SMT1201 ENGINEERING MATEHEMATICS - III

(Common to ALL branches except BIO GROUPS, CSE & IT)

II YEAR III SEMISTER 2015 BATCH ONWARDS

Unit III

FOURIER TRANSFORMS

The infinite Fourier transform - Sine and Cosine transform - Properties - Inversion theorem - Convolution theorem - Parseval's identity - Finite Fourier sine and cosine transform.

Fourier Transform

Complex Fourier Transform (Infinite)

Let f(x) be a function defined (-∞,∞) and be piece-wise continuous in each finite partial interval then the complex Fourier transform of f(x) is defined by

$$\mathcal{F}\{f(x)\} = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

Inverse Fourier Transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

Properties of Fourier Transform

1. Linear Property

F{af(x) + bg(x)} = a F{f(x)} + b F{g(x)} where F is the Fourier transform

2. Shifting Theorem

If $\mathcal{F}\{f(x)\} = F(s) then\, \mathcal{F}\{f(x-a)\} = e^{isa}F(s)$

3. Change of Scale property

$$\mathcal{F}\{f(x)\} = F(s)\ then\ \ \mathcal{F}\{f(ax)\} = \frac{1}{|a|}\,F\left(\frac{s}{a}\right)\ where\ a\ \neq 0$$

4. Modulation Theorem

$$\mathcal{F}\{f(x)\} = F(s) then\, \mathcal{F}\{f(x)cosax\} = \frac{1}{2}[F(s-a) + F(s+a)]$$

Convolution Theorem

Definition

The convolution of two functions f(x) and g(x) is defined as

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt$$

Fourier Transform of Convolution of two functions

The Fourier Transform of Convolution of f(x) and g(x) is the product of their Fourier Transform

$$f(x) * g(x) = F(s) \,.\, G(s) = \mathcal{F}\{f(x)\} \,.\, \mathcal{F}\{g(x)\}$$

Perseval's Identity

If F(s) is the Fourier transform of f(x) then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

Infinite Fourier Cosine Transform

$$\mathcal{F}_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) Cos\ sx\ dx$$

Inverse Fourier Cosine Transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) Cos\ xs\ ds$$

Infinite Fourier Sine Transform

$$\mathcal{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\ Sin\ sx\ dx$$

Inverse Fourier Sine Transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\ Sin\ xs\ ds$$

Properties of Fourier Sine and Cosine Transforms

1 Linear Property

$$\mathcal{F}_c\{af(x) \pm b\ g(x)\} = a\mathcal{F}_c\{f(x)\} \pm b\mathcal{F}_c\{g(x)\}$$

$$\mathcal{F}_s\{af(x) \pm b\ g(x)\} = a\mathcal{F}_s\{f(x)\} \pm b\mathcal{F}_s\{g(x)\}$$

2 Modulation Property

$$\mathcal{F}_s\{f(x)Sinax\} = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$$

$$\mathcal{F}_s\{f(x)Cosax\} = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$$

$$\mathcal{F}_c\{f(x)Sinax\} = \frac{1}{2}[F_s(a+s) + F_s(a-s)]$$

$$\mathcal{F}_c\{f(x)Cosax\} = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

3 $\mathcal{F}_c\{f(ax)\} = \frac{1}{a}\, F_c\left(\frac{s}{a}\right)$

4 $\mathcal{F}_s\{f(ax)\} = \frac{1}{a}\, F_s\left(\frac{s}{a}\right)$

5 Identities

If F_C (s) and G_c(s) are the Fourier cosine transforms and F_s(s) and G_s(s) are the transforms of f(x) and g(x) respectively then

i) $\int_{-\infty}^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} F_c(s)\, G_c(s)ds$

ii) $\int_{-\infty}^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} F_s(s)\, G_s(s)ds$

iii) $\int_0^{\infty} |f(x)|^2 dx = \int_0^{\infty} |F_c(s)|^2 ds = \int_0^{\infty} |F_s(s)|^2 ds$

PROBLEMS

Problem 1 If the Fourier transform of $f(x)$ is $F(s)$ then, what is Fourier transform of $f(ax)$?

Solution:

Fourier transform of $f(x)$ is

$$F(s)=F(f(x))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F(f(ax))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{isx}dx$$

Put $t=ax$

$dt=adx$

$$F(f(ax))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{is(t/a)}\frac{dt}{a}$$

$$=\frac{1}{a}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist/a}dt$$

$$=F(f(ax))=\frac{1}{a}.F\left(\frac{s}{a}\right).$$

Problem 2 Find the Fourier sine transform of e^{-3x}.

Solution:

$$F_s(f(x))=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin sx\ dx$$

$$F_s(e^{-3x})=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-3x}\sin sx\ dx$$

$$=\sqrt{\frac{2}{\pi}}\left\{\frac{e^{-3x}}{s^2+9}(-3\sin sx-s\cos x)\right\}_0^{\infty}$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+9}\right)\left[\because\int e^{ax}\sin bx\ dx=\frac{e^{ax}}{a^2+b^2}[a\sin bx-b\cos bx]\right].$$

Problem 3 Find the Fourier sine transform of $f(x)=e^{-ax}$, $a>0$. Hence deduce that $\int_{0}^{\infty}\frac{x\sin\alpha x}{1+x^2}dx=\frac{\pi}{2}e^{-\alpha}$.

Solution:

$$F_s(f(x))=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin sx\,dx$$

$$F_s(e^{-ax})=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-ax}\sin sx\ dx$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

By inverse Sine transform, we get

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_s(s)\sin sx\ ds$$

$$=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\sin sx\ ds$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{s\sin sx}{s^2+a^2}ds$$

$$\frac{\pi}{2}f(x) = \int_0^\infty \frac{s\sin sx}{s^2+a^2}ds$$

$$\frac{\pi}{2}e^{-ax} = \int_0^\infty \frac{s\sin sx}{s^2+a^2}ds$$

Put $a=1, \quad x=\alpha$

$$\frac{\pi}{2}e^{-\alpha} = \int_0^\infty \frac{s\sin sx}{s^2+1}ds$$

Replace 's' by 'x'

$$\int_0^\infty \frac{s\sin sx}{1+x^2}dx = \frac{\pi}{2}e^{-\alpha}.$$

Problem 4 Prove that $F_C\left[f(x)\cos ax\right] = \frac{1}{2}\left[F_C(s+a)+F_C(s-a)\right]$.

Solution:

$$F_c(s) = F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\ dx$$

$$F_c\left[f(x)\cos ax\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\left[\frac{\cos(a+s)x+\cos(a-s)x}{2}\right]dx$$

$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx\right\} + \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx\right\}$$

$$= \frac{1}{2}\left[F_c(s+a)+F_c(s-a)\right].$$

Problem 5 Find the Fourier cosine transform of $f(x) = \begin{cases}\cos x, & 0<x<a\\ 0, & x\geq a\end{cases}$.

Solution:

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^a \cos x\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a\left[\frac{\cos(s+1)x+\cos(s-1)x}{2}\right]dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)x}{s+1}+\frac{\sin(s-1)x}{s-1}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)a}{s+1}+\frac{\sin(s-1)a}{s-1}\right], \text{ provided } S\neq 1,\ S\neq -1.$$

Problem 6 Find $F_C\left(xe^{-ax}\right)$ and $F_S\left(xe^{-ax}\right)$.

Solution:

$$F_c\left(xe^{-ax}\right) = \frac{d}{ds}F_s\left[f(x)\right]$$

$$F_c\left(xe^{-ax}\right) = \frac{d}{ds}F_s\left[e^{-ax}\right]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx\right]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{\left(s^2+a^2\right)^2}\right].$$

$$F_s\left[xe^{-ax}\right] = -\frac{d}{ds}\left[F_c e^{-ax}\right]\left(\therefore F_s\left(xf(x)\right) = -\frac{d}{ds}\left(F_c\left(f(x)\right)\right)\right)$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx\right]$$

$$= -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{2as}{\left(s^2+a^2\right)^2}\right].$$

Problem 7 If $F(s)$ is the Fourier transform of $f(x)$, then prove that the Fourier transform of $e^{ax} f(x)$ is $F(s+a)$.

Solution:

$$F(s) = F\left(f(x)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left(e^{iax}f(x)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(a+s)x} f(x)dx$$

$$= F(s+a).$$

Problem 8 Find the Fourier cosine transform of $e^{-2x} + 3e^{-x}$.

Solution:

Let $f(x) = e^{-2x} + 3e^{-x}$

$$F_c\left(f(x)\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[e^{-2x}+3e^{-x}\right] = \sqrt{\frac{2}{\pi}}\left\{\int_0^\infty e^{-2x}\cos sx\,dx + \int_0^\infty 3e^{-x}\cos sx\,dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2}{s^2+4} + \frac{3}{s^2+1}\right].$$

Problem 9 State convolution theorem.

Solution:

If F(s) and G(s) are Fourier transform of $f(x)$ and $g(x)$ respectively, Then the Fourier transform of the convolutions of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

i.e. $F\left[f(x) * g(x)\right] = F\left[f(x)\right]F\left[g(x)\right]$

Problem 10 Derive the relation between Fourier transform and Laplace transform.

Solution:

Consider $f(t)\begin{cases} e^{-xt}g(t), & t>0 \\ 0 \quad , & t<0 \end{cases} \qquad -(1)$

The Fourier transform of $f(x)$ is given by

$$F\left[f(t)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-xt}g(t)e^{ist}dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(is-x)t} g(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-pt} g(t)\, dt \text{ where } p = x - is$$

$$= \frac{1}{\sqrt{2\pi}} L(g(t)) \left[\therefore L\left(f(t) = \int_{0}^{\infty} e^{-st} f(t)\, dt \right) \right]$$

$\therefore$ Fourier transform of $f(t) = \frac{1}{\sqrt{2\pi}} \times$ Laplace transform of g(t) where g(t) is defined by (1).

Problem 11 Find the Fourier sine transform of $\frac{1}{x}$.

Solution:

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$$

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{1}{x} \sin sx\, dx$$

Let $sx = \theta$

$sdx = d\theta;\ \theta : 0 \to \infty$

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{s}{\theta} \sin\theta \frac{d\theta}{s}$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{\sin\theta}{\theta}\, d\theta \left[\therefore \int_{0}^{\infty} \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}}.$$

Problem 12 Find $f(x)$ if its sine transform is $e^{-as}, a > 0$.

Solution:

$$F_s(f(x)) = F(s)$$

Given that $F_s(f(x)) = e^{-as}$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(s) \sin x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-as} \sin sx\, ds$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-as}}{a^2 + s^2} (-a \sin sx - x \cos sx) \right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{x}{a^2 + x^2} \right).$$

Problem 13 Using Parseval's Theorem find the value of $\int_{0}^{\infty} \frac{x^2}{(x^2+a^2)^2} dx, a > 0$. Find the Fourier transform of $e^{-a|x|}, a > 0$.

Solution:

Parseval's identify is $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$

Result : $F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + a^2} \right]$

$$\int_{-\infty}^{\infty} \left|\left(e^{-ax}\right)\right|^2 dx = \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right)^2 ds$$

$$2\int_{0}^{\infty} \left(e^{-ax}\right)^2 dx = 2\int_{0}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right)^2 ds$$

$$\left(\frac{e^{-2ax}}{-2a}\right)_0^{\infty} = \frac{2}{\pi}\int_0^{\infty} \frac{s^2}{\left(s^2+a^2\right)^2} ds$$

$$i.e., \int_0^{\infty} \frac{s^2}{\left(s^2+a^2\right)^2} ds = \frac{\pi}{2}\left(\frac{0+1}{2a}\right) = \frac{\pi}{4a}.$$

$$\therefore \int_0^{\infty} \frac{x^2}{\left(x^2+a^2\right)^2} dx = \frac{\pi}{4a}.$$

Problem 14 Find the Fourier sine transform of $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$.

Solution:

The Fourier sine transform of $f(x)$ is given by $F_s\left(f(x)\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^1 \sin sx \, dx + \int_1^{\infty} 0 \sin sx \, dx \right\} = \sqrt{\frac{2}{\pi}} \left[\frac{-\cos sx}{s}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left\{\frac{-\cos s}{s} + \frac{1}{s}\right\} = \sqrt{\frac{2}{\pi}} \left[\frac{1}{s} - \frac{\cos s}{s}\right].$$

Problem 15 Find the Fourier transform of $e^{-a|x|}$, a > 0

Solution:

$$F\left(f(x)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \left(\cos sx + i \sin sx\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \cos sx \, dx \quad \left[\therefore \int_{-\infty}^{\infty} e^{-a|x|} \sin sx \, dx = 0, \text{ odd function}\right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \cos sx \, dx$$

$$F\left(e^{-a|x|}\right) = \frac{2}{\sqrt{2\pi}} \left(\frac{a}{a^2+s^2}\right).$$

$$F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a}0\,dx+\int_{-a}^{-a}\left(a-|x|\right)e^{isx}dx+\int_{a}^{\infty}0\,dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{-a}\left(a-|x|\right)\left(\cos sx+i\sin sx\right)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{-a}\left(a-|x|\right)\cos sx\,dx+0 \qquad \left[\therefore \int a\sin sx\;\&\;|x|\sin sx \text{ are } odd\;\; functions\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[2\int_{0}^{a}\left(a-x\right)\cos sx\;dx\right]$$

$$f(x)=\frac{2}{\sqrt{2\pi}}\left[\left(a-x\right)\left(\frac{\sin sx}{s}\right)-\left(-1\right)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[0-\frac{\cos sx}{s^2}+\frac{1}{s^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos as}{s^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2\frac{as}{2}}{s^2}\right] \qquad -(1)$$

By inverse Fourier transform $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)\;e^{-isx}\;ds$.

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2\frac{as}{2}}{s^2}\right]e^{-isx}ds \text{ Put } x=0$$

$$f(0)=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2\frac{as}{2}}{s^2}ds$$

$$\frac{\pi a}{4}=\int_0^{\infty}\frac{\sin^2\left(\frac{as}{2}\right)}{s^2}ds$$

Put $a=2$

$$\frac{\pi}{2}=\int_0^{\infty}\frac{\sin^2 s}{s^2}ds$$ [$\therefore$ s is a dummy variable, we can replace it by ‘t’]

i.e $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$.

Problem 17 **(i)** Prove that $e^{\frac{-x^2}{2}}$ is self – reciprocal with respect to Fourier transform.

(ii) Find the Fourier transform of $f(x)=\begin{cases}1, & |x|\le a\\ 0, & |x|>a\end{cases}$. Hence evaluate $\int_0^{\infty}\frac{\sin s}{s}ds$.

Solution:

(i) $f(x)=e^{-x^2/2}$

$F(s)=F\left(f(x)\right)$

$$F(s) = F(f(x)) = \frac{1}{\sqrt{2\pi}} f(x) e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2 + isx + \frac{i^2 s^2}{2} - \frac{i^2 s^2}{2}}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x - is)^2}{2}} e^{\frac{-s^2}{2}}\ dx$$

Let $y = \frac{x - is}{\sqrt{2}}$ $\quad x = \infty \Rightarrow y = \infty$

$dx = \sqrt{2} dy$ $\quad x = -\infty \Rightarrow y = -\infty$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-y^2} e^{-s^2/2} \sqrt{2}\ dy$$

$$= \frac{2e^{-s^2/2}}{\sqrt{\pi}} \int_0^{\infty} e^{-y^2}\ dy$$

$$= \frac{2e^{-s^2/2}}{\sqrt{\pi}} \int_0^{\infty} e^{-y^2}\ dy = \frac{2e^{-s^2/2}}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} \qquad \left[\int_0^{\infty} e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2} \right]$$

$F(s) = e^{-s^2/2}$ *i.e.* $e^{-x^2/2}$ is self reciprocal hence proved.

(ii). Fourier transform of $f(x)$ is

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} = \int_{-a}^{a} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} = \int_{-a}^{a} (\cos sx - i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} = \int_{0}^{a} \cos sx\, dx \qquad (\therefore \sin sx \text{ is } an\, odd\ fn.)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx\ dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{\sin\ sx}{s} \right]_0^a$$

$$F(s) = \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{\sin\ as}{s} \right]$$

By inverse Fourier transforms,

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin as}{s}(\cos sx-i\sin sx)dx$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin as}{s}\cos sx\,dx-(0)\left[\because\frac{\sin as}{s}\sin sx \text{ is } odd\right]$$

$$=\frac{2}{\pi}\int_{0}^{\infty}\left(\frac{\sin as}{s}\right)\cos sx\,ds$$

$put\;\; a=1,\;\; x=0$

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin s}{s}ds$$

$$\frac{\pi}{2}\times 1=\int_{0}^{\infty}\frac{\sin s}{s}ds\quad(\because f(x)=1,-a\le x\le a)$$

$$\therefore\int_{0}^{\infty}\frac{\sin s}{s}ds=\frac{\pi}{2}.$$

Problem 18. Find the Fourier cosine transform of $f(x)$ defined as

$$f(x)=\begin{cases} x \;\; for & 0<x<1 \\ 2-x \;\; for & 1<x<2 \\ 0 \;\; for & x>2 \end{cases}$$

Solution. By definition of Fourier Transform

$$\begin{aligned} F[f(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\;dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_0^1 x\cos sx\;dx+\int_1^2(2-x)\cos sx\;dx\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[\left(x\frac{\sin sx}{s}-\frac{\cos sx}{s^2}\right)_0^1+\left((2-x)\frac{\sin sx}{s}+\frac{\cos sx}{s^2}\right)_1^2\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[\left(\frac{\sin s}{s}-0-\frac{\cos s-\cos 0}{s^2}\right)+\left(0-(1)\frac{\sin s}{s}+\frac{\cos 2s-\cos s}{s^2}\right)\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{\cos 2s-2\cos s+1}{s^2}\right] \end{aligned}$$

Problem 19

Find the Fourier transform of $f(x) = \begin{cases} e^{ikx}, & a < x < b; \\ 0, & x < a. \text{ and } x > b \end{cases}$

Solution.

$$\begin{aligned} F[f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{ikx} e^{isx}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{i(k+s)x}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(k+s)x}}{i(k+s)} \right]_a^b \\ &= \frac{i}{(k+s)\sqrt{2\pi}} \left[e^{i(k+s)b} - e^{i(k+s)a} \right] \end{aligned}$$

Problem 20. State and Prove convolution theorem on Fourier transforms

Statement: The Fourier transforms of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

$$F\left(f(x) * g(x) \right) = F\left[f(x) \right] F\left[g(x) \right]$$

Proof:

$$F(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t)\, dt \;\; e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t) e^{isx}\, dx\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} g(x-t) e^{isx}\, dx \right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) F\left(g(x-t) \right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} F\left(g(t) \right) dt \qquad \left[\therefore f\left(g(x-t) \right) = e^{ist} F\left(g(t) \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist}\, dt\, G(s) \qquad \left[\because F\left(g(t) \right) = G(s) \right]$$

$$= F(f*g) = F(s).G(s). \qquad \left[\because F\left(f(t) \right) = F(s) \right].$$

Problem 21

Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| \ge a \end{cases}$ and hence evaluate

(i) $\int_0^{\infty} \left(\frac{\sin t - t\cos t}{t^3} \right) dt$ **(ii)** $\int_0^{\infty} \left(\frac{\sin t - t\cos t}{t^3} \right)^2 dt$

Solutions

Fourier transform of $f(x)$ is

$$F(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 + \int_{-a}^{a}(a^2 - x^2)e^{isx}dx + 0\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}(a^2 - x^2)(\cos sx + i\sin x)dx\right]$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{a}(a^2 - x^2)\cos sx\,dx \left[\therefore (a^2 - x^2)\sin sx \text{ is } an\ odd\ fn.\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[(a^2 - x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{2a\cos as}{s^2} + \frac{2a\sin as}{s^3}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{2a\cos as + 2\sin as}{s^3}\right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right] \qquad -(1)$$

By inverse Fourier transforms,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}dx$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)(\cos sx - i\sin sx)ds$$

$$f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin as - as\cos as}{s^3}\cos sx\,dx \qquad (\therefore \text{ the second terms is on odd function })$$

Put $a = 1$

$$f(x) = \frac{2}{\pi}\times 2\int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}\cos sx\,ds \qquad \left[f(x) = \begin{cases}1 - x^2, & |x| \le 1\\ 0, & |x| \ge 1\end{cases}\right]$$

Put $x = 0$

$$f(0) = \frac{4}{\pi}\int_{0}^{\infty}\frac{\sin s - s\cos s}{s^3}ds \qquad \left[\begin{aligned} f(0) &= 1 - 0\\ &= 1\end{aligned}\right]$$

$$1 = \frac{4}{\pi}\int_{0}^{\infty}\frac{\sin s - s\cos s}{s^3}ds$$

$$\frac{\pi}{4} = \int_{0}^{\infty}\frac{\text{sint - t cost}}{t^3}\text{ dt}.$$

Hence (i) is proved. Using Parseval's identify

$$\int_{-\infty}^{\infty}|F(s)|^2 ds = \int_{-\infty}^{\infty}|f(x)|^2\ dx$$

$$\int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)\right]^2 ds = \int_{-a}^{a}\left|a^2 - x^2\right| dx$$

$$\int_{-\infty}^{\infty}\frac{8}{\pi}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \int_{-1}^{1}\left(1-x^2\right)^2 dx$$

$$2\times\frac{8}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_{0}^{1}\left(1-x\right)^2 dx$$

$$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x + \frac{x^5}{4} - \frac{2x^3}{3}\right]_0^1$$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{16}\times 2\left(\frac{8}{15}\right) = \frac{\pi}{15}$$

Put $a = 1$

Put s = t

$$\int_{0}^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}.$$ Hence (ii) is proved.

Problem: 22

Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$ and hence find the value of

(i) $\int_{0}^{\infty}\frac{\sin^2 t}{t^2}dt$. **(ii)** $\int_{0}^{\infty}\frac{\sin^4 t}{t^4}dt$.

Solution:

The Fourier transform of $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{is}\ dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx\, dx \left[\because (1-|x|) \sin sx \text{ is an odd } fn.\right]$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ (1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{\cos sx}{s^2}\right) \right\}_0^1$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ -\frac{\cos s}{s^2} + \frac{1}{s^2} \right\}$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right] - (1)$$

(i) By inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right] (\cos ssx - \mathrm{i} \sin sx) (by\ (1))$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2} \right) \cos sx\, ds \ \ (\therefore \text{Second term is odd})$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\frac{1-\cos s}{s^2} \right) \cos sx\, ds$$

$$Put x = 0$$

$$1-|0| = \frac{2}{\pi} \int_0^{\infty} \left(\frac{1-\cos s}{s^2} \right) ds$$

$$\int_0^{\infty} \left(\frac{1-\cos s}{s^2} \right) ds = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{2\sin^2(s/2)}{s^2} ds = \frac{\pi}{2}$$

$$put \quad t = s/2 \qquad ds = 2dt$$

$$\int_0^{\infty} \frac{2\sin^2 t}{(2t)^2} 2dt = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}.$$

(ii) Using Parseval's identity.

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}} \left(\frac{1-\cos s}{s^2} \right) \right]^2 ds = \int_{-1}^{1} (1-|x|)^2 dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2} \right)^2 ds = \int_{-1}^{1} (1-|x|)^2 dx$$

$$\frac{4}{\pi} \int_0^{\infty} \left(\frac{1-\cos s}{s^2} \right)^2 ds = 2\int_0^{1} (1-x)^2 dx$$

$$\frac{4}{\pi} \int_0^{\infty} \left(\frac{2\sin^2\left(\frac{s}{2}\right)}{s^2} \right)^2 ds = \left[2\left(\frac{1-x}{-3} \right)^3 \right]_0^1$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin^2\left(\frac{s}{2}\right)}{s^2}\right)^4 ds = \frac{2}{3}; Let\, t = s/2, dt = \frac{ds}{2}$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin t}{2t}\right)^4 2\,dt = \frac{2}{3}$$

$$\frac{16}{16\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{1}{3}$$

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$

Problem 23 **(i)** Find the Fourier sine transform of $e^{-|x|}$. Hence prove that $\int_0^\infty \left(\frac{x\sin\alpha x}{1+x^2}\right)dx = \frac{\pi}{2}e^{-\alpha}$, $\alpha > 0$.

(ii) Find the Fourier sine transform of $e^{-ax}\,(a>0)$. Hence find (a) $F_C\left(xe^{-ax}\right)$ and (b) $F_S\left(\frac{e^{-ax}}{x}\right)$.

Solution:

(i) $F_s\left(f(x)\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$F_s\left(e^{-|x|}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-|x|}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left(\frac{s}{1+s^2}\right)\sin sx\,dx$$

Result: $\int_0^\infty e^{ax}\sin bx dx = \frac{b}{a^2+b^2}$

By Fourier sine inversion formula, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin\ sx\,ds$$

$$e^{-x} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{s}{1+s^2}\right)\sin\ sx\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{s\sin sx}{s^2+1}ds$$

$$\int_0^\infty \frac{s\sin sx}{s^2+1}ds = \frac{\pi}{2}e^{-x} \qquad put\ x = a$$

$$\int_0^\infty \frac{s\sin sa}{1+s^2}ds = \frac{\pi}{2}e^{-a}$$

Replace S by x

$$\int_0^\infty \frac{x\sin ax}{1+x^2}dx = \frac{\pi}{2}e^{-a}.$$

(ii) . $F_s\left(f(x)\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$F_s\left(e^{-ax}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left\{\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right\}=\sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right).......(1)$$

By Property

$$F_s\left[x\ f(x)\right]=-\frac{d}{ds}\left[F_c\left(f(x)\right)\right]$$

$$F_c\left[x\ f(x)\right]=\frac{d}{ds}F_s\left(f(x)\right)$$

(a) To Find $F_c\left[xe^{-ax}\right]$

$$F_c\left[xe^{-ax}\right]=\frac{d}{ds}F_s\left(e^{-ax}\right)$$

$$=\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{a^2+s^2-2s^2}{\left(a^2+s^2\right)^2}\right]$$

$$F_c\left[x.f(x)\right]=\sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{\left(a^2+s^2\right)^2}\right]$$

(b) To find $F_s\left[\frac{e^{-ax}}{x}\right]$

$$F_s\left(f(x)\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-at}}{t}\sin st\ dt \qquad -(1)$$

$$F(s) \qquad =\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-at}}{t}\sin st\ dt$$

Diff. on both sides w.r to ‘s’ we get

$$\frac{d}{ds}\left(F(s)\right)=\frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-at}}{t}\sin st\ dt \quad \left[\therefore \int_0^\infty e^{-ax}\cos bx\ dx=\frac{a}{b^2+a^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\partial}{\partial s}\left\{\frac{e^{-at}}{t}\sin st\right\}dt$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{te^{-at}\cos st}{t}dt$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-at}\cos st\ \ dt$$

$$\frac{d}{ds}F(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-at}\cos st\ \ dt \quad =\sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

Integrating w.r. to ‘s’ we get

$$F(s)=\sqrt{\frac{2}{\pi}}\int\frac{a}{s^2+a^2}ds+c$$

$$=\sqrt{\frac{2}{\pi}}\,a.\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)+c$$

But $F(s)=0$ When $s=0 \therefore c=0$ from (1)

$$\therefore F(s)=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right).$$

Problem 24 **(i)** Find the Fourier transform of $e^{-a^2x^2}$ Hence prove that $e^{\frac{-x^2}{2}}$ is self reciprocal with respect to Fourier Transforms.

(ii) Find the Fourier cosine transform of x^{n-1}. Hence deduce that $\frac{1}{\sqrt{x}}$ is self-reciprocal under cosine transform. Also find $F\left(\frac{1}{\sqrt{|x|}}\right)$.

Solution:

(i) $F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{is}\,dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(a^2x^2)+isx}\,dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(a^2x^2-isx)}\,dx \qquad -(1)$

Consider $a^2x^2 - isx$

$= (ax)^2 - 2(ax)\frac{(is)}{2a} + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2$

$= \left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2} \qquad -(2)$

Sub: (2) in (1) ,We get

$F\left[f(x)\right] = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{-\left[\left(ax-\frac{is}{2a}\right)+\frac{s^2}{4a^2}\right]}dx$

$= \sqrt{\frac{1}{2\pi}}\, e^{-\frac{s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2}dx$

$= \sqrt{\frac{1}{2\pi}}\, e^{-\frac{s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-t^2}\frac{dt}{a} \qquad Let\ t = ax - \frac{is}{2a},\ dt = adx$

$F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2\pi}}e^{-\frac{s^2}{4a^2}}\sqrt{\pi} \qquad \left[\therefore \int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}\right]$

$= \frac{1}{a\sqrt{2}}e^{-\frac{s^2}{4a^2}} \qquad -(3)$

$Put\ a = \frac{1}{\sqrt{2}}\ in\ (3)$

$F\left[e^{-x^2/2}\right] = e^{-S^2/2}$

$\therefore e^{-S^2/2}$ is self reciprocal with respect to Fourier Transforms.

(ii). $F_c\left(f(x)\right) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx$

$F_c\left(x^{n-1}\right) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} x^{n-1}\cos sx\ dx \qquad -(1)$

We know that $\Gamma(n) = \int_0^{\infty} e^{-y}y^{n-1}dy$

Put $y = ax$, we get $\int_0^{\infty} e^{-ax}(ax)^{n-1}adx = \Gamma(n)$

$$\int_0^\infty e^{-ax}x^{n-1}dx = \frac{\Gamma(n)}{a^n}$$

Put a = is

$$\therefore \int_0^\infty e^{-\text{is}\,x}x^{n-1}dx = \frac{\Gamma(n)}{(\text{is})^n}$$

$$\int_0^\infty (\cos sx - i\sin sx)\,x^{n-1}dx = \frac{\Gamma(n)}{s^n}i^{-n}$$

$$= \frac{\Gamma(n)}{s^n}\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right)^{-n}$$

$$= \frac{\Gamma(n)}{s^n}\left[\cos\frac{n\pi}{2} - \text{i}\sin\frac{n\pi}{2}\right]$$

Equating real parts, we get

$$\int_0^\infty x^{n-1}\cos sx dx = \frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \quad -(2)$$

Using this in (1) we get

$$F_c\left(x^{n-1}\right) = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2}$$

Put $n = \frac{1}{2}$

$$F_c\left(\frac{1}{\sqrt{x}}\right) = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}\cos\frac{\pi}{4}$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{s}}\,\frac{1}{\sqrt{2}} \qquad \left[\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{1}{\sqrt{s}}$$

Hence $\frac{1}{\sqrt{x}}$ is self-reciprocal under Fourier cosine transform

To find $F\left\{\frac{1}{\sqrt{1x1}}\right\}$

$$F\left\{\frac{1}{\sqrt{|x|}}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e\frac{\text{is}\,x}{\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{1}{\sqrt{x}}(\cos sx + \text{is}\,insx)\,dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{x}}\cos sx\,dx \quad [\therefore The \sec ond\; term\, odd]$$

Put n =1/2 in (2), we get

$$\int_0^\infty \frac{\cos sx}{\sqrt{x}}\,dx = \frac{\Gamma 1/2}{\sqrt{s}}\cos\frac{\pi}{4}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\,\frac{1}{\sqrt{2}} = \frac{\sqrt{\pi}}{\sqrt{2s}}$$

$$\therefore F\left\{\frac{1}{\sqrt{|x|}}\right\} = \sqrt{\frac{2}{\pi}} \times \frac{\sqrt{\pi}}{\sqrt{2s}} = \frac{1}{\sqrt{s}}.$$

Problem 25 **(i)** Find $f(x)$ if its Fourier sine Transform is $\frac{e^{-as}}{s}$.

(ii) Using Parseval's Identify for Fourier cosine and sine transforms of e^{-ax}, evaluate

(a). $\int_0^\infty \frac{1}{(a^2+x^2)^2}dx$ **(b).** $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}dx$

Solution:

(i) Let $F_s(f(x)) = \frac{e^{-as}}{s}$

Then $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx\ dx \qquad -(1)$

$\therefore \frac{df}{dx} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\cos sx\, ds = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+x^2}$

$\therefore F(x) = \sqrt{\frac{2}{\pi}}a\int \frac{dx}{a^2+x^2}$

$= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)+c \qquad -(2)$

At $x=0$, $f(0)=0$ using (1)

(2) => $f(0) = \sqrt{\frac{2}{\pi}}\tan^{-1}(0)+c \quad \therefore c=0$

Hence $f(x) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)$.

(ii) (a) To find $\int_0^\infty \frac{dx}{(a^2+x^2)^2}$

$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\, dx$

$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$

$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right) \qquad -(1)$

By Parseval's identify.

$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(s)|^2 ds$

$\int_0^\infty e^{-2ax}dx = \int_0^\infty\left[\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\right]^2 ds, \text{ from } (1)$

$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}a^2\int_0^\infty \frac{ds}{(a^2+s^2)^2}$

$\frac{1}{2a} = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{a^2+s^2}$

i.e $\int_0^\infty \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3}$ [Replace s by x]

(b) To find $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\ dx$

$$F_s\left(e^{-ax}\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}\left(-a\sin sx-s\cos sx\right)\right]_0^{\infty}$$

$$F_s\left(e^{-ax}\right)=\sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)$$

By parseval's identify

$$\int_0^{\infty}\left|f(x)^2\right|dx=\int_0^{\infty}\left|F_s\left(f(x)\right)\right|^2 ds$$

$$\int_0^{\infty}\left(e^{-ax}\right)^2 dx=\frac{2}{\pi}\int_0^{\infty}\left(\frac{s}{a^2+s^2}\right)^2 ds$$

$$i.e\int_0^{\infty}\frac{s}{\left(a^2+s^2\right)}ds=\frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^{\infty}=\frac{\pi}{2}\times\frac{1}{2a}$$

$$\int_0^{\infty}\frac{x^2}{\left(x^2+a^2\right)^2}dx=\frac{\pi}{4a}\qquad (\text{Re } place\ 's'\ by\ 'x').$$

Problem 26 **(i).** Find the Fourier cosine transform of $e^{-ax}\cos ax$

(ii). Evaluate **(a).** $\int_0^{\infty}\frac{1}{\left(x^2+1\right)\left(x^2+4\right)}dx$ **(b).** $\int_0^{\infty}\frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx$, using Fourier cosine and sine transform.

Solution:

(i) $F_c\left(f(x)\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\ dx$

$$F_c\left(e^{-ax}\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos sx\ dx$$

$$F_c(s)=\sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

By Modulation Theorem,

$$F_c\left[f(x)\cos ax\right]=\frac{1}{2}\left[F_c(a+s)+F_c(a-s)\right]$$

$$F_c\left[e^{-ax}\cos ax\right]=\frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\left\{\frac{a}{a^2+(a+s)^2}+\frac{a}{a^2+(a-s)^2}\right\}\right]$$

$$=\frac{1}{2}\times\sqrt{\frac{2}{\pi}}\times a\left\{\frac{a^2+(a-s)^2+a^2+(a+s)^2}{\left[a^2+(a+s)^2\right]\left[a^2+(a-s)^2\right]}\right\}.$$

$$=\frac{a}{\sqrt{2\pi}}\left[\frac{4a^2+2s^2}{s^4+4a^4}\right]$$

$$F_c\left[e^{-ax}\cos ax\right]=\frac{2a}{\sqrt{2\pi}}\left[\frac{2a^2+s^2}{s^4+4a^4}\right].$$

(ii) (a) Let $f(x)=e^{-x}$ and $g(x)=e^{-2x}$

$$F_c\left(e^{-x}\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{s^2+1}\left(-\cos x+s\sin sx\right)\right]_0^{\infty}$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1}\right] \qquad -(1)$$

$$F_c\left(e^{-2x}\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2x}\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{2}{s^2+4}\right) \qquad -(2)$$

$$\therefore \int_0^\infty f(x)\,g(x)\,dx=\int_0^\infty F_c\left(f(x)\right)F_c\left(g(x)\right)ds$$

$$\int_0^\infty e^{-x}e^{-2x}dx=\frac{2}{\pi}\int_0^\infty\left(\frac{1}{s^2+1}.\frac{1}{s^2+4}\right)ds\left(from(1)\&(2)\right)$$

$$\int_0^\infty e^{-3x}dx=\frac{4}{\pi}\int_0^\infty\frac{ds}{\left(s^2+1\right)\left(s^2+4\right)}ds$$

$$\int_0^\infty\frac{ds}{\left(s^2+1\right)\left(s^2+4\right)}=\frac{\pi}{4}\left[\frac{e^{-3x}}{-3}\right]_0^\infty=\frac{\pi}{4}\left(\frac{1}{3}\right)$$

$$\int_0^\infty\frac{ds}{\left(s^2+1\right)\left(s^2+4\right)}=\frac{\pi}{12}.$$

(b) To find $\int_0^\infty \frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx.$

Let

$$\left(f(x)\right)=e^{-ax},\, g(x)=e^{-bx}$$

$$F_s\left(g(x)\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx=\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right) \qquad -(1)$$

$$F_s\left(g(x)\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\sin sx=\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+b^2}\right) \qquad -(2)$$

$$\int_0^\infty f(x)g(x)\,dx=\int_0^\infty F_s\left[f(x)\right].F_s\left[g(x)\right]ds \;\; From\,(1)\text{ and}(2)$$

$$\int_0^\infty e^{-ax}e^{-bx}dx=\frac{2}{\pi}\int_0^\infty\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}ds$$

$$\int_0^\infty\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}ds=\frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

$$i.e\int_0^\infty\frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx=\frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty=\frac{\pi}{2(a+b)}.$$

Problem 27 **(i).** Find Fourier transform of $e^{-a|x|}$ and hence deduce that

(a). $\int_0^\infty \frac{\cos xt}{a^2+t^2}dt = \frac{\pi}{2a}e^{-a|x|}$ **(b).** $F\left[xe^{-a|x|}\right] = i\sqrt{\frac{2}{\pi}}\frac{2as}{\left(s^2+a^2\right)^2}$.

(ii) . Find Fourier cosine transform of $e^{-ax}\sin ax$.

Solution:

(i) Fourier transform of $f(x)$ is

$$F\left(f(x)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\left(\cos sx + i\sin sx\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\cos sx\ dx \qquad \left[\therefore e^{-a|x|}\sin sx \text{ is } odd\ fn.\right]$$

$$F\left[e^{-a|x|}\right] = \frac{2}{\sqrt{2\pi}}\left[\frac{a}{a^2+s^2}\right] = F(s) \qquad -(1)$$

(a) Using Fourie rinverse transform,

$$e^{-a|x|} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\left(\cos sx + i\sin sx\right)ds$$

$$= \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{\cos sx}{a^2+s^2}ds + 0 \qquad \left[\therefore \frac{\sin sx}{s^2+a^2} \text{ is } an\ odd\ fn.\right]$$

$$= \frac{2a}{\pi}\int_{-\infty}^{\infty}\frac{\cos xt}{a^2+t^2}dt \qquad \left(\text{Re}\,place\,'s'\ by\ 't'\right)$$

$$\frac{\pi}{2a}e^{-a|x|} = \int_{-\infty}^{\infty}\frac{\cos xt}{a^2+t^2}dt.$$

(b) . Find Fourier cosine transform of $e^{-ax}\sin ax$.

To prove $F\left[xe^{-a|x|}\right] = i\sqrt{\frac{2}{\pi}}\frac{2as}{\left(s^2+a^2\right)^2}$

Property:

$$F\left[x^n f(x)\right] = (-i)^n\frac{d^nF}{ds^n}$$

$$F\left[x\,f(x)\right] = -i\frac{dF(s)}{ds}$$

$$F\left[xe^{-a|x|}\right] = \frac{-i\,dF\left(e^{-a|x|}\right)}{ds}$$

$$= -i\frac{d}{ds}\left(\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\right)$$

$$= -ia\sqrt{\frac{2}{\pi}}\left(\frac{-2s}{\left(a^2+s^2\right)^2}\right) = i\sqrt{\frac{2}{\pi}}\frac{2as}{\left(a^2+s^2\right)^2}.$$

(ii) Find the Fourier cosine transform of $e^{-ax}\sin ax$.

$$F_c\left(f(x)\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[e^{-ax}\sin ax\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin ax\,\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^\infty e^{-ax}\left[\sin(s+a)x - \sin(s-a)x\right]dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{s+a}{a^2+(s+a)^2} - \frac{s-a}{a^2+(s+a)^2}\right\} \qquad \left[\therefore \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{\left(a^2+(s+a)^2\right)(s+a)-(s-a)\left(a^2+(s+a)^2\right)}{\left(a^2+(s+a)^2\right)\left(a^2+(s-a)^2\right)}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{2a^2s+s^3-2as^2+2a^3+as^2-2a^2s-2s^2-s^3-2as^2+2a^3+s^2a+2sa^2}{4a^4+2a^2s^3-4a^3s+2a^2s^2+s^4-2as^3+4a^3s+2as^2-4a^2s^2}\right\}$$

$$= \frac{2}{\sqrt{2\pi}}\left\{\frac{2a^3-as^2}{4a^4+s^4}\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{a\left(2a^2-s^2\right)}{4a^4+s^4}\right).$$

Problem 28 (i). State and Prove Parseval's Identity in Fourier Transform.

(ii). Find Fourier cosine transform of e^{-x^2}

Solution:

(i) Parseval's identity:

Statement: If $F(s)$ is the Fourier transform of $f(x)$, then $\int_{-\infty}^{\infty}\left|f(x)\right|^2dx = \int_{\infty}^{\infty}\left|F(s)\right|^2$

Pr*oof by convolution theorem* $F\left[f*g\right] = F(s)G(s)$

$$f*g = F^{-1}\left[F(s)G(s)\right]$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)G(s)e^{isx}\,ds \quad -(1)$$

Put $x=0$ and $g(-t) = \overline{f(t)}$, then it follows that $G(s) = \overline{F(s)}$

$\therefore (1)$ *becomes*

$$\int_{-\infty}^{\infty}\left[f(t)\overline{f(t)}\right]dt = \int_{-\infty}^{\infty}\left[F(s)\overline{F(s)}\right]ds$$

$$i.e. \int_{-\infty}^{\infty}\left|f(t)\right|^2 dt = \int_{-\infty}^{\infty}\left|F(s)\right|^2 ds$$

$$i.e. \int_{-\infty}^{\infty}\left|f(t)\right|^2 dx = \int_{-\infty}^{\infty}\left|F(s)\right|^2 ds$$

(ii)

$$F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[e^{-x^2}\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-x^2}\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\times\frac{1}{2}\int_{-\infty}^{\infty}e^{-x^2}\cos sx\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2}\cos sx\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2}\quad \text{RP of}\quad e^{-isx}dx$$

$$=\text{R.P of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2}\,e^{\text{is}x}\,dx$$

$$=\text{R.P of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2+\text{is}x}dx$$

$$=\text{R.P of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2+\text{is}x}\,\frac{e^{\frac{s^2}{4}}}{e^{\frac{s^2}{4}}}dx$$

$$=\text{R.P of }e^{-s^2/4}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2+isx+s^2/4}dx$$

$$=\text{R.P of }e^{-s^2/4}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-(x-is/2)^2}dx$$

Put $\frac{x-\text{is}}{2}=t\quad dx=dt$

When $t=-\infty \qquad y=-\infty$

$t=\infty \qquad y=\infty$

$$F_c\left[f(x)\right]=R.P\ \text{ of }\frac{e^{-s^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-t^2}dt$$

$$\text{R.P.of }\frac{e^{-s^2/4}}{\sqrt{2\pi}}\times\sqrt{\pi}\qquad\qquad \left[\therefore\int_{-\infty}^{\infty}e^{-t^2}dt=\sqrt{\pi}\right]$$

$$=\frac{e^{-s^2/4}}{\sqrt{2}}$$

$$\Rightarrow F_c\left[e^{-x^2}\right]=\frac{e^{-s^2/4}}{\sqrt{2}}.$$

Problem 29 (i). Find the Fourier transform of $\frac{\sin \text{ax}}{x}$ and hence prove that $\int_{-\infty}^{\infty}\frac{\sin^2 ax}{x^2}dx=4a\pi$.

(ii). Find $f(x)$, if the Fourier transform of $F(s)$ is $\frac{2\sin 3(s-2\pi)}{(s-2\pi)}$.

Solution:

(i) $F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{\text{is}x}\,dx$

$$F\left[\frac{\sin ax}{x}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{\sin ax}{x}\right]e^{\text{is}x}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{\sin ax}{x}\right](\cos sx+\sin x)dx$$

$$=\sqrt{\frac{2}{\pi}}\times\pi$$

$$F\left[\frac{\sin ax}{x}\right]=\sqrt{2\pi} \qquad -(1)$$

$$\left[\sqrt{\frac{2}{\pi}}\int_0^\infty\left[\frac{\sin(a+s)x}{x}+\frac{\sin(a-s)x}{x}\right]dx=\left\{\sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]\right\}\right] \text{ if } a+s>0 \,\&\, a-s>0$$

$$\text{if } a+s>0 \,\&\, a-s<0 \text{ or } a+s<0 \,\&\, a-s>0$$

By Parseval's identity

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx=\int_{-\infty}^{\infty}|F(s)|^2\,ds$$

$$\int_{-\infty}^{\infty}\frac{\sin^2 ax}{x^2}\,dx=\int_{-a}^{a}\left|\sqrt{2\pi}\right|^2 ds=2\pi[s]_{-a}^{a}=2\pi(a+a)=4\pi a$$

$$i.e. \int_{-\infty}^{\infty}\frac{\sin^2 ax}{x^2}\,dx=4\pi a.$$

(ii) Let us find $F^{-1}\left\{\frac{2\sin 3s}{s}\right\}$

$$F^{-1}\left\{\frac{2\sin 3s}{s}\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin 3s}{s}e^{-isx}\,ds$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin 3s}{s}(\cos sx+i\sin sx)\,ds=\frac{1}{\pi}\int_0^\infty\frac{2\sin 3s\cos sx}{s}\,ds$$

(By the property of odd and even function)

$$=\frac{1}{\pi}\int_0^\infty\left\{\frac{\sin(3+x)s}{s}+\frac{\sin(3-x)}{s}\right\}ds$$

$$=\frac{1}{\pi}\left[\int_0^\infty\frac{\sin(3+x)s}{s}\,ds+\int_0^\infty\frac{\sin(3-x)s}{s}\,ds\right]$$

$$=\begin{cases}\frac{1}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] \text{ if } 3+x>0 \,\&\, 3-x>0\\ \qquad\qquad \text{if } 3+x>0 \,\&\, 3-x<0 \text{ or}\\ 0 \qquad\qquad 3+x<0 \,\&\, 3-x>0\end{cases}$$

$$\left[\therefore \int_0^\infty\frac{\sin mx}{x}\,dx=\frac{\pi}{2} \text{ or } -\frac{\pi}{2} \text{ according as } m>0 \text{ or } m<0 .\right]$$

$$=\begin{cases}1 & \text{if } -3<x<3\\ 0 & \text{if } x<-3 \text{ or } x>3\end{cases}$$

$$=\begin{cases}1 & \text{if } |x|<3\\ 0 & \text{if } |x|>3\end{cases} \qquad -(1)$$

By the shitting property, $F\left\{e^{iax}f(x)\right\}=F(s-a)$

$$e^{iax}f(x)=F^{-1}\left\{F(s-a)\right\}$$

Thus $F^{-1}\left\{\frac{2\sin[3(s-2\pi)]}{s-2\pi}\right\}=e^{i2\pi}F^{-1}\left(\frac{2\sin 3s}{s}\right)$

$$=e^{i2\pi}\times\begin{cases}1 \text{ if } |x|<3\\ 0 \text{ if } |x|>3\end{cases}$$

$$=e^{i2\pi}\times\begin{cases}e^{i2\pi x} & \text{if } |x|<3\\ 0 & \text{if } |x|>3\end{cases} \quad .$$

Finite Fourier Transform

Let f(x) be function defined on (0,l). Suppose f(x) is sectionally continuous, then the Finite Fourier sine Transform of f(x) is s function on the set of integers

$$\bar{\mathcal{F}}_s\{f(x)\} = \int_0^l f(x)\sin\frac{p\pi x}{l}\,dx = \bar{F}_s(p)\ where\ p \in Z$$

Also the Finite Fourier Cosine Transform is defied by

$$\overline{\mathcal{F}_c}\{f(x)\} = \int_0^l f(x)\cos\frac{p\pi x}{l}\,dx = \bar{F}_c(p)\ where\ p \in Z$$

Inversion formula for sine transform

$$f(x) = \frac{2}{l}\sum_{p=1}^{\infty}\bar{F}_s(p)\ \sin\frac{p\pi x}{l}$$

Inversion formula for cosine transform

$f(x) = \frac{1}{l}\bar{F}_c(0) + \frac{2}{l}\sum_{p=1}^{\infty}\bar{F}_c(p)\ cos\ \frac{p\pi x}{l}$ Where $\bar{F}_c(0) = \int_0^l f(x)dx$

Problem 1) Find the Finite Fourier Cosine and Sign Transform of f(x) = x^2

Solution:

$$\bar{\mathcal{F}}_s\{x^2\} = \int_0^l x^2\sin\frac{p\pi x}{l}\,dx$$

$$= \left[x^2\left(-\frac{cos\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - 2x\left(-\frac{sin\frac{p\pi x}{l}}{\left(\frac{p\pi}{l}\right)^2}\right) + 2\left(\frac{cos\frac{p\pi x}{l}}{\left(\frac{p\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \left[-x^2\frac{l}{p\pi}cos\frac{p\pi x}{l} - 2x\left(\frac{l}{p\pi}\right)^2 sin\frac{p\pi x}{l} + 2\left(\frac{l}{p\pi}\right)^3 cos\frac{p\pi x}{l}\right]_0^l$$

$$\bar{\mathcal{F}}_s\{x^2\} = -\frac{l^3}{p\pi}(-1)^p + \frac{l^3}{p^3\pi^3}[(-1)^p - 1]\ if\ p \neq 0$$

$$\bar{\mathcal{F}}_c\{x^2\} = \int_0^l x^2\cos\frac{p\pi x}{l}\,dx$$

$$= \left[x^2\left(-\frac{sin\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - 2x\left(-\frac{cos\frac{p\pi x}{l}}{\left(\frac{p\pi}{l}\right)^2}\right) + 2\left(\frac{sin\frac{p\pi x}{l}}{\left(\frac{p\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \left[x^2 \frac{l}{p\pi} sin\frac{p\pi x}{l} + 2x\left(\frac{l}{p\pi}\right)^2 cos\frac{p\pi x}{l} + 2\left(\frac{l}{p\pi}\right)^3 sin\frac{p\pi x}{l} \right]_0^l$$

$$\bar{\mathcal{F}}_c\{x^2\} = -\frac{2l^3}{p^2\pi^2}(-1)^p \ if\ p \neq 0$$

Exercise 1) Find the finite fourier sine and cosine transform of f(x) = e^{ax} in (0, *l*).

--x-- THE END --x--