LINEAR TIME INVARIANT DISCRETE TIME SYSTEMS

LTI-DT systems – Characterization using difference equation – Properties of convolution and interconnection of LTI Systems – Causality and Stability of LTI Systems – Impulse response, Convolution Sum and Frequency response – Computation of Impulse response and Transfer function using Z Transform.

LTI – DT Systems:

A DT System which satisfies Linearity and time invariance property is called LTI DT systems. LTI systems comprise a very important class of systems, and they can be described by a standard mathematical formalism.

Characterization using difference equation:

Systems described by constant-coefficient, linear difference equations are LTI systems. In exploring this fact, it is important to keep in mind that our default setting is that all signals are defined for –∞< n < ∞. The difference equation is a formula for computing an output sample at time n based on past and present input samples and past output samples in the time domain. The difference equation is as follows:

$$\begin{aligned} y(n) &= b_0\, x(n) + b_1\, x(n-1) + \cdots + b_M\, x(n-M) \\ &\quad - a_1\, y(n-1) - \cdots - a_N\, y(n-N) \\ &= \sum_{i=0}^{M} b_i\, x(n-i) - \sum_{j=1}^{N} a_j\, y(n-j) \end{aligned}$$

Where x(n) is the input signal and y(n) is the output signal and constants $b_i, i = 0, 1, 2, \ldots, M$, $a_i, i = 1, 2, \ldots, N$ are called the coefficients. We have a system whose input and output signals are related by

$$y[n] + ay[n-1] = bx[n]\ ,\ -\infty < n < \infty$$

where a and b are real constants. This is called afirst-order, constant-coefficient, linear difference equation. Given an input signal x[n], this can be viewed as an equation that must be solved for y[n] for each input signal x[n] thereis a unique solution for the output signal y[n].

$$y[n] = \sum_{k=-\infty}^{\infty} (-a)^{n-k} bx[k]$$

Example 1: Find the solution to the following difference equation by using the z-transformx(k+2)+3x(k+1)+2x(k)=0, x(0)=0,x(1)=1

Solution:

Take the z-transform of both side of the equation, we get $z^2 X(z) - z^2 x(0) - z \cdot x(1) + 3z \cdot X(z) - 3z \cdot x(0) + 2X(z) = 0$

Substituting in the initial conditions and simplifying gives

$$X(z) = \frac{z}{z^2 + 3z + 2} = \frac{z}{(z+1)(z+2)} = \frac{z}{z+1} - \frac{z}{z+2}$$

TaketheinverseZtransformofaboveequationweget,

$$x(k) = Z^{-1}[X(z)] = Z^{-1}\left[\frac{z}{z-(-1)}\right] - Z^{-1}\left[\frac{z}{z-(-2)}\right] = (-1)^k - (-2)^k, \quad k = 0, 1, 2, \ldots$$

Example 2: Using the z-transform to solve the following difference equation$x(k + 2) + 0.4x(k +1) - 0.32x(k) = u(k)$, where $x(0) = 0$ and $x(1) = 1$. The input $u(k)$ is a unit step input, i.e. $u(k) = 1$, for $k \geq 0$.

Solution:

Take the z-transform of the difference equation we get

$$z^2 X(z) - z^2 x(0) - z \cdot x(1) + 0.4 \cdot z \cdot X(z) - 0.4 \cdot z \cdot x(0) - 0.32 \cdot X(z) = \frac{z}{z-1}$$

Substituting the initial conditions and simplifying, we obtain

$$X(z) = \frac{z^2}{(z-1)(z^2 + 0.4z - 0.32)} = \frac{z^2}{(z-1)(z+0.8)(z-0.4)}$$

The partial fraction expansion of the solution X(z) is

$$X(z) = 0.926\frac{z}{z-1} - 0.3704\frac{z}{z+0.8} - 0.5556\frac{z}{z-0.4}$$

The corresponding time sequence can be obtained by taking the inverse z-transform of the above equation:

$x(k) = 0.926 - 0.3704(-0.8)^k - 0.5556(0.4)^k$, for $k = 0, 1, 2, \ldots$

Convolution Sum:

To each LTI system there corresponds a signal h[n] such that the input-output behavior of the system is described by

∞

y[n] = ∑ x[k]h[n — k]
k=—∞
This expression is called the convolution sum representation for LTI systems. In addition, the shifting property easily shows that h[n] is the response of the system to a unit-pulse input signal.

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=\sum_{k=-\infty}^{\infty} g[k]h[n-k]=h[n]$$

Thus the input-output behavior of a discrete-time, linear, time-invariant system is completely described by the unit-pulse response of the system. If h[n] is known, then the response to anyinput can be computed from the convolution sum.

The system response to this inputsignal is given by

$$\begin{aligned}\hat{y}[n] &= \sum_{k=-\infty}^{\infty} \hat{x}[k]\,h[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[k-n_o]\,h[n-k]\end{aligned}$$

To rewrite this expression, change the summation index from k to 1 = k — N, to obtain

$$\begin{aligned}\hat{y}[n] &= \sum_{l=-\infty}^{\infty} x[l]\,h[n-n_o-l] \\ &= y[n-n_o]\end{aligned}$$

The convolution representation for linear, time-invariant systems can be developed by adopting a particular representation for the input signal and then enforcing the properties of linearity and time invariance on the corresponding response

Example, if n = 3, then the right side is evaluated by the sifting property to verify

$$\sum_{k=-\infty}^{\infty} x[k]h[3-k] = x[3]$$

We can use this signal representation to derive an LTI system representation as follows. The response of an LTI system to a unit pulse input, x[n] = u[n], is given the special notation y[n] = h[n]. Then by time invariance, the response to a k—shifted unit pulse, u[n] = δ[n — k] = h[n — k]. Furthermore, by linearity, the response to a linear combination of shifted unit pulses is the linear combination of the responses to the shifted unit pulses. That is, the response to x[n], as written above, is

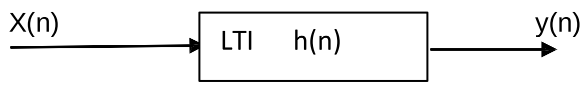
∞

$$y[n] = \sum_{k=-\infty} x[k]\, h[n - k]$$

Thus have arrived at the convolution sum representation for LTI systems. The convolution representation follows directly from linearity and time invariance An alternate expression for the convolution sum is obtained by changing the summation variable from k to 1 = n - k:

$$y[n] = \sum_{k=-\infty}^{\infty} h[l]\, x[n - l]$$

It is clear from the convolution representation that if the unit-pulse response of an LTI system is known, then we can compute the response to any other input signal by evaluating the convolution sum. Indeed, we specifically label LTI systems with the unit-pulse response in drawing block diagrams, as shown below



Example: convolve the following signals using matrix method x(n)={ 1 1 2 2}, h(n)={1 2 3

4} Ans: y(n)={1 3 7 12 14 14 8}

Properties of Convolution – Interconnections of DT LTI Systems

Convolution of two signals given by

$$y[n] = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

For any n, the value of y[n] in general depends on all values of the signals x[n] and h[n], y[n] = x[n]* h[n], for example, y[2] = x[2]* h[2] .

- Commutativity: Convolution is commutative. That is, x(n) * h(n) = h(n) * x(n)

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]\ , \text{ for all } n$$

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{q=\infty}^{\infty} x[n-q]h[q]$$

$$= \sum_{q=-\infty}^{\infty} h[q]x[n-q] = h(n) * x(n)$$

Using this result, there are two different ways to describe in words the role of the unit-pulse response values in the input-output behavior of an LTI system. The value of h[n – k] determineshow the n^{th} value of the output signal depends on the k^{th} value of the input signal. Or, the value of h[q] determines how the value of y[n] depends on the value of x[n – q].

- Associativity:Convolution is associative. That is, (x* h1 * h2))[n] = ((x* h1)* h_2)[n]
- Distributivity: Convolution is distributive (with respect to addition). That is, (x* (h1 + h_2))[n]= (x* h_i)[n]+ (x* h_2)[n]

For any constant b,((bx)* h)[n] = b (x* h)[n]

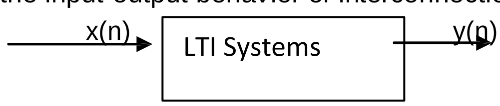
- Shifting Property:This is simply a restatement of the time-invariance property. For any integer n_o , if i[n] = x[n — n_o], then

$$h[n] = (x* h)[n — n_o]$$

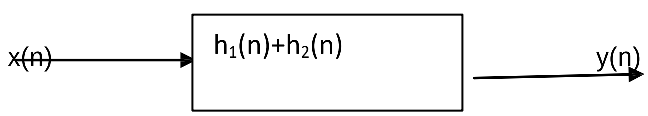
- Identity: It is worth noting that the "star" operation has the unit pulse as an identity element. Namely,
(x* S)[n] = x[n]

This can be interpreted in system-theoretic terms as the fact that the identity system, y[n] = x[n] has the unit-pulse response h[n] = δ[n] .Also we can write (δ* δ)[n]= g[n]. The unit pulse is the unit-pulse response of the system whose unit-pulse response is a unit pulse.

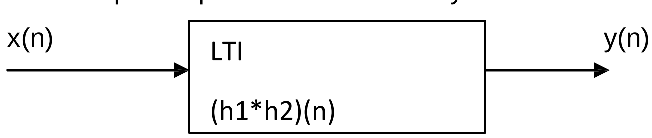
These algebraic properties of the mathematical operation of convolution lead directly to methods for describing the input-output behavior of interconnections of LTI systems. For example,



has the same input-output behavior as the system



both have the same input-output behavior as the system



Transfer Function and Impulse Response Sequence

The transfer function for the continuous-time system relates the Z transform of the continuous-time output to that of the continuous-time input. For discrete-time systems, the transfer function relates the z-transform of the output at the sample instance to that of the sampled input.Consider a linear time- invariant discrete-time system characterized by the following lineardifference equation:

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \cdots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + b_2 u(k-2) + \cdots + b_n u(k-n)$$

where u(k) and y(k) are the system input and output, respectively, at the kth sample instances. If we take the z-transform and by using the time shift property of the z-transform, we obtain

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_n z^{-n} Y(z) = b_0 U(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z) + \cdots + b_n z^{-n} U(z)$$

or

$$\left(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}\right) \cdot Y(z) = \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}\right) \cdot U(z)$$

which can be written as

$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \cdot U(z) = G(z) \cdot U(z)$$

where

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$

Consider the response of the linear discrete-time system described by Equation, initially atrest (y(k) = 0, k < 0), when the input u(k) is the Kronecker delta function δ0(k), i.e.

$$u(k) = \delta_0(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

Since

$$U(z) = \mathsf{Z}\left[u(k)\right] = \mathsf{Z}\left[\delta_0(k)\right] = 1$$

then

$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} = G(z)$$

Thus, G(z) is the z-transform of the response of the system to the Kronecker delta function input. The function G(z) is called the transfer function of the discrete-time system. In the above derivation, the role of the Kronecker delta function in discrete-time system is similar to that of the unit impulse function (the Dirac delta function) in continuous-time systems.The inverse transform of G(z) as given by Eq.

$$g(k) = Z^{-1}[G(z)] = Z^{-1}\left[\frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}\right]$$

is called the impulse response function (sequence).The system described by the difference equation

$$y(k+n) + a_1 y(k+n-1) + a_2 y(k+n-2) + \cdots + a_n y(k)$$
$$= b_0 u(k+n) + b_1 u(k+n-1) + b_2 u(k+n-2) + \cdots + b_n u(k)$$

where the system is initially at rest (y(k) = 0, k < 0) and the input u(k) = 0, for k < 0, can berepresented by the transfer function G(z).

Example Consider the difference equationy(k + 2) + a y(k +1) + a y(k) = b u(k + 2) + b u(k +1) + b u(k). Assuming that the system is initially at rest and u(k) = 0 for k < 0, find the transfer function.

Solution:

The z-transform of the difference equation is

$$\left[z^2Y(z) - z^2y(0) - z\cdot y(1)\right] + a_1\left[zY(z) - z\cdot y(0)\right] + a_2Y(z)$$
$$= b_0\left[z^2U(z) - z^2u(0) - z\cdot u(1)\right] + b_1\left[zU(z) - z\cdot u(0)\right] + b_2U(z)$$

Collect common terms

$$\left(z^2 + a_1 z + a_2\right)\cdot Y(z) = \left(b_0 z^2 + b_1 z + b_2\right)\cdot U(z)$$
$$+z^2\left[y(0) - b_0u(0)\right] + z\left[y(1) + a_1y(0) - b_0u(1) + b_1u(0)\right]$$

Hence

$$Y(z) = \frac{b_0z^2 + b_1z + b_2}{z^2 + a_1z + a_2}\cdot U(z) + \frac{\left[y(0) - b_0u(0)\right]z^2 + \left[y(1) + a_1y(0) - b_0u(1) + b_1u(0)\right]z}{z^2 + a_1z + a_2}$$

To determine the initial conditions y(0) and y(1), we substitute k = −2 into the original difference equation and obtain

$$y(0) + a_1 y(-1) + a_2 y(-2) = b_0 u(0) + b_1 u(-1) + b_2 u(-2),$$

which implies

$$y(0) = b_0 \cdot u(0)$$

By substitute k = −1 into the original difference equation and obtain

$$y(1) + a_1 y(0) + a_2 y(-1) = b_0 u(1) + b_1 u(0) + b_2 u(-1),$$

which impliesy(1) =-a y(0)+ b u(1)+ b u(0)

By substituting Equations we get

$$Y(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} \cdot U(z)$$

Hence, if both y(k) and u(k) are zero for k < 0, then the system's input and output are related by the above Equation.The transfer function G(z) = Y(z) / U(z) can be written as

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}.$$

The above Equation is the same transfer function for the system described by the difference equation

y(k) + a y(k − 1) + a y(k − 2) = b u(k) + b u(k − 1) + b u(k −2).

Causality and Stability of LTI Systems

A DT system is said to be a causal if the output of the system at any time depends only on the present input, past input but does not depend on future input and output

Ex: y(n)=x(n), x(n-1), x(n-2)….

A system is said to be stable if and only if every bounded input produces a bounded output condition for stability is given by

$$\sum_{-\infty}^{\infty} |h(n)| < \infty$$

Computation of Impulse response and Transfer function using Z Transform.

Example: Assuming that the system is initially at rest, find the impulse response of the following discrete time system:

y(k + 3) = 2u(k + 3) − u(k + 2) + 4u(k +1) + u(k). Find the transfer function

Solution:

Transfer function of the system can be written as

$$G(z) = \frac{2z^3 - z^2 + 4z + 1}{z^3} = 2 - z^{-1} + 4z^{-2} + z^{-3}$$

The impulse response of the system with zero initial condition is then the inverse z-transform of the pulse transfer function,

$$g(k) = Z^{-1}[G(z)] = Z^{-1}\left[2 - z^{-1} + 4z^{-2} + z^{-3}\right] = 2 \cdot \delta_0(k) - \delta_0(k-1) + 4 \cdot \delta_0(k-2) + \delta_0(k-3)$$

Henceg(0) = 2, g(1) = −1, g(2) = 4, g(3) = 1, g(k) = 0, for k >3

Frequency Response of Discrete-Time Systems

In order for systems to possess a steady-state response to a sinusoidal input, in must be stable (all the poles of the transfer function must lie within the unit circle of the complex z plane). Let the system of interest be defined by

$$G(z) = \frac{Y(z)}{U(z)} = \frac{N(z)}{(z - p_1)(z - p_2)\cdots(z - p_n)}$$

where pi are the complex poles of the system. We further assume that the system is stable, i.e. pi <1 for all i.

Let the input to the system be a cosine sequence of radian frequency ω, i.e.

$$u(k) = A\cos(\omega kT) = \frac{A}{2}\left(e^{j\omega kT} + e^{-j\omega kT}\right)$$

The corresponding z-transform of the input sequence is

$$U(z) = \frac{A}{2}\left(\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right)$$

Substituting the input equations the output Y(z) is given by

$$Y(z) = G(z) \cdot U(z) = \frac{N(z)}{(z - p_1)(z - p_2)\cdots(z - p_n)} \cdot \frac{A}{2}\left(\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right)$$

A partial fraction expansion of the above equation can be written as

$$Y(z) = B\frac{z}{z-e^{j\omega T}} + C\frac{z}{z-e^{-j\omega T}} + \sum_{i=1}^{n} D_i \frac{z}{z-p_i}$$

Each term in the summation on the right hand side of equation yields a time domain sequence of the form Di (pi) k, which if pi <1 will vanish when k gets larger and hence does not contribute to the steady-state response. The coefficients B and C in Eq. can be evaluated by the following formula

$$B = \frac{z-e^{j\omega T}}{z} Y(z)\Big|_{z=e^{j\omega T}} \quad \text{and} \quad C = \frac{z-e^{-j\omega T}}{z} Y(z)\Big|_{z=e^{-j\omega T}}$$

j T

Substituting Y(Z) in the above formula gives

$$B = \frac{z-e^{j\omega T}}{z} Y(z)\Big|_{z=e^{j\omega T}} = \frac{A}{2}\left[1 + \frac{z-e^{j\omega T}}{z-e^{-j\omega T}}\right] G(z)\Big|_{z=e^{j\omega T}} = \frac{A}{2} G\left(e^{j\omega T}\right)$$

$$C = \frac{z-e^{-j\omega T}}{z} Y(z)\Big|_{z=e^{-j\omega T}} = \frac{A}{2}\left[\frac{z-e^{-j\omega T}}{z-e^{j\omega T}} + 1\right] G(z)\Big|_{z=e^{-j\omega T}} = \frac{A}{2} G\left(e^{-j\omega T}\right)$$

Thus the steady state response YSS(s) is

$$Y_{SS}(z) = \frac{A}{2}\left[G\left(e^{j\omega T}\right)\frac{z}{z-e^{j\omega T}} + G\left(e^{-j\omega T}\right)\frac{z}{z-e^{-j\omega T}}\right]$$

Since G(z) is a rational function of the complex variable z, G(e $^{j\omega T}$) is a complex number that can be written in polar forma as

$$G\left(e^{j\omega T}\right) = \left|G\left(e^{j\omega T}\right)\right| \cdot e^{j\angle G\left(e^{j\omega T}\right)} = \left|G\left(e^{j\omega T}\right)\right| \cdot e^{j\phi}$$

where φ is the phase angle of the complex number G(e jωT). With similar reasoning, G(e $^{-j\omega T}$) will have the same magnitude and conjugate phase angle as G(e $^{j\omega T}$), i.e.

$$G\left(e^{-j\omega T}\right) = \left|G\left(e^{-j\omega T}\right)\right| \cdot e^{j\angle G\left(e^{-j\omega T}\right)} = \left|G\left(e^{j\omega T}\right)\right| \cdot e^{-j\phi}$$

Substituting the values, the steady-state response can be written as

$$Y_{SS}(z) = \frac{A}{2} \cdot \left|G\left(e^{j\omega T}\right)\right| \cdot \left[e^{j\phi}\frac{z}{z-e^{j\omega T}} + e^{-j\phi}\frac{z}{z-e^{-j\omega T}}\right]$$

Taking inverse z-transform of the above equation, we can obtain the time sequence of the steadystate sinusoidal response to be

$$y(k)=\frac{A}{2}\cdot\left|G\left(e^{j\omega T}\right)\right|\cdot\left[e^{j\phi}\left(e^{j\omega T}\right)^{k}+e^{-j\phi}\left(e^{-j\omega T}\right)^{k}\right]=A\cdot\left|G\left(e^{j\omega T}\right)\right|\cdot\frac{1}{2}\left(e^{j(\omega kT+\phi)}+e^{-j(\omega kT+\phi)}\right)$$

the Euler identity, the above equation can be further simplified and the steady-state

sinusoidal response isUsing

$$y(k)=A\cdot\left|G\left(e^{j\omega T}\right)\right|\cdot\cos\left(\omega kT+\phi\right),\ \text{where}\ \phi=\angle G\left(e^{j\omega T}\right).$$

From the above Equationwe see that, similar to the continuous-time case, the steady-state response of the system G(z) to a sinusoidal input is still sinusoidal with the same frequency but scaled in amplitude and shifted in phase. The amplitude of the steady-state response is scaled by a factor of $G|e^{jwt|}$,which will be referred to as the system gain associated with G(z) at frequency ω . The complex function of ω, G(ejωT), is called the frequency response function of the system G(z). The frequency response function of a system can be obtained by replacing the z-transform complex variable z with $e^{j\omega T}$, i.e.

$$G\left(e^{j\omega T}\right)=G(z)\big|_{z=e^{j\omega T}}=G\left(\cos(\omega T)+j\sin(\omega T)\right).$$

As in the continuous-time case, we are usually interested in the magnitude and phasecharacteristics of this function as a function of frequency. It is interesting to note that the DC gain of the system corresponds to the magnitude of the frequency response function at ω = 0,

DC Gain= $G\left(e^{j\omega T}\right)\Big|_{\omega=0}=G(z)\big|_{z=1}=G(1)$

This is slightly different from the continuous-time case where the DC gain is evaluated bysubstituting the Laplace variable s by 0.

Example: Find the frequency response for the discrete-time system described by the following difference equation:

$y(k) = e^{-2T} y(k-1) + u(k)$, where T =π/5

Solution:

The impulse transfer function of the system can be found by taking the z-transform of thedifference equation and assuming zero initial conditions

$$Y(z)=e^{-2T}z^{-1}Y(z)+U(z)$$

which implies

$$G(z)=\frac{Y(z)}{U(z)}=\frac{1}{1-e^{-2T}z^{-1}}=\frac{z}{z-e^{-2T}}$$

The frequency response of the system is

$$G(e^{j\omega T}) = \frac{e^{j\omega T}}{e^{j\omega T} - e^{-2T}}$$