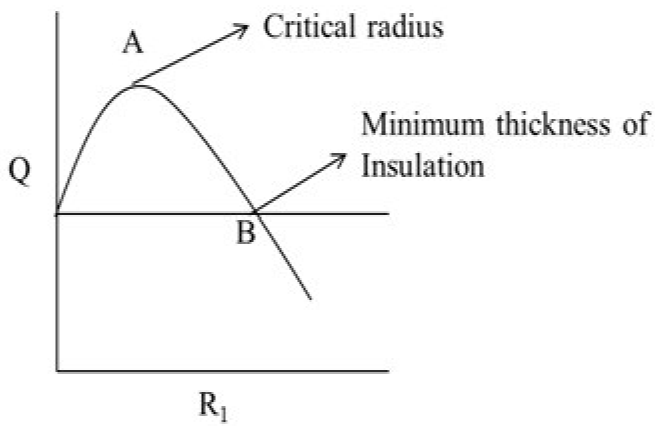

Fig 27.3 Thickness of insulation Vs. heat flux

Since, the overall heat transfer resistance should be minimum at critical radius, we have

$$\left.\frac{dU}{dR_1}\right|_{Critical\ radius} = 0 \qquad ($$

or

$$\frac{1}{k_I R_I} - \frac{1}{R_I^2 h_a} = 0 \qquad (27.33)$$

which provides the critical radius of insulation

$$R_I\big|_{Critical\ radius} = R_c = \frac{k_I}{h_a} \qquad (27$$

If the overall resistance U is differentiated twice with respect to the radius of insulation R1, we obtain

$$\frac{d^2U}{dR^2{}_I} = -\frac{1}{k_I R_I{}^2} + \frac{2}{R_I{}^3 h_a}$$

At critical radius of insulation the value of the right hand side in Equation of (27.35) is positive as shown below

$$\frac{d^2U}{dR_I^2} = \frac{-h_a^2}{k^2} + \frac{2h_a^2}{h_a k^3} = \frac{h_b^2}{k^3} > 0$$

Since, the value of second derivative is always positive, verifies that U is minimum at critical radius of insulation. Thus, the heat loss is maximum at critical radius of insulation given by Equation (27.33).

2.7 Derivation of equation of energy

In this section, we derive the equation of energy by using Axiom-4, which states that energy is conserved. The equation of total energy may be further divided into two parts. First is the equation of mechanical energy and second is the equation of thermal energy. The equation of mechanical energy is derived from equation of motion. The equation of thermal energy is derived by subtracting the equation of mechanical energy from the equation of total energy. Later, the equation of thermal energy is modified in temperature explicit form, which may be used for obtaining temperature profile

Consider a stationary control volume of dimension Δx, Δy and Δz. The fluid is flowing with a velocity $\underset{\sim}{v}$, which has components vx, vy and vz in x, y and z directions respectively, as shown in the Fig.(28.1).

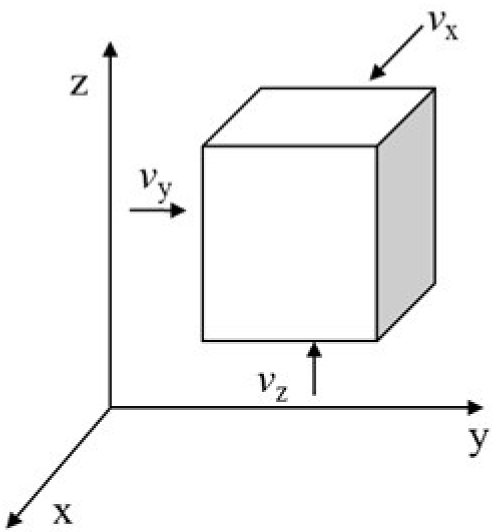

Fig 28.1 Cubical control volume

The total energy consists of potential energy $\hat{G}$, Internal energy $\hat{U}$ and kinetic energy $\frac{1}{2}v^2$. Since the control volume is fixed in the space, the change in potential energy is negligible. Therefore, the energy balance may be written as .

$$\begin{pmatrix}\text{Rate of accumulation of internal}\\ \text{\& kinetic energy in control volume}\end{pmatrix} = \begin{pmatrix}\text{Rate of net change of internal}\\ \text{\& kinetic energy by convection}\end{pmatrix} + \begin{pmatrix}\text{N}\\ \text{ad}\end{pmatrix}$$

$$+\begin{pmatrix}\text{Work done on the system against vario}\\ \text{forces(Pressure, Gravity and Shear)}\end{pmatrix}$$

$$+\begin{pmatrix}\text{Rate of heat addition}\\ \text{by some heat sources}\end{pmatrix}$$

Now, we take each term in equation (28.1) and write them separately as given below.

(1) Rate of accumulation of internal and kinetic energy

The rate of accumulation of internal and kinetic energy in the control may be written as

$$\frac{\partial}{\partial t}\left[(\rho\Delta x\Delta y\Delta z)\left(\frac{1}{2}v^2+\hat{U}\right)\right]$$

where, $\hat{U}$ is the internal energy per unit mass of the system.

(2) Rate of net change of kinetic and internal energy by convection

Net inflow of kinetic and internal energy by convection may be written as

$$\left[(\rho v_x \Delta y \Delta z)\left(\frac{1}{2}v^2+\hat{U}\right)\right]_x-\left[(\rho v_x \Delta y \Delta z)\left(\frac{1}{2}v^2+\hat{U}\right)\right]_{x+\Delta x}$$

$$+\left[(\rho v_y \Delta x \Delta z)\left(\frac{1}{2}v^2+\hat{U}\right)\right]_y-\left[(\rho v_y \Delta x \Delta z)\left(\frac{1}{2}v^2+\hat{U}\right)\right]_{y+\Delta y}$$

$$+\left[(\rho v_z \Delta y \Delta x)\left(\frac{1}{2}v^2+\hat{U}\right)\right]_z-\left[(\rho v_z \Delta x \Delta y)\left(\frac{1}{2}v^2+\hat{U}\right)\right]_{z+\Delta z}$$

(3) Rate of heat addition by conduction

As shown in Fig. (28.2), heat flux, $\underset{\sim}{q}$, has three components qx, qy and qz respectively. Therefore, the net heat addition by conduction is given below

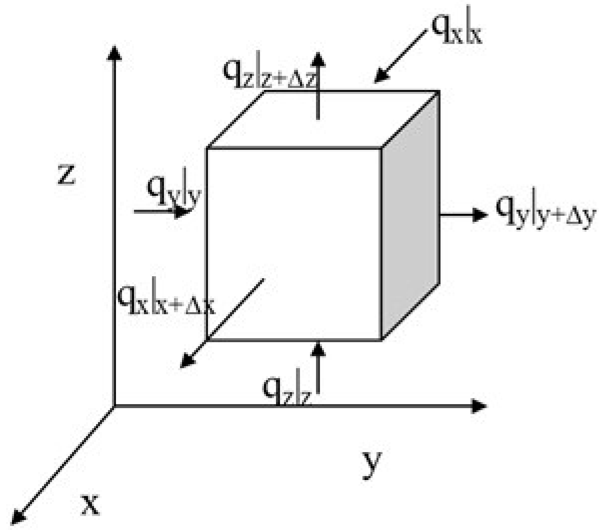

Fig 28.2 Cubical control volume with heat transfer

$$\left(q_x \Delta y \Delta z\right)\big|_x - \left(q_x \Delta y \Delta z\right)\big|_{x+\Delta x} + \left(q_y \Delta x \Delta z\right)\big|_y - \left(q_y \Delta x \Delta z\right)\big|_{y+\Delta y} + \left(q_z \Delta x \Delta y\right)\big|_z - \left(q_z \Delta x \Delta y\right)\big|_{z+\Delta z}$$

(4) Rate work done on the system

Work done on the system is defined as a scalar product of force and displacement vectors. If $\underset{\sim}{F}$ is a force and $\partial \underset{\sim}{l}$ is the displacement then work done delta w, is

$$\partial w = \underset{\sim}{F} \, . \, \partial \underset{\sim}{l}$$

or rate of work done is given by

$$\frac{\partial w}{\partial t} = \underset{\sim}{F} \cdot \frac{\partial \underset{\sim}{l}}{\partial t}$$

or

$$= \underset{\sim}{F} . \underset{\sim}{v}$$

or

$$= F_x v_x + F_y v_y + F_z v_z$$

Now, consider the forces acting on a fluid element. These forces are

- Gravity force
- Pressure force
- Shear force

Therefore, we need to consider work done by these force on the control volume separately.

1. Rate of work done against Gravity forces

By following Equation (28.6), the work done against gravity force may be written as

$$= \rho\Delta x\Delta y\Delta z g_x v_x + \rho\Delta x\Delta y\Delta z g_y v_y + \rho\Delta x\Delta y\Delta z g_z v_z$$

2. Rate of work done by pressure forces

The pressure forces always act in the opposite direction to the outer normal of a plane. It is a compressible force. Therefore, the work done on the control volume by pressure forces may be calculated as follows
Rate of work done by x directed pressure forces:

$$= \left(v_x P\right)\big|_x \Delta y\Delta z - \left(v_x P\right)\big|_{x+\Delta x} \Delta y\Delta z$$

Rate of work done by y directed pressure forces:

$$= \left(v_y P\right)\big|_y \Delta x\Delta z - \left(v_y P\right)\big|_{y+\Delta y} \Delta x\Delta z$$

Rate of work done by z directed pressure forces:

$$= \left(v_z P\right)\big|_z \Delta x\Delta y - \left(v_z P\right)\big|_{z+\Delta z} \Delta x\Delta y$$

Derivation of equation of energy

3. Rate of Work done by shear forces

There are nine components of shear stress tensor. Three of these act on x directed face, similarly the net three acts on the y directed face and remaining three act on z directed faces (shown in

Fig. 29.1) as discussed earlier.

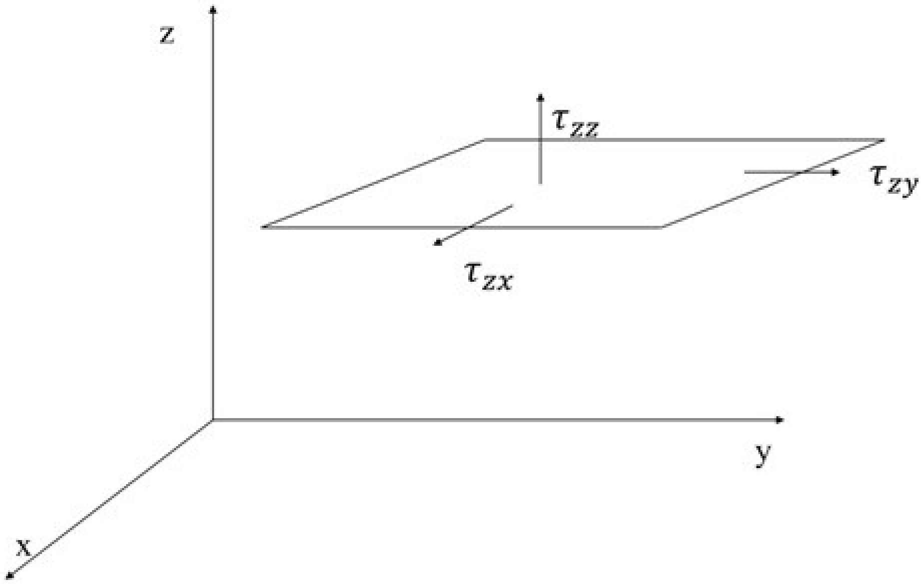

Fig.29.1 Shear stresses, acting on z directed plane

Therefore, the work done by shear forces on the control volume, acting on x directed plane may be calculated as

$$\left(\tau_{xx}v_x+\tau_{xy}v_y+\tau_{xz}v_z\right)\Big|_{x+\Delta x}\Delta y\Delta z-\left(\tau_{xx}v_x+\tau_{xy}v_y+\tau_{xz}v_z\right)\Big|_{x}\Delta y\Delta z$$

Similarly, the work done by shear forces on the control volume, acting on y directed plane may be calculated as

$$\left(\tau_{yx}v_x+\tau_{yy}v_y+\tau_{yz}v_z\right)\Big|_{y+\Delta y}\Delta x\Delta z-\left(\tau_{yx}v_x+\tau_{yy}v_y+\tau_{yz}v_z\right)\Big|_{y}\Delta x\Delta z$$

and the work done by shear force on the control volume, acting on z directed plane may be calculated as

$$\left(\tau_{zx}v_x+\tau_{zy}v_y+\tau_{zz}v_z\right)\Big|_{z+\Delta z}\Delta x\Delta y-\left(\tau_{zx}v_x+\tau_{zy}v_y+\tau_{zz}v_z\right)\Big|_{z}\Delta x\Delta y$$

5. Rate of Heat addition by Heat source or sink

If any heat source/sink is present in the control volume which generates the heat as Sc per unit volume then heat generated in the control volume may be written as.

$$= Sc\Delta x\Delta y\Delta z$$

We now substitute all terms given above in Eqution (28.1) and then divide by ΔxΔyΔz. After taking the limits Δx,Δy and Δz going to zero, we obtain the following equation

$$\frac{\partial}{\partial t}\left[\rho\left(\hat{U}+\frac{v^2}{2}\right)\right]=-\left[\frac{\partial}{\partial x}\left\{\rho v_x\left(\hat{U}+\frac{v^2}{2}\right)\right\}+\frac{\partial}{\partial y}\left\{\rho v_y\left(\hat{U}+\frac{v^2}{2}\right)\right\}+\frac{\partial}{\partial z}\left\{\rho v_z\left(\hat{U}+\frac{v^2}{2}\right)\right\}\right]-\left[\frac{\partial}{\partial x}q\right.$$

$$\rho\left(v_x g_x + v_y g_y + v_z g_z\right)-\left[\frac{\partial}{\partial x}(Pv_x)+\frac{\partial}{\partial y}(Pv_y)+\frac{\partial}{\partial z}(Pv_z)\right]$$

$$+\left[\frac{\partial}{\partial x}\left(\tau_{xx}v_x+\tau_{xy}v_y+\tau_{xz}v_z\right)+\frac{\partial}{\partial y}\left(\tau_{yx}v_x+\tau_{yy}v_y+\tau_{yz}v_z\right)+\frac{\partial}{\partial z}\left(\tau_{zx}v_x+\tau_{zy}v_y+\tau_{z}\right.\right.$$

In Equation (29.5), the stress tensor $\underset{\sim}{\tau}$ was taken as shear forces. To change it into momentum flux, we replace all components of $\underset{\sim}{\tau}$ with a minus sign. In addition, if we rewrite the Equation (29.8) in vector and tensor form , we obtain the following result for equation of energy.

$$\frac{\partial}{\partial t}\left[\rho\left(\hat{U}+\frac{v^2}{2}\right)\right]=-\underset{\sim}{\nabla}.\left[\rho\left(\hat{U}+\frac{v^2}{2}\right)\right]\underset{\sim}{v}-\underset{\sim}{\nabla}.\underset{\sim}{q}+\rho\underset{\sim}{v}.\underset{\sim}{g}-\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right)-\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right)+Sc$$

We may further simplify Equation (29.6) by combining the internal energy and kinetic energy terms as shown below. If

$$s=\hat{U}+\frac{v^2}{2}$$

Then, Equation (29.6) may be written as

$$\frac{\partial(\rho s)}{\partial t}+\underset{\sim}{\nabla}.\rho s\underset{\sim}{v}=-\underset{\sim}{\nabla}.\underset{\sim}{q}+\rho\underset{\sim}{v}.\underset{\sim}{g}-\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right)-\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right)+Sc$$

The left hand side of Equation (29.8) may be modified as shown below

$$\frac{\partial(\rho s)}{\partial t} + \underset{\sim}{\nabla}.(\rho s \underset{\sim}{v})$$

or

$$\frac{\rho \partial s}{\partial t} + \frac{s \partial \rho}{\partial t} + s\underset{\sim}{\nabla}.(\rho \underset{\sim}{v}) + (\rho \underset{\sim}{v})\underset{\sim}{\nabla} s$$

or

$$\rho\left[\frac{\partial s}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla} s\right] + s\left[\frac{\partial \rho}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla}\rho \underset{\sim}{v}\right]$$

But,

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}.\rho \underset{\sim}{v} = 0$$

Therefor, Equation (29.10) simplifies to,

$$= \rho\left[\frac{\partial s}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla} s\right]$$

or

$$= \rho \frac{Ds}{Dt}$$

Therefore, Equation (29.08) simplifies to

$$\frac{\rho Ds}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{q} - \underset{\sim}{\nabla}.(p\underset{\sim}{v}) + \rho \underset{\sim}{v}.\underset{\sim}{g} - \underset{\sim}{\nabla}.(\underset{\sim}{\tau}.\underset{\sim}{v}) + Sc$$

or

$$\frac{\rho D\left(\hat{U}+v^2/2\right)}{Dt} = -\nabla.q - \nabla.(Pv) + \rho v.g - \nabla.(\tau.v) + S_c$$

Above equation represents the equation of total energy in terms of substantial derivative. Since observer is moving with fluid in the case of substantial derivative, the convective terms are not presents in the above equation.

Derivation of equation of energy
In previous lecture, we had derived the equation of energy which may be further divided into two parts

1. Equation of mechanical energy

2. Equation of thermal energy

Equation of mechanical energy
For understanding the nature of mechanical energy, consider a simple case of a single particle moving in one direction as shown in Fig. 30.1. Assume the particle has mass m and is located at height h from a reference plane and moving upward with velocity v. Gravity is the only force working on the particle.

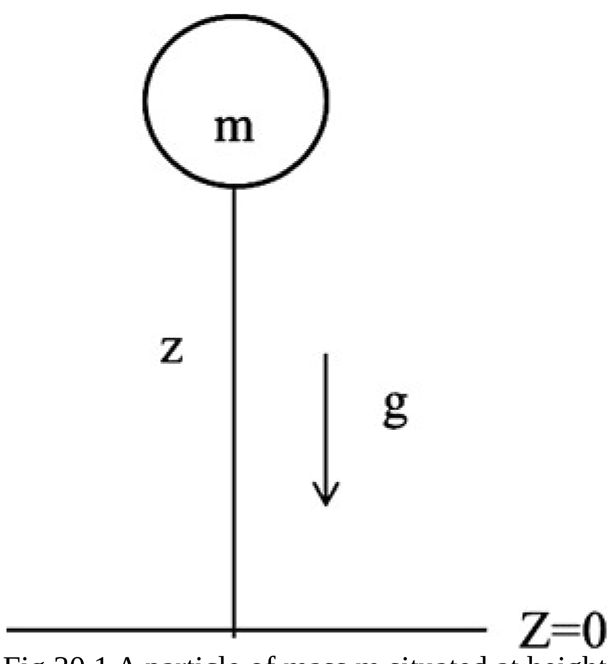

Fig 30.1 A particle of mass m situated at height z

Starting with Newton’s second law of motion, we have
Force = mass x acceleration
where

$$\underset{\sim}{F} = m\underset{\sim}{a} \quad (30.1)$$

or

$$\underset{\sim}{F} = m\frac{d\underset{\sim}{v}}{dt} \quad (3$$

By taking dot product of equation (30.2) with velocity , we find that

$$\underset{\sim}{v}.\left[\underset{\sim}{F} = m\frac{d\underset{\sim}{v}}{dt}\right]$$

or

$$\underset{\sim}{v}.\underset{\sim}{F} = m\underset{\sim}{v}.\frac{d\underset{\sim}{v}}{dt}$$

Using vector identity, we have

$$\frac{d\left(\underset{\sim}{v}.\underset{\sim}{v}\right)}{dt} = \underset{\sim}{v}.\frac{d\underset{\sim}{v}}{dt} + \frac{d\underset{\sim}{v}}{dt}.\underset{\sim}{v} = 2\underset{\sim}{v}.\frac{d\underset{\sim}{v}}{dt}$$

or

$$\underset{\sim}{v}.\frac{d\underset{\sim}{v}}{dt} = \frac{1}{2}\frac{d\left(v^2\right)}{dt}$$

where, v is the magnitude of the velocity vector $\underset{\sim}{v}$.
By substituting Equation (30.6) in Equation (30.4), we obtain

$$\underset{\sim}{v}.\underset{\sim}{F} = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

or,

$$v_1F_1 + v_2F_2 + v_3F_3 = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

For the example given above, we have ,
F1 = 0, F2 = 0, F3=-mg .... (30.8)

and

ν1 = 0, ν2 = 0, ν3 = ν .... (30.9)

Thus, Equation (30.7), reduces to

$$-vmg = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

Substitute ν = (dz/dt). Thus we obtain,

$$-mg\frac{dz}{dt} = m\frac{d\left(\frac{v^2}{2}\right)}{dt}$$

Since, m and g are constants. We may rewrite above equation as,

$$-\frac{d(mgz)}{dt} = \frac{d\left(m\frac{v^2}{2}\right)}{dt}$$

or

$$\frac{d}{dt}\left(mgz + m\frac{v^2}{2}\right) = 0$$

Thus $$mgz + m\frac{v^2}{2} = constant$$

......

First term in above equation is the potential energy and second term represents the kinetic energy. Therefore, the above equation states that the sum of kinetic and potential energy remains constant. This is the equation of mechanical energy for a particle and similar equation may be derived for fluids as shown below.

Equation of mechanical energy for fluids

The equation of motion for a fluid is equivalent to Newton's second law of motion for solid bodies. Therefore, to derive the equation of mechanical energy for fluids we take the dot product of velocity with equation of motion for fluids. i.e.,

$$\underset{\sim}{v}.\left[\rho\frac{D\underset{\sim}{v}}{Dt} = \rho\underset{\sim}{g} - \underset{\sim}{\nabla}P - \underset{\sim}{\nabla}.\underset{\sim}{\tau}\right]$$

As before,

$$\underset{\sim}{v}.\frac{D\underset{\sim}{v}}{Dt} = \frac{D\left(\frac{v^2}{2}\right)}{Dt}$$

(Note: substantial derivatives behave like normal derivatives.). Thus,

$$\rho\frac{D}{Dt}\left(\frac{v^2}{2}\right) = \rho\underset{\sim}{g}.\underset{\sim}{v} - \underset{\sim}{v}.\left(\underset{\sim}{\nabla}P\right) - \underset{\sim}{v}.\left(\underset{\sim}{\nabla}.\underset{\sim}{\tau}\right)$$

The following vector and tensor identities may be used for simplifying Equation (30.14)

$$\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) = P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) + \underset{\sim}{v}.\left(\underset{\sim}{\nabla}P\right)$$

and if $\underset{\sim}{\tau}$ is a second order symmetric tensor then we also have

$$\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right) = \underset{\sim}{v}.\underset{\sim}{\nabla}.\underset{\sim}{\tau} + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}$$

Thus, we obtain

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right) = \underset{\sim}{v}.\left(\rho \underset{\sim}{g}\right) - \left[\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) + \left(-P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)\right)\right] - \left[\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right) + \left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right)\right]$$

or

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right) = \underset{\sim}{v}.\left(\rho \underset{\sim}{g}\right) - \underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right) + P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) - \underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right) + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}$$

Equation (30.18) is called the equation of mechanical energy for fluids. Significance of each term is given below.

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2}\right)\begin{Bmatrix} Rate\ of\ change\ of \\ kinetic\ energy \\ per\ unit\ volume \end{Bmatrix} = \underset{\sim}{v}.\left(\rho \underset{\sim}{g}\right)\begin{Bmatrix} work\ done\ by\ gravity \\ force\ on\ the\ system \end{Bmatrix}$$

$$-\underset{\sim}{\nabla}.\left(P\underset{\sim}{v}\right)\begin{Bmatrix} work\ done\ by\ pressure \\ force\ on\ the\ system \end{Bmatrix} + P\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right)\begin{Bmatrix} reversible\ conversion\ of \\ kinetic\ energy\ into \\ the\ internal\ energy \end{Bmatrix}$$

$$-\underset{\sim}{\nabla}.\left(\underset{\sim}{\tau}.\underset{\sim}{v}\right)\begin{Bmatrix} work\ done\ by\ viscous \\ forces\ on\ system \end{Bmatrix} + \underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\begin{Bmatrix} irreversible\ conversion\ of \\ kinetic\ energy\ in\ to\ the\ hea \end{Bmatrix}$$

As discussed earlier, the equation of thermal energy can be derived by subtracting the equation of mechanical energy (Equation (30.18)) from the equation of total energy (Equation (29.15)), i.e.,

$$\begin{pmatrix} Equation\ of\ thermal \\ energy \end{pmatrix} = \begin{pmatrix} Equation\ of \\ energy \end{pmatrix} - \begin{pmatrix} Equation\ of\ mechanical \\ energy \end{pmatrix}$$

thus

$$\rho \frac{D(\hat{U})}{Dt} = -\nabla.q - \nabla.(Pv) + S_c + \nabla.(Pv) - P(\nabla.v) - (\tau : \nabla v)$$

or

$$\rho \frac{D(\hat{U})}{Dt} = -\nabla.q + S_c - P(\nabla.v) - (\tau : \nabla v)$$

The significance of each term in equation of thermal energy, Equation (31.21) is given below

$$\rho \frac{D(\hat{U})}{Dt}\begin{Bmatrix}\text{Rate of change of} \\ \text{internal energy} \\ \text{per unit volume}\end{Bmatrix} = -\nabla.q\begin{Bmatrix}\text{Heat transferred} \\ \text{by conduction}\end{Bmatrix} + S_c\begin{Bmatrix}\text{Heat gener} \\ \text{by source}\end{Bmatrix}$$

$$-P(\nabla.v)\begin{Bmatrix}\text{reversible conversion of} \\ \text{kinetic energy into} \\ \text{the internal energy}\end{Bmatrix} - (\tau : \nabla v)\begin{Bmatrix}\text{irreversible conversion of} \\ \text{kinetic energy in to the he}\end{Bmatrix}$$

Here $(\tau : \nabla v)$, is known as the viscous heat dissipation and the significance of this will be discussed later.

Equation of mechanical energy of fluids and its interpretation

If we consider a special case of non-viscous fluid, where the shear stress is zero, Equation (30.14) simplifies as shown below

$$\rho \frac{D\left(\frac{v^2}{2}\right)}{Dt} = -v.\nabla P + \rho(v.g)$$

Here, the gravity may be represented by gradient of a scalar quantity Φ, or

$$g = -\nabla \hat{\phi}$$

Then, Equation (30.23) may be rewritten as

$$\rho \frac{D\left(\frac{v^2}{2}\right)}{Dt} = -\underset{\sim}{v}.\underset{\sim}{\nabla} P + \rho\left(-\underset{\sim}{v}.\underset{\sim}{\nabla}\hat{\phi}\right)$$

Further, if we assume that pressure and gravity do not depend on time. Thus, we have

$$\frac{\partial P(t)}{\partial t} = 0$$

and

$$\rho \frac{\partial \phi}{\partial t} = 0$$

After substituting these values in Equation (30.25), we obtain

$$\rho \frac{D\left(\frac{v^2}{2}\right)}{Dt} = -\underset{\sim}{v}.\underset{\sim}{\nabla} P - \frac{\partial P}{\partial t} + \rho\left(-\underset{\sim}{v}.\underset{\sim}{\nabla}\hat{\phi}\right) - \rho\frac{\partial \hat{\phi}}{\partial t}$$

which may be further simplified as

$$\frac{D\left(\frac{v^2}{2}\right)}{Dt} = \frac{D\left(\frac{P}{\rho}\right)}{Dt} - \frac{D\hat{\phi}}{Dt}$$

or

$$\rho \frac{D}{Dt}\left(\frac{v^2}{2} + \frac{p}{\rho} + \hat{\phi}\right) = 0$$

which leads to

$$\left(\frac{v^2}{2} + \frac{p}{\rho} + \hat{\phi}\right) = constant$$

The above equation is called Bernoulli's equation. This states that the sum of the kinetic energy, pressure and potential energy heads is constant for a non-viscous fluid.

Comparisons of mechanical and thermal energy

Equations (30.18) and (30.20) show that the heat flux by conduction, $\underset{\sim}{q}$, is not present in the equation of mechanical energy and its contribution shows up only in the equation of thermal energy. Similarly, the heat addition by a heat so appears only in the equation of thermal energy. Also, as is shown letter the term $\left(-\underset{\sim}{\tau} : \nabla \underset{\sim}{v}\right)$ is always positive and appears with a minus sign in the equation of mechanical energy while it appears with a positive sign in the equation of thermal energy. Indicating the mechanical energy always degrades and converts into thermal energy. All these facts indicate that theses equations are consistence with the second law of thermodynamics which specifies that for an isolated system work can be converted into heat but not vice versa .

Temperature explicit form of the equation of thermal energy

Internal energy, $\hat{U}$, is a state function which may be written in the terms of two intensive variables. If the intensive variables are temperature and volume, we may write $\hat{U}(T,\hat{V})$ and

$$d\hat{U} = \left(\frac{\partial \hat{U}}{\partial T}\right)_{\hat{V}} dT + \left(\frac{\partial \hat{U}}{\partial \hat{V}}\right)_T d\hat{V} \qquad \textbf{(31.1)}$$

where the symbol "^" represents the value of the property per unit mass or per unit mole.

From thermodynamics for fluids

$$d\hat{U} = C_v\, dT + \left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right] d\hat{V} \qquad \textbf{(31.2)}$$

For example, if fluid is an ideal gas then substituting the ideal gas law in the right hand side of Equation (31.2), we obtain

$$P\hat{V} = RT$$

or

$$\left(\frac{dP}{dT}\right)_{\hat{V}} = \frac{R}{\hat{V}}$$

Thus,

$$d\hat{U} = C_v dT + \left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right] = C_v dT + \left[-P + \frac{TR}{\hat{V}}\right] = 0 \qquad \textbf{(31.3)}$$

For real fluids, Equation (31.2) may be written in the form of the substantial derivative,i.e.,

$$\frac{D\hat{U}}{Dt} = C_v \frac{DT}{Dt} + \left[-P + T\left(\frac{\partial p}{\partial T}\right)_{\hat{V}}\right]\frac{D\hat{V}}{Dt} \qquad \textbf{(31.4)}$$

Furthermore, the specific volume $\hat{V}$ is the inverse of the density ρ, or

$$\hat{V} = \left(\frac{1}{\rho}\right) \qquad \textbf{(31.5)}$$

Therefore,

$$\frac{D\hat{V}}{Dt} = \frac{D\left(\frac{1}{\rho}\right)}{Dt} = \frac{-1}{\rho^2}\frac{D\rho}{Dt} \qquad \textbf{(31.6)}$$

Thus, Equation (31.4) reduces to

$$\frac{D\hat{U}}{Dt} = C_v \frac{DT}{Dt} + \left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right]\left(\frac{-1}{\rho^2}\frac{D\rho}{Dt}\right)$$

or

$$\rho\frac{D\hat{U}}{Dt} = \rho C_v \frac{DT}{Dt} + \left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right]\left(\frac{-1}{\rho}\frac{D\rho}{Dt}\right) \quad \textbf{(31.7)}$$

However, the equation of thermal energy is

$$\rho\frac{D(\hat{U})}{Dt} = -\nabla.q + Sc - P(\nabla.v) - (\tau : \nabla v) \quad \textbf{(31.}$$

From Equations (31.7) and (31.8), we obtain the temperature explicit form of equation of thermal energy as

$$\rho C_v \frac{DT}{Dt} = -\nabla.q - p(\nabla.v) + \frac{1}{\rho^2}\left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right]\frac{D\rho}{Dt} + (-\tau : \nabla v) + Sc \quad \textbf{(31.9}$$

Using the equation of continuity $\left(\frac{D\rho}{Dt} + \rho(\nabla.v) = 0\right)$, in above equation, we finally obtain

$$\rho C_v \frac{DT}{Dt} = -\nabla.q - P(\nabla.v) - \frac{1}{\rho}\left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right]\left[\rho(\nabla.v)\right] + (-\tau : \nabla v) + Sc \quad \textbf{(31.}$$

which simplifies the equation (31.10) leads to

$$\rho C_v \frac{DT}{Dt} = -\underset{\sim}{\nabla}.\underset{\sim}{q} - T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}} \left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) + \left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right) + Sc$$

or

$$\rho C_v \left[\frac{\partial T}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla} T\right] = -\underset{\sim}{\nabla}.\underset{\sim}{q} - T\left(\frac{\partial p}{\partial T}\right)_{\hat{V}} \left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) + \left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right) + Sc \qquad \textbf{(3}$$

Equation (31.11) represents the equation of thermal energy in terms of temperatures for a real fluid. Expended form of Equation (31.11) in cartesian, cylindrical and spherical coordinate system is given in Appendix-5. Some limiting cases of Equation (31.11) are discussed below.

Case 1: heat conduction in solids

In solids, all velocities are zero and Equation (31.11) simplifies to

$$\rho C_v \frac{\partial T}{\partial t} = -\underset{\sim}{\nabla}.\underset{\sim}{q} + Sc \qquad \textbf{(3}$$

where, the heat flux $\underset{\sim}{q}$ may be estimated by the Fourier's law of heat conduction. If *k* is a constant then

$$\underset{\sim}{q} = -k\underset{\sim}{\nabla} T$$

or,

$$\rho C_v \frac{\partial T}{\partial t} = +k\nabla^2 T + Sc$$

Equation (31.13) is also known as the Fourier's second law of heat conduction.

Case 2 Heat transfer in fluids with constant ρ and k

For constant density, equation of continuity reduces to

$$\nabla.\underset{\sim}{v} = 0$$

In addition, the heat capacity at constant volume and constant pressures are the same. Thus,

$$C_v = C_p$$

For this case, the equation of thermal energy may be simplified to

$$\rho C_v \left[\frac{\partial T}{\partial t} + \underset{\sim}{v}.\nabla T \right] = k\nabla^2 T + \left(-\underset{\sim}{\tau} : \nabla \underset{\sim}{v}\right) + Sc$$

or

$$\frac{\partial T}{\partial t} + \underset{\sim}{v}.\nabla T = \frac{k}{\rho C_p} \nabla^2 T + \frac{\left(-\underset{\sim}{\tau} : \nabla \underset{\sim}{v}\right)}{\rho C_p} + \frac{Sc}{\rho C_p}$$

Equation (31.18) represents the equation of thermal energy for constant density and heat conductivity. It may be noted that Equation (31.18) has a similar form as the equation of motion for constant density and viscosity (Navior Stoke equation), i.e.,

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \rho.\nabla \underset{\sim}{v} = \frac{\mu}{\rho} \nabla^2 \underset{\sim}{v} - \frac{\nabla p}{\rho} + \underset{\sim}{g}$$

Both equations show the similarities between momentum and heat transport. The detail form of Equation (31.18) for different coordinate system is given in the Appendix -5

2.8 Viscous heat dissipation / viscous heating

The viscous heat dissipation term $\left(-\underset{\sim}{\tau} : \nabla \underset{\sim}{v}\right)$, in the equation of thermal energy represents the conversion of mechanical energy into thermal energy due to viscous dissipation. This term is always positive. For Newtonian fluids, $\underset{\sim}{\tau}$ may be calculated by using the Newton's law viscosity. The details are not shown here but may be found elsewhere, i.e.,

$$\underset{\sim}{\tau} = -\mu \nabla \underset{\sim}{v}$$

or

$$-\underset{\sim}{\tau} : \nabla \underset{\sim}{v} = \mu \phi_v$$

where, Φ_v is a scalar quantity and the value of Φ_v for different coordinate system is given in the Appendix -6. As may be seen from this table that all terms present in the expression from Φ_v are positive. Thus, this viscous dissipation leads to increase the value of thermal energy and raises the temperature of fluid.

Significance of Viscous dissipation / heating

As discussed in previous lecture, the viscous dissipation leads to rise in the temperature of fluid. Here, we solve a problem which help us to understand the significance of this term in terms of rise in temperature.

Consider a fluid flowing under laminar conditions between two parallel plates which are kept at same temperature T0 as shown in Fig. (32.1). The incoming fluid, at z = 0, have the same temperature T0. Assuming steady state, determine the increase in temperature of fluid due to viscous heat dissipation.

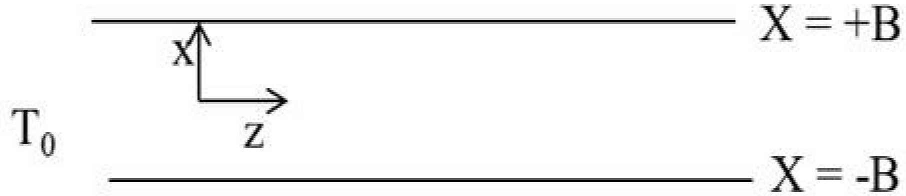

Fig 32.1 Viscous flow between two parallel plates

Assumptions

- Density ρ, viscosity μ and thermal conductivity k are constants.
- System is in steady state.

- Flow is laminar (simple shear flow) and fully developed.
- Newton’s law of viscosity is applicable.

Fluid is flowing in z direction only and the Non-zero velocity components are
vx = 0, vy = 0, vz = vz(x)
the velocity profile vz may be easily obtained for a Newtonian fluid as given below

$$v_z = v_{z,max}\left[1 - \frac{x^2}{B^2}\right] \quad (32.1)$$

where $v_{z,max}$ is the maximum velocity of fluid.

At steady state, the heat produced by viscous dissipation is removed from both plates to keep the temperature of the plates at T_0. Thus, temperature of the fluid increases until the heat generation by viscous dissipation matches the heat remove from the plates and the temperature is no longer a function of z coordinate. For this fully developed region, the temperature is a function of *x* coordinate only, i.e.,

$$T = T(x) \quad (32.2)$$

For this case, the equation of thermal energy may be simplified as follows

$$0 = k\frac{d^2T}{dx^2} + \mu\varphi_v \quad (32.3)$$

Substituting the value of viscous dissipation in the equation of thermal energy, we have

$$0 = k\frac{d^2T}{dx^2} + \mu\left(\frac{dv_z}{dx}\right)^2 \quad (32.4)$$

The velocity gradient in equation (32.4) may be estimated from Equation (32.1) as

$$\frac{dv_z}{dx} = v_{z\,max}\left(\frac{-2x}{b^2}\right) \tag{32.5}$$

Therefore, the Equation (32.4) finally simplifies as

$$0 = k\frac{d^2T}{dx^2} + \mu\frac{4x^2}{B^4}v_{z,max}^2$$

or

$$\frac{d^2T}{dx^2} = -\left(\frac{4\mu}{k}\frac{v_{z,max}^2}{B^4}\right)x^2 = Ax^2 \tag{32.6}$$

where, A is a constant.

Integrating Equation (32.6), we finally obtain the temperature profile as

$$T = \frac{Ax^4}{12} + c_1x + c_2 \tag{32.7}$$

where c_1 and c_2 are the constants of integrations. These constants may be determined by using the following boundary conditions,

$$x = \pm B, T = T_0 \tag{32.8}$$

or

$$T_0 = \frac{AB^4}{12} \pm c_1B + c_2 \tag{32.9}$$

which leads to

$$c_1 = 0 \tag{32.10}$$

and

$$c_2 = T_0 - \frac{AB^4}{12} \tag{32.11}$$

Thus, the rise in temperature of fluid is given by the following equation by substituting the value of c_1 and c_2 in Equation (32.7), we finally obtain,

$$T - T_0 = \frac{-AB^4}{12}\left[1 - \left(\frac{x}{B}\right)^4\right]$$

or

$$T - T_0 = \frac{\mu v_{z,max}^2}{3k}\left[1 - \left(\frac{x}{B}\right)^4\right] \tag{32.12}$$

Equation (32.12) shows that the rise in temperature is maximum at $x = 0$ and is given by

$$\left(T - T_0\right)_{max} = \frac{\mu v_{z,max}^2}{3k} \tag{32.13}$$

For example, if water is flowing at a maximum velocity *100 ft/sec*, then we find,

$$\left(T - T_0\right)_{max} = \frac{\mu v_{z,max}^2}{3k} = \frac{100^2}{3}\frac{\mu_{water\ 25^0 C}}{k_{water\ 25^0 C}} \approx 1^\circ F \tag{32.14}$$

Thus, even for such large velocities, the rise in temperature is only about 1^0F. For highly viscous fluids, such as polymer solutions with viscosity *1000 Cp*, the temperature rise may be

$$(T - T_0)_{max} \approx 1000^\circ F \quad \textbf{(32.15)}$$

Therefore, we may conclude that the viscous dissipation may be important for highly viscous fluids, for low viscosity fluid like air, the viscous dissipation term may be safely neglected.

2.9 Transpiration Cooling/ heating

Transpiration cooling / heating is used to reduce or enhance the heat transfer rates by a convective flow of fluid in or opposite direction of the actual heat transfer. Additional convective flow provides forced convection to produce the desired effect. A classic example of the transpiration cooling is the design of a storage tank for liquefied gas nitrogen or oxygen. Both liquefied gas have cryogenic boiling points. The required thickness of insulation material may be quite high in order to reduce heat gained by conduction from surrounding at atmospheric temperature to very low temperature in the tank. Here, we may use transpiration cooling to reduce the heat transfer. Design of a cryogenic storage tank is shown below

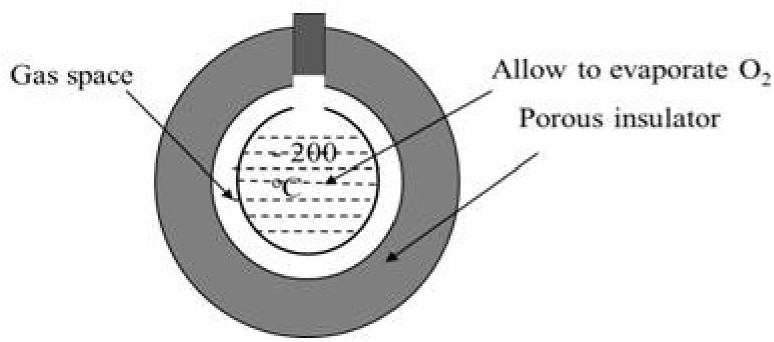

Fig 3 Use of transpiration cooling in liquefied gas container

Here, the cryogenic liquid is stored in a spherical container, which is surrounded by a porous insulation, as shown in Fig. (32.2). Some amount of stored liquid is allowed to vaporize. This vaporized gas fills the space between liquid gas container and the porous insulation. It provides the cold jacket to the container and reduces the heat transfer. As this gas starts to diffuse through pores, it provides a convective transpiration cooling and prevents the heat transferred from outside to the liquid container.

Here, we solve a simpler example of transpiration cooling. Assume the radius of two concentric porous spheres are kR and R as shown in Fig. (32.3). The temperature of inner sphere is T_k and outer sphere is T_0. The inner sphere is refrigerated. An air stream at temperature T_k with the mass flow rate w is forced from the inner sphere to outer sphere for maintaining the internal sphere temperature at T_k. Determine the heat flux with and without transpiration cooling(w = 0).

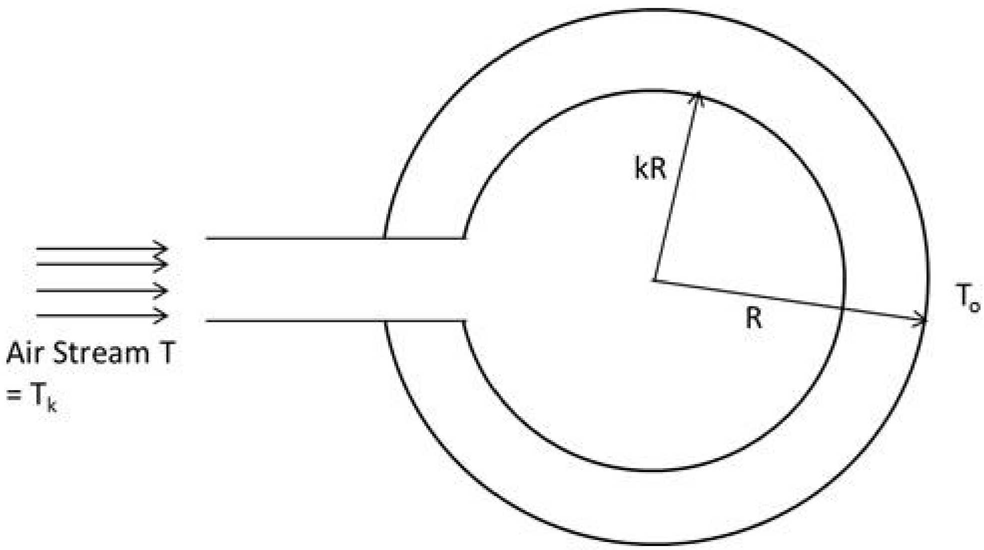

Fig 32.3 Transpiration cooling between two concentric spherical shell

Assumptions

- Density ρ, viscosity μ & thermal conductivity k are constant.
- System is in steady state.
- Flow is laminar (simple shear flow) and fully developed.

Air flows only in the radial direction. Therefore, the non-zero velocity components are

$$v_\theta = v_\phi = 0 \;\&\; v_r = v_r(r) \tag{32.16}$$

Assume, the temperature is changing only in radial direction, or,

$$T = T(r) \tag{32.17}$$

The equation of continuity in spherical coordinates provides,

$$\frac{1}{r}\frac{d}{dr}(\rho r^2 v_r) = 0$$

By integrating above equation, we obtain

$$\rho r^2 v_r = constant \tag{32.18}$$

Here, the mass flow rate may be calculated as

$$w = 4\pi r^2 \rho v_r \tag{32.19}$$

Thus,

$$v_r = \frac{w}{4\pi\rho r^2} \tag{32.20}$$

Since, the viscosity of air is very low, we may neglect the viscous dissipation term from the equation of thermal energy, or

$$\rho C_p v_r \frac{dT}{dr} = k\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)\right] \tag{32.21}$$

Substituting the value of v_r from Equation (32.20) in above equation, we obtain

$$\rho C_p \frac{w}{4\pi\rho r^2}\frac{dT}{dr} = \frac{k}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr} \right)$$

or

$$\frac{d}{dr}\left(r^2 \frac{dT}{dr} \right) = \frac{C_p w}{4\pi k}\frac{dT}{dr} = R_0 \frac{dT}{dr} \qquad \textbf{(32.22)}$$

where,

$$R_0 = \frac{C_p w}{4\pi k} \qquad \textbf{(32.23)}$$

Equation (32.22) may be solved as follows

$$\frac{dT}{dr} = Y \qquad \textbf{(32.2}$$

Thus, Equation (32.22) may be rewritten as,

$$\frac{d}{dr}\left(r^2 Y\right) = R_0 Y \qquad \textbf{(32.2}$$

or

$$r^2 \frac{dY}{dr} + 2rY = R_0 Y$$

or

$$\frac{dY}{Y} = \frac{(R_0 - 2r)dr}{r^2}$$

which may be integrated to gives the following expression,

$$ln\, Y + ln\, r^2 = \frac{-R_0}{r} + c_1 = ln\left(Yr^2\right)$$ (32

Here, c_1 is the integral constant. Thus, we have

$$Y = \frac{c_2}{r^2} e^{-(R_0/r)} = \frac{dT}{dr}$$ (3

where, $c_2 = e^{c_1}$

By integrating Equation (32.27), we finally obtain

$$T = c_2 \int \frac{e^{-(R_0/r)}}{r^2} dr + c_3$$ (

Here, c_3 is another integration constant.

Boundary conditions for this problem are

at
$r = R,\ T = T_0$

and at
$r = KR, T = T_k$ (

Finally evaluating c_2 and c_3 , we obtain the temperature profile between $r = KR$ to R,

$$\frac{T - T_0}{T_k - T_0} = \frac{e^{-R_0/r} - e^{-R_0/R}}{e^{-R_0/KR} - e^{-R_0/R}}$$

Now, we may calculate heat loss from inner sphere by first calculating the temperature gradient from above equation then multiplying it by the surface area of the sphere. Thus,

$$Q = (area\ of\ inner\ sphere) \times (heat\ flux)\Big|_{r=kR}$$

or

$$Q = -(4\pi r^2 q_r)\Big|_{r=kR}$$

where, $q_r = -k\dfrac{dT}{dr}$

Here, '−' sign is used as the outer normal of the inner sphere is in negative r direction. Thus,

$$Q = \frac{4\pi k R_0 (T_0 - T_k)}{e^{(R_0/KR)(1-K)} - 1}$$

where

$$R_0 = \frac{w_r C_p}{4\pi k}$$

Now, consider the situation when transpiration cooling in not used. Therefore, the velocity of air stream v_r is zero. In this case, the equation of thermal energy may be written as

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)\right] = 0$$

By integrating Equation (32.35) and using the above boundary conditions, we obtain

$$\frac{T-T_0}{T_k-T_0}=\frac{K}{1-K}\left(\frac{1}{r}-\frac{1}{R}\right)$$

We may once again calculate the heat loss from inner sphere by following the same procedure as earlier, thus,

$$Q_0=\frac{4\pi KR(T_0-T_k)}{1-K}$$

Here, Q_0 is the heat loss without transpiration cooling. The transpiration cooling efficiency may now be defined as

$$\varepsilon=\frac{Q_0-Q}{Q_0}=1-\frac{\varphi}{e^{\varphi}-1}$$

where,

$$\varphi=\frac{R_0(1-K)}{KR}$$

Relation between the transpiration efficiency and *Φ* is shown in Fig. (32.4). As may be seen from this figure, the transpiration efficiency increases with increase in the value of *Φ* or mass flow rate, *w*. For very large values of *Φ*, transpiration efficiency approaches to one, which implies that heat transfer from outer sphere to inner sphere, Q, is negligible.

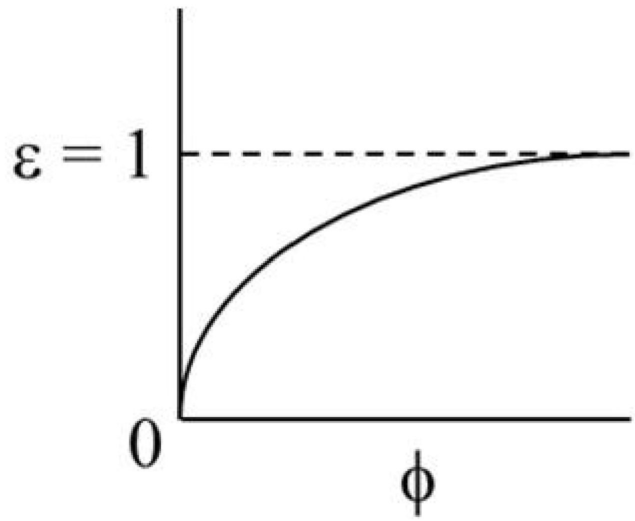

Fig 32.4 Efficiency of transpiration cooling

2.10 Heat conduction in a rectangular fin

Fins are used to enhance the heat transfer from any surface by increasing the heat transfer surface area. Due to heat conduction, this additional surface is at lower and lower temperatures as more and more surface area is added. Thus, the rate of heat transfer decreases as we move away from the original surface and efficiency of the fin is reduced. The effectiveness of a fin is defined as the ratio of actual heat transfer through the fin and heat transfer when the whole fin surface is available at the same temperature as that of the original surface. A simple rectangular fin is shown in Fig. (33.1). The wall temperature is T_w and the ambient temperature is T_a. Dimensions of fin are as shown in Fig. (33.1). Formulate the problem and determine the temperature profile. Finally also calculate the efficiency of the fin.

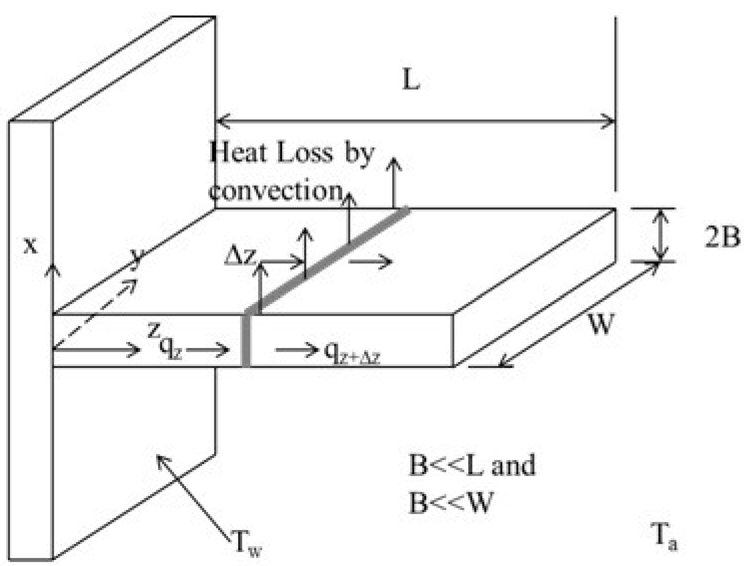

Fig 33.1 Rectangular fin

Assumption

- System is at steady state.
- Fin has a constant heat conductivity.
- The outside heat transfer coefficient is h_a.

We may solve this problem for two different cases.

Case 1: when L, W and B are of the same order of magnitude

In this case the temperature is a function of *x*, *y* and *z* coordinates, i.e.,

$$T = T(x, y, z)$$

and the equation of thermal energy may be written as

$$-\underset{\sim}{\nabla} . \underset{\sim}{q} = 0$$

After substituting the Fourier's law of conduction, we obtain

$$k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] = 0$$

The above equation may needs to be solved subject to the following boundary conditions,

$$z = 0, \ T = T_W$$

$$z = L, \ q_z\big|_{z=L} = h_a(T\big|_{z=L} - T_a)$$

$$x = +B, \ q_z\big|_{x=B} = h_a(T\big|_{x=B} - T_a)$$

$$x = -B, \ q_z\big|_{x=-B} = h_a(T\big|_{x=-B} - T_a)$$

$$y = 0, \ q_y\big|_{y=w} = h_a(T\big|_{y=w} - T_a)$$

$$y = w, \ q_y\big|_{y=w} = h_a(T\big|_{y=0} - T_a)$$

and it is obvious that Equation (33.3) can not be solved analytically and requires computational methods.

Case 2: Now, consider a case when thickness of the fin is negligible in comparison with height and width, i.e., L & W >> 2B

In this case, the heat flux in y direction is small and we may write temperature as function of *x* and *z* coordinates only,

$$T = T\left(x, z\right)$$

In this case, the equation of thermal energy may be simplified as

$$k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}\right]=0$$

This may also require numerical solutions. To find simple analytical solution, we may further assume that the average temperature, T_{avg} , at any cross section of fin is a function of *z* coordinate only, i.e.,

$$T_{avg} = T_{avg}\left(z\right)$$

Thus, we are reducing the original two dimensional into a one-dimensional problem to obtain an approximate solution. This assumption is reasonably valid since the thickness of the fin is very small and the variation of temperature in *x* direction may be averaged out. Since, we have reduced a two dimensional problem to a one-dimensional problem by averaging the temperature, the equation of thermal energy is no longer applicable. We may use the shell energy balance approach to find a appropriate solution. For the sake of convenience,T_{avg} is now replaced by T . The control volume is a strip of the fin with thickness *2B*, length *Δz* and width *W* as shown in Fig. (33.2).

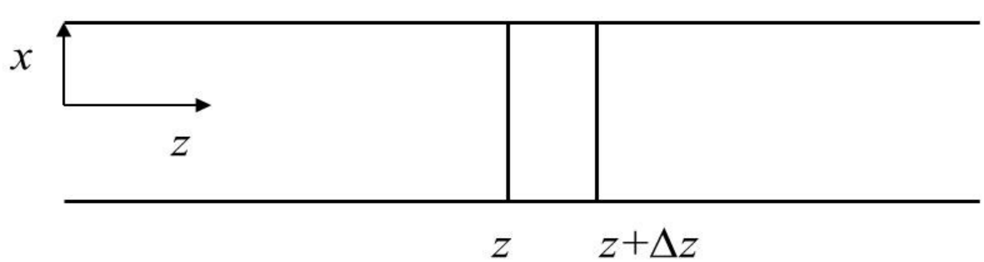

Fig 33.2 Control volume for a rectangular fin

Heat entering the control volume by conduction

$$= q_z \big|_z \times 2BW$$

Heat leaving the control volume by conduction

$$= q_z \big|_{z+\Delta z} \times 2BW$$

Heat loss from upper and lower surfaces of the control volume

$$= 2h_a\left(T - T_a\right) W\Delta z$$

Thus, energy balance may be written as

$$(q_z\big|_z - q_z\big|_{z+\Delta z})\, 2BW - 2h_a\left(T - T_a\right) W\Delta z = 0$$

After dividing the Equation (33.17) by *2BWΔz* and taking the limit $\Delta z \to 0$, we obtain

$$\frac{dq_z}{dz} = -\frac{h_a}{B}(T - T_a)$$

Fourier's law may be applied here as

$$q_z = -k\frac{dT}{dz}$$

and finally, we obtain

$$\frac{d^2T}{dz^2} = \frac{h_a}{kB}(T - T_a)$$

The Boundary conditions may be written as

at

$$z=0,\ T=T_w$$

and at

$$z=L, \quad -k\left.\frac{dT}{dz}\right|_{z=L}=h_a\left(\left.T\right|_{z=L}-T_a\right)$$

However, for a sufficiently long fin $\left(\left.T\right|_{z=L}-T_a\right)$ is small and we may assume that at

$$z=L, \left.\frac{dT}{dz}\right|_{z=L}=0$$

By applying the above boundary conditions in Equation (33.20), we obtain the following solution

$$\frac{(T-T_a)}{(T_w-T_a)}=\cosh N\varepsilon-(\tanh N)\sinh N\varepsilon$$

where

$$N=\frac{hL^2}{KB}$$ and

$$\varepsilon=\frac{z}{L}$$

This equation may be arranged as

$$\theta=\frac{\cosh N(1-\varepsilon)}{\cosh N}$$

with

$$\theta = \frac{T - T_a}{T_w - T_a}$$

Going back to the original two dimensional problem given in Equation (33.11), even without assuming anything further, we may still obtain the same results as given by Equation (33.26), provided *B* is very small. Thus, the fin temperature is the function of both *x* and *z* coordinates, i.e., $T(x,z)$, and the equation of thermal energy may be written as

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_z}{\partial z} = 0$$

If *2B* is small (from the definition of a differential), we may approximate the first term as

$$\frac{\partial q_x}{\partial x} = \lim_{B \to 0} \frac{q_x\big|_{x=B} - q_x\big|_{x=-B}}{2B}$$

By applying the boundary conditions given in the Equations (33.6) and (33.7), we get,

$$\frac{\partial q}{\partial x} \cong \frac{h_a(T - T_a) - \left(-h_a(T - T_a)\right)}{2B}$$

or

$$\frac{\partial q}{\partial x} \cong \frac{h_a}{B}(T - T_a)$$

where the partial derivative is replaced by the total derivative. By substituting the Equation (33.31) in (33.28), we again obtain

$$\frac{dq_z}{dz} + \frac{h_a(T - T_a)}{B} = 0$$

where the partial derivative is replaced by the total derivative. After using the Fourier's law of heat conduction, we obtain the following differential equation

$$-k\frac{d^2T}{dz^2} + \frac{h_a}{B}(T - T_a) = 0$$

Equation (33.33) has the same form as Equation (33.20) and therefore the solutions is also the same

Efficiency of the rectangular fin

As state above , the efficiency of fin may be defined as,

$$\eta = \frac{Actual\ rate\ of\ heat\ loss\ from\ fin}{Rate\ of\ heat\ loss\ from\ an\ isothermal\ fin\ at\ T_w}$$

Substituting the values, we get

$$\eta = \frac{\int_0^W \int_0^L h_a\ (T - T_a) dz\ dy}{\int_0^W \int_0^L h_a\ (T_W - T_a) dz\ dy}$$

which may be simplified by using the above dimensionless numbers, defined in Equation (33.25) and (33.27), as

$$\eta = \frac{\int_0^W \theta d\varepsilon}{\int_0^W d\varepsilon}$$

The Equation (33.35) leads to the following simple solution

$$\eta = \frac{1}{\cosh N}\left(-\frac{1}{N} \sinh N(1-\varepsilon) \right)\Bigg|_0^1$$

which may be further simplified as

$$\eta = \frac{\tan h\, N}{N}$$

As may be noted from above equation η depends only on N. A simple plot for η vs N is shown below

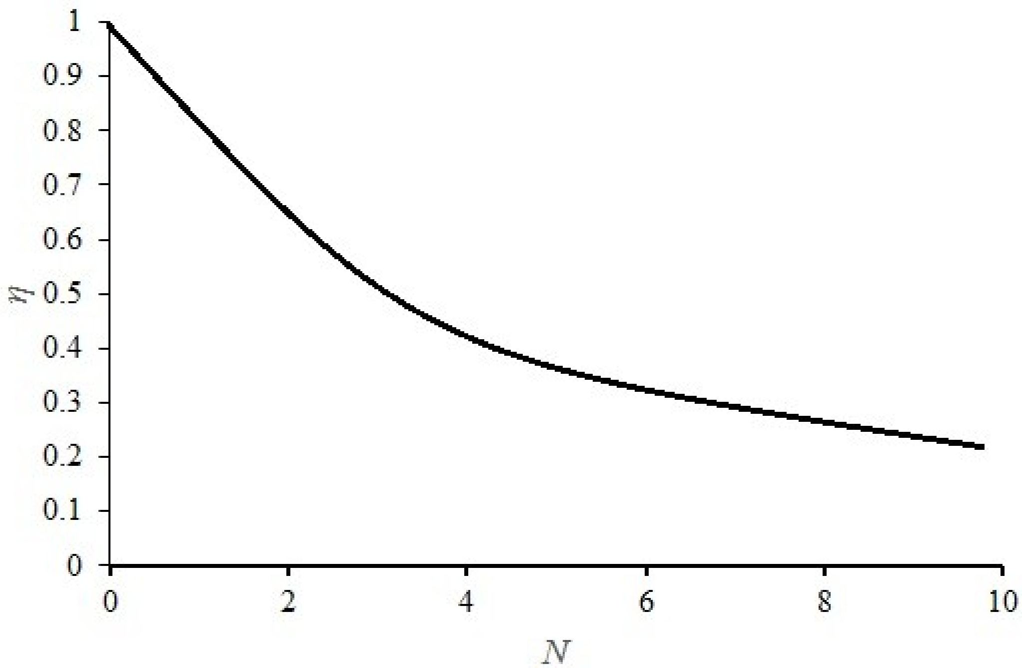

Fig 33.3 Efficiency of fin vs N

Fig. 4 shows that long fins have lesser efficiency. The optimum fin length may be found by optimizing the enhancement in heat transfer and the cost of this additional surface area. In the real life, the fins need not to be rectangular in shape and the different shaped fins may be designed for enhanced fin efficiency.

Fourier's Law of Heat Conduction

$$\underset{\sim}{q} = -k\underset{\sim}{\nabla}T$$

Cartesian coordinates(x,y,z)

$$q_x = -k\frac{\partial T}{\partial x}$$

$$q_y = -k\frac{\partial T}{\partial y}$$

$$q_z = -k\frac{\partial T}{\partial z}$$

Cylindrical coordinates (r,θ,z)

$$q_r = -k\frac{\partial T}{\partial r}$$

$$q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial \theta}$$

$$q_z = -k\frac{\partial T}{\partial z}$$

Spherical coordinate (r, θ,Φ)

$$q_r = -k\frac{\partial T}{\partial r}$$

$$q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial \theta}$$

$$q_\phi = -k\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

Equation of Thermal Energy in Terms of heat flux

$$\rho\hat{c}_{\hat{v}}\left[\frac{\partial T}{\partial t} + \underset{\sim}{v}.\underset{\sim}{\nabla}T\right] = -\underset{\sim}{\nabla}.\underset{\sim}{q} - T\left(\frac{\partial p}{\partial T}\right)_{\hat{v}}\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) + \left(-\underset{\sim}{\tau} : \underset{\sim}{\nabla}\underset{\sim}{v}\right) + Sc$$

Cartesian coordinates(x,y,z)

$$\rho\hat{c}_{\hat{v}}\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = -\left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right] - T\left(\frac{\partial p}{\partial T}\right)_{\hat{v}}\left(\underset{\sim}{\nabla}.\underset{\sim}{v}\right) - (\underset{\sim}{\tau} : \underset{\sim}{\nabla}$$

Cylindrical coordinates (r,θ,z)

$$\rho\hat{c}_{\hat{V}}\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}\right)=-\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rq_r\right)+\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{\partial q_z}{\partial z}\right]-T\left(\frac{\partial p}{\partial T}\right)_{\hat{V}}\left(\nabla.v\right)$$

Spherical coordinate (r, θ,Φ)

$$\rho\hat{c}_{\hat{V}}\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)=-\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2q_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(q_0\sin\theta\right)+\right.$$

$$-T\left(\frac{\partial p}{\partial T}\right)_{\hat{V}}\left(\nabla.v\right)-\left(\tau:\nabla v\right)+Sc$$

Equation of Energy for Pure Newtonian Fluid with Constant ρ and k

$$\frac{\partial T}{\partial t}+v.\nabla T=\frac{k}{\rho\hat{c}_p}\nabla^2T+\frac{\mu\phi_v}{\rho\hat{c}_p}+\frac{Sc}{\rho\hat{c}_p}$$

Cartesian coordinates(x,y,z)

$$\left(\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}\right)=\frac{k}{\rho\hat{c}_p}\left[\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}+\frac{\partial^2T}{\partial z^2}\right]+\frac{\mu\phi_v}{\rho\hat{c}_p}+\frac{Sc}{\rho\hat{c}_p}$$

Cylindrical coordinates (r,θ,z)

$$\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}\right)=\frac{k}{\rho\hat{c}_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2T}{\partial \theta^2}+\frac{\partial^2T}{\partial z^2}\right]+\frac{\mu\phi_v}{\rho\hat{c}_p}+\frac{Sc}{\rho\hat{c}_p}$$

Spherical coordinate (r, θ,Φ)

$$\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)=\frac{k}{\rho\hat{c}_p}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)\right.$$

$$+\frac{\mu\phi_v}{\rho\hat{c}_p}+\frac{Sc}{\rho\hat{c}_p}$$

Appendix - 6

Dissipation Function for Newtonian Fluids

Cartesian coordinates(x,y,z)

$$\phi_v = 2\left(\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right) + \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]^2 + \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]^2 + \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]^2$$

Cylindrical coordinates (r,θ,z)

$$\phi_v = 2\left(\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right) + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]^2$$

$$-\frac{2}{3}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}\right]^2$$

Spherical coordinate (r, θ,Φ)

$$\phi_v = 2\left(\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r + v_\theta \cot\theta}{r}\right)^2\right) + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial}{}\right.$$

$$+\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]^2 + \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]^2$$

$$-\frac{2}{3}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right]^2$$