UNIT 2 COMPLEX INTEGRATION .

Cauchy’s integral theorem - Cauchy’s integral formula - problems - Taylor’s and Laurent’s series - Singularities - Poles and Residues - Cauchy’s residue theorem and problems.

UNIT- II COMPLEX INTEGRATION

DEFINITIONS:

Simple curve: A curve C is said to be simple if it does not cross itself. 

Simple closed curve: A simple curve which does not have end points is called a simple closed curve. 

Multiple curve: A curve which intersects itself at some point is called a multiple curve. 

Multiple closed curve: A multiple curve which is closed is called multiple closed curve. 

Simply connected Region: A region R is said to be simply connected if any closed curve which lies in R can be shrunk to a point without leaving R. 

Multiply connected Region: A region which is not simply connected is called multiply connected. 

Cauchy's Integral Theorem (or) Cauchy's Fundamental Theorem:

Statement: If $f(z)$ is analytic and $f'(z)$ is continuous inside and on a simple closed curve C, then $\int_C f(z)\,dz = 0$

Extension of Cauchy's Integral theorem for Multiply connected region

Let C be a simple closed curve and $C_1, C_2, \dots C_n$ be a finite number of simple closed curves which have no points in common and which are lying inside C.

If a function $f(z)$ is analytic in the region consisting of all points within and on C except for the points interior to $C_1, C_2, \ldots C_n$ then,

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz$$

where all the integrals are taken in the anticlockwise direction.

Cauchy's Integral Formula:

Statement: Let $f(z)$ be analytic everywhere within and on a simple closed curve C, taken in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z-z_0}.$$

Cauchy's Integral Formula for the derivatives:

If a function $f(z)$ is analytic within and on a simple closed curve C and z_0 is any point lying in it, then $f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^2}$.

Note: $f''(z_0) = \frac{2!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^3}$, $f'''(z_0) = \frac{3!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^4}$

In general, $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$

PROBLEMS.

1. Evaluate $\int_C \frac{dz}{z-a}$ where C is the simple closed curve and $z=a$ is a point (a) inside C (b) outside C.

Soln: The function $\frac{1}{z-a}$ is analytic everywhere except at $z=a$

(a) If $z=a$ lies inside C, then by Cauchy's integral formula, $f(a) = \frac{1}{2\pi i}\int_C \frac{dz}{z-a} \Rightarrow \int_C \frac{dz}{z-a} = 2\pi i\, f(a)$

where $f(z)=1$ & so $f(a)=1$

$$\therefore \boxed{\int_C \frac{dz}{z-a} = 2\pi i}$$

(b) If $z=a$ lies outside C, then the function $\frac{1}{z-a}$ is analytic everywhere inside C and so by Cauchy's Integral theorem, $\boxed{\int_C f(z)\,dz = 0}$

2. Evaluate $\int_C \frac{z^2+5}{z-3}\,dz$ on the circle $|z|=4$.

Solution: The function $\frac{z^2+5}{z-3}$ is analytic everywhere except at the point $z=3$ which lies inside $|z|=4$.

Hence by Cauchy's Integral formula, $f(3) = \frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz$

where $f(z) = z^2+5$ and so $f(3) = 3^2+5 = 14$

$$\therefore \boxed{\int_C \frac{z^2+5}{z-3}\,dz = 2\pi i \times 14 = 28\pi i}$$

3. Evaluate $\int_C \frac{\cos z\, dz}{z(z^2+8)}$ where C is the square whose sides are the lines $x = \pm 2$ and $y = \pm 2$.

Solution: The function $\frac{\cos z}{z(z^2+8)}$ is analytic everywhere except at the points $z = 0, \pm 2\sqrt{2}\, i$. Of these points only $z=0$ lies inside C. Thus by Cauchy's Integral formula,

$$\int_C \frac{\cos z\, dz}{z(z^2+8)} = 2\pi i\, f(0) \text{ where } f(z) = \frac{\cos z}{z^2+8}$$

& hence $f(0) = \frac{\cos 0}{0+8} = \frac{1}{8}$.

ie, $$\int_C \frac{\cos z\, dz}{z(z^2+8)} = \int_C \frac{\left(\cos z / z^2+8\right) dz}{z} = 2\pi i \times \frac{1}{8} = \frac{\pi i}{4}$$

$$\Rightarrow \boxed{\int_C \frac{\cos z\, dz}{z(z^2+8)} = \frac{\pi i}{4}}$$

4. Find the value of the integral $\int_C \frac{(z+4)dz}{z^2+2z+5}$ where C is $|z+1+i| = 2$.

Solution: The function $\frac{z+4}{z^2+2z+5}$ is analytic everywhere except at the points given by $z^2+2z+5=0$

(ie) $z = -1 \pm 2i$. Of these points, the point $z = -1-2i$ lies inside C since, $|z+1+i| = |(-1-2i)+1+i| = |-i| = 1 < 2$

Thus by Cauchy's integral formula,

$$\int_C \frac{(z+4)\, dz}{z^2+2z+5} = \int_C \frac{(z+4)/z-(-1+2i)\, dz}{[z-(-1-2i)]} = \int_C \frac{\left(z+4/z+1-2i\right) dz}{z+1+2i} = 2\pi i f(-1-2$$

where $f(z) = \frac{z+4}{z+1-2i}$ and so $f(-1-2i) = \frac{-1-2i+4}{-1-2i+1-2i} = \frac{3-2i}{-4i}$

$$= \frac{-3}{4i} + \frac{1}{2} = \frac{1}{2} + \frac{3i}{4}$$

$$\therefore \boxed{\int_C \frac{(z+4)\,dz}{z^2+2z+5} = 2\pi i \times \left(\frac{1}{2} + \frac{3i}{4}\right) = \pi i - \frac{3\pi}{2}}$$

5. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ where C is the circle $|z| = 3$.

Solution: The function $\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$ is analytic everywhere except at the points $z=1$ and $z=2$. Since both the points lies inside C, by Extension of Cauchy's theorem for multiply connected region,

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = \int_{C_1} \frac{\left(\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right)}{z-1}\,dz + \int_{C_2} \frac{\left(\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\right)}{z-2}\,dz$$

where C_1 is a circle around $z=1$ and C_2 is a circle around $z=2$. Applying Cauchy's Integral formula separately for each of the integrals on the RHS of the above equation, we have,

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = 2\pi i\, f_1(1) + 2\pi i\, f_2(2) \quad \text{where}$$

$$f_1(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\ ; \quad f_1(1) = \frac{\sin \pi + \cos \pi}{1-2} = \frac{-1}{-1} = 1$$

$$f_2(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\ ; \quad f_2(2) = \frac{\sin 4\pi + \cos 4\pi}{2-1} = \frac{0+1}{1} = 1$$

$$\Rightarrow \boxed{\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = 2\pi i + 2\pi i = 4\pi i}$$

6. Find the value of $\int_C \frac{dz}{(z^2+4)^3}$ where C is the circle $|z-i|=2$

Solution: The function $\frac{1}{(z^2+4)^3}$ is analytic everywhere except at the points $z^2+4=0 \Rightarrow z=\pm 2i$. Of these two points only the point $z=2i$ lies inside since $|z-i| = |2i-i| = (i) = 1 < 2$. Thus, by Cauchy's integral formula for the derivatives,

$$\int_C \frac{dz}{(z^2+4)^3} = \int_C \frac{dz/(z+2i)^3}{(z-2i)^3} = \frac{2\pi i\, f''(2i)}{2!} = \pi i\, f''(2i)$$

where $f(z) = \frac{1}{(z+2i)^3}$, $f'(z) = \frac{-3}{(z+2i)^4}$, $f''(z) = \frac{12}{(z+2i)^5}$

and so $f''(2i) = \frac{12}{(4i)^5} = \frac{3}{4^4 i}$

$$\Rightarrow \boxed{\int_C \frac{dz}{(z^2+4)^3} = \pi i \times \frac{3}{4^4 i} = \frac{3\pi}{256}}$$

TAYLOR'S SERIES :

If a function $f(z)$ is analytic inside a circle 'C' with centre at z_0 and radius R_0 (ie) $|z-z_0|<R_0$ then at each point 'z' inside C, $f(z)$ has the series representation,

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z-z_0)^n}{n!} f^{(n)}(z_0) + \cdots$$

The expansion given above is called Taylor's series expansion of $f(z)$ about the point $z = z_0$

Note: If $z_0 = 0$, we get $f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \cdots + z^n \frac{f^n(0)}{n!} + \cdots$ which is Taylor's expansion of $f(z)$ about the origin and it is known as Maclaurin's series expansion of $f(z)$.

LAURENT'S SERIES:

Let C_1 and C_2 be concentric circles of radii R_1 and R_2 with centre at z_0. If $f(z)$ is analytic throughout an annular region $R_1 < |z - z_0| < R_2$ then at each point "z" in the annular region $f(z)$ has the series representation, $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)\, dz}{(z-z_0)^{n+1}}$ $(n = 0, 1, 2, \ldots)$, $b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)\, dz}{(z-z_0)^{-n+1}}$ $(n = 1, 2, 3, \ldots)$

Note 1: $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is called the regular or analytic part of $f(z)$ and $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ is the principal part of $f(z)$

2. If the principal part is zero, Laurent's series reduces to Taylor's series.

Note 3: For getting the series, we use Binomial theorem, to obtain the series — either Taylor's or Laurent's.

$$\left.\begin{aligned}(1+x)^{-1} &= 1-x+x^2-x^3+\cdots\\(1-x)^{-1} &= 1+x+x^2+x^3+\cdots\\(1+x)^{-2} &= 1-2x+3x^2-4x^3+\cdots\\(1-x)^{-2} &= 1+2x+3x^2+4x^3+\cdots\end{aligned}\right\}\ \text{provided } |x|<1$$

PROBLEMS:

1. Expand $\cos z$ into a Taylor's series expansion about the point $z=\pi/2$.

Solution: Taylor's series about the point $z=\pi/2$ is,

$$f(z) = f(\pi/2) + (z-\pi/2)\,f'(\pi/2) + \frac{(z-\pi/2)^2}{2!}\,f''(\pi/2) + \cdots$$

$f(z) = \cos z$ $\qquad$ $f(\pi/2) = \cos\pi/2 = 0$

$f'(z) = -\sin z$ $\qquad$ $f'(\pi/2) = -\sin\pi/2 = -1$

$f''(z) = -\cos z$ $\qquad$ $f''(\pi/2) = 0$

$f'''(z) = \sin z$ $\qquad$ $f'''(\pi/2) = 1$

$f^{IV}(z) = \cos z$ $\qquad$ $f^{IV}(\pi/2) = 0$

$f^{(V)}(z) = -\sin z$ $\qquad$ $f^{(V)}(\pi/2) = -1$

$$\Rightarrow \cos z = -\frac{(z-\pi/2)}{1!} + \frac{(z-\pi/2)^3}{3!} - \frac{(z-\pi/2)^5}{5!} + \cdots$$

2. Show that when $|z+1|<1$, $z^{-2} = \sum_{n=0}^{\infty} (n+1)(z+1)^n$. (5)

Solution: Let $z+1=u$, then $z^{-2} = \frac{1}{z^2} = \frac{1}{(u-1)^2}$ and

the region becomes $|u|<1$

$$\Rightarrow \frac{1}{(u-1)^2} = \frac{1}{(1-u)^2} = (1-u)^{-2} = 1+2u+3u^2+4u^3+\cdots$$

since $|u|<1$

$$= 1+2(z+1)+3(z+1)^2+4(z+1)^3+\cdots$$

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} (n+1)(z+1)^n$$

3. Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ as a Laurent's series

in (i) $|z|<2$ (ii) $2<|z|<3$ (iii) $|z|>3$.

Solution:

Since the numerator degree and denominator degree are equal, dividing the numerator by denominator

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{(-5z-7)}{(z+2)(z+3)}.$$ Now applying

partial fractions, $\frac{-5z-7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

Solving we get $A=3$, $B=-8$.

$$\Rightarrow \frac{z^2-1}{(z+2)(z+3)} = 1+\frac{3}{z+2} - \frac{8}{z+3}$$

(i) $|z|<2 \Rightarrow |z|<3$

$$\frac{z^2-1}{(z+2)(z+3)} = 1+\frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+z/3\right)}$$

$$\frac{z^2-1}{(z+2)(z+3)} = 1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1+\frac{3}{2}\left[1-\frac{z}{2}+\left(\frac{z}{2}\right)^2-\left(\frac{z}{2}\right)^3+\cdots\right] - \frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2 - \cdots\right]$$

$$= -\frac{1}{6} + z\left(\frac{-3}{2^2}+\frac{8}{3^2}\right) + z^2\left(\frac{3}{2^3}-\frac{8}{3^3}\right) + z^3\left(\frac{-3}{2^4}+\frac{8}{3^4}\right) + \cdots$$

(ii) $2<|z|<3$ $\quad \left[\Rightarrow |z|>2 \ \& \ |z|<3\right]$

$$\frac{z^2-1}{(z+2)(z+3)} = 1+\frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3(1+z/3)}$$

$$= 1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right] - \frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right]$$

(iii) $|z|>3$ $\quad \left[\Rightarrow |z|>2\right]$

$$\frac{z^2-1}{(z+2)(z+3)} = 1+\frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right] - \frac{8}{z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\cdots\right]$$

$$= 1+\frac{1}{z}(3-8)+\frac{1}{z^2}(-3\cdot 2+8\cdot 3)+\frac{1}{z^3}(3\cdot 2^2-8\cdot 3^2)+\frac{1}{z^4}(-3\cdot 2^3+8\cdot 3^3)+\cdots$$

DEFINITIONS.

Zero of a function: A point $z = z_0$ is said to be a zero of $f(z)$ if $f(z_0) = 0$. z_0 is said to be a zero of $f(z)$ of order 'm' if $f(z) = (z-z_0)^m \phi(z)$ where $\phi(z_0) \neq 0$.

Singular Point: The point $z = z_0$ is said to be a singular point of a function $f(z)$ if $f(z)$ is not analytic at z_0.

eg (1) $\frac{1}{z(z+2)}$ has singular points at $z=0$ & $z=-2$.

(2) $\frac{1}{\sin z}$ has singular points at $z = \pm n\pi$ $(n = 0, 1, 2, \ldots)$

Isolated singularity: If $z = z_0$ is a singular point of the function $f(z)$ and if there exists a neighbourhood of z_0 containing no other singular points of $f(z)$ then the point $z = z_0$ is said to be an isolated singularity of $f(z)$

eg: $\frac{1}{z(z-1)}$ has singular points $z=0$ & $z=1$ which are isolated singularities.

Removable Singularity:

Let $z = z_0$ be an isolated singularity of $f(z)$, then $z = z_0$ is called a removable singularity if $f(z)$ has no negative powers of $z - z_0$ in the Laurent's series expansion of $f(z)$ about the point $z = z_0$.

(ie) $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$

eg, $f(z) = \frac{\sin z}{z} = \frac{1}{z}\left\{ z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \right\} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$

which has no negative powers of $z - 0$ where $z=0$ is the singular point.

Pole:

Let $z=z_0$ be an isolated singularity of $f(z)$. Then $z=z_0$ is called a pole if $f(z)$ has finite number of negative powers of $z-z_0$ in the Laurent's series expansion of $f(z)$ about the point $z=z_0$.

(ie) $$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \cdots + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

where $b_m \neq 0$.

Here the pole $z=z_0$ is said to be a pole of order 'm' or a pole of multiplicity 'm'.

Note: If $m=1$, then z_0 is a pole of order one or simple pole
If $m=2$, then z_0 is a pole of order two or a double pole and so on.

eg

$f(z) = \frac{e^{2z}}{(z-1)^3}$. Here $z=1$ is a singular point.

$$\therefore \frac{e^{2z}}{(z-1)^3} = \frac{e^{2(z-1+1)}}{(z-1)^3} = \frac{e^{2(z-1)}\cdot e^2}{(z-1)^3} = \frac{e^2}{(z-1)^3}\left\{1 + 2(z-1) + \frac{[2(z-1)]^2}{2!} + \cdots\right\}$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{(z-1)} + \cdots$$ which implies that $z=1$ is a pole of order '3'.

Essential Singularity:

Let $z=z_0$ be an isolated singularity of $f(z)$. Then $z=z_0$ is called an essential singularity if $f(z)$ has infinitely many negative powers of $z-z_0$ in the Laurent's series expansion of $f(z)$ about $z=z_0$.

(iii) eg, $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n}$. Here $z=0$ is an essential singularity.

Residue of a function:

Let C be any closed positively oriented curve around z_0 then the residue of $f(z)$ at $z=z_0$ is given to be,

$$\boxed{\text{Res.}\left[f(z)\right]_{z=z_0} = \frac{1}{2\pi i}\int_C f(z)\,dz}$$

But $\text{Res.}\,f(z)\Big]_{z=z_0}$ = coefficient of $\frac{1}{z-z_0}$ in the Laurent's series expansion of $f(z)$ about $z=z_0$.

Result 1:

Residue of a function at a simple pole is $\lim_{z\to z_0}(z-z_0)f(z)$

Result 2:

Residue of a function at a pole of order 'm' is

$$\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

CAUCHY's RESIDUE THEOREM:

If a function $f(z)$ is analytic at all points inside and on a simple closed curve C except for a finite number of points $z_1, z_2, \ldots z_n$ inside C

then,

$$\boxed{\int_C f(z)\,dz = 2\pi i \times \left[\text{Sum of the residues of } f(z) \text{ at } z_1, z_2, \ldots z_n\right]}$$

RESULT: If we can find a positive integer 'n' such that

$$\lim_{z \to z_0} (z-z_0)^n f(z) \neq 0$$ then $z = z_0$ is a pole of order 'n' for $f(z)$.

PROBLEMS:

Find the poles of the following functions and the residues at each pole.

(i) $$\frac{z^2}{(z-1)(z-2)^2}$$ (ii) $$\frac{1-e^{2z}}{z^4}.$$

Solution

(i) $z = 1$ and $z = 2$ are singular points of the function $$\frac{z^2}{(z-1)(z-2)^2}.$$

Since $$\lim_{z \to 1} \cancel{(z-1)} \frac{z^2}{\cancel{(z-1)}(z-2)^2} = \frac{1}{1} \neq 0$$

$z = 1$ is a pole of order 'one'.

Since $$\lim_{z \to 2} \cancel{(z-2)^2} \frac{z^2}{(z-1)\cancel{(z-2)^2}} = \frac{4}{1} \neq 0$$

$z = 2$ is a pole of order two.

$$\text{Res. } f(z) \text{ at } z = 1 = \lim_{z \to 1} \cancel{(z-1)} \frac{z^2}{\cancel{(z-1)}(z-2)^2} = 1$$

Res $f(z)$ at $z=2$ is $\frac{1}{(2-1)!} \underset{z\to 2}{lt} \frac{d}{dz}\left[\cancel{(z-2)^2} \cdot \frac{z^2}{(z-1)\cancel{(z-2)^2}}\right]$

$$= \frac{1}{1!} \underset{z\to 2}{lt} \frac{(z-1)(2z)-z^2}{(z-1)^2} = \underset{z\to 2}{lt} \frac{z^2-2z}{(z-1)^2} = 0.$$

(ii) since $\underset{z\to 0}{lt}\ z^3 \frac{1-e^{2z}}{z^4} = \underset{z\to 0}{lt} \frac{1-e^{2z}}{z} = \underset{z\to 0}{lt} \frac{-2e^{2z}}{1} \neq 0.$

$z=0$ is a pole of order '3'.

Res. $f(z)$ at $z=0$ is $\frac{1}{(3-1)!} \underset{z\to 0}{lt} \frac{d^2}{dz^2}\left[z^3 \cdot \frac{1-e^{2z}}{z^4}\right]$

$$= \frac{1}{2!} \underset{z\to 0}{lt}\left[\frac{2-2e^{2z}+4ze^{2z}-4z^2e^{2z}}{z^3}\right] = \frac{1}{2!}\left(\frac{-16}{6}\right) = \frac{-8}{6}$$

Residue $= \frac{-4}{3}$

2. Using cauchy Residue theorem, evaluate,

$\int_C \frac{dz}{z^3(z-1)}$ where C is $|z|=2$.

<u>Solution:</u> The function $f(z) = \frac{1}{z^3(z-1)}$ has poles at $z=1$ which is a simple pole and at $z=0$ which is a pole of order '3'.

Res. $f(z)$ at $z=1$ is $\underset{z\to 1}{lt} \cancel{(z-1)} \frac{1}{z^3\cancel{(z-1)}} = 1$

Res. $f(z)$ at $z=0$ is $\frac{1}{(3-1)!} \underset{z \to 0}{\text{lt}} \frac{d^2}{dz^2}\left[\cancel{z^3} \cdot \frac{1}{\cancel{z^3}(z-1)}\right]$

$$= \frac{1}{2!} \underset{z \to 0}{\text{lt}} \frac{2}{(z-1)^3} = \frac{1}{2} \cdot (-2)$$

$$= -1$$

$\therefore \int_C \frac{dz}{z^3(z-1)} = 2\pi i \times$ [Res. of $f(z)$ at $z=0$ + Res. of $f(z)$ at $z=1$] by Cauchy Residue theorem,

$$\Rightarrow \int_C \frac{dz}{z^3(z-1)} = 2\pi i \times [-1+1] = 0.$$

$$\Rightarrow \boxed{\int_C \frac{dz}{z^3(z-1)} = 0.}$$