ANALYSIS OF CONTINUOUS TIME SIGNAL

Continuous Time Fourier Transform

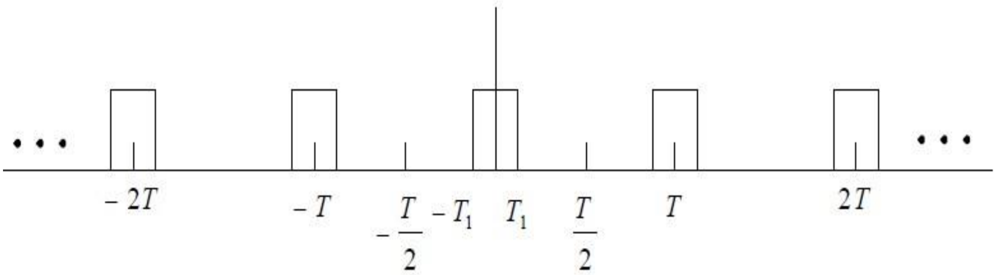
Any continuous time periodic signal x(t) can be represented as a linear combination of complex exponentials and the Fourier coefficients (or spectrum) are discrete. The Fourier series can be applied to periodic signals only but the Fourier transform can also be applied to non-periodic functions like rectangular pulse, step functions, ramp function etc. The Fourier transform of Continuous Time signals can be obtained from Fourier series by applying appropriate conditions.

The Fourier transform can be developed by finding Fourier series of a periodic function and the tending T to infinity.

Representation of Aperiodic signals:

StartingfromtheFourierseriesrepresentationforthecontinuous-time periodic square wave:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}, \qquad 2.1$$



The Fourier coefficients a_k for this square wave are $$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}.$$

2.2

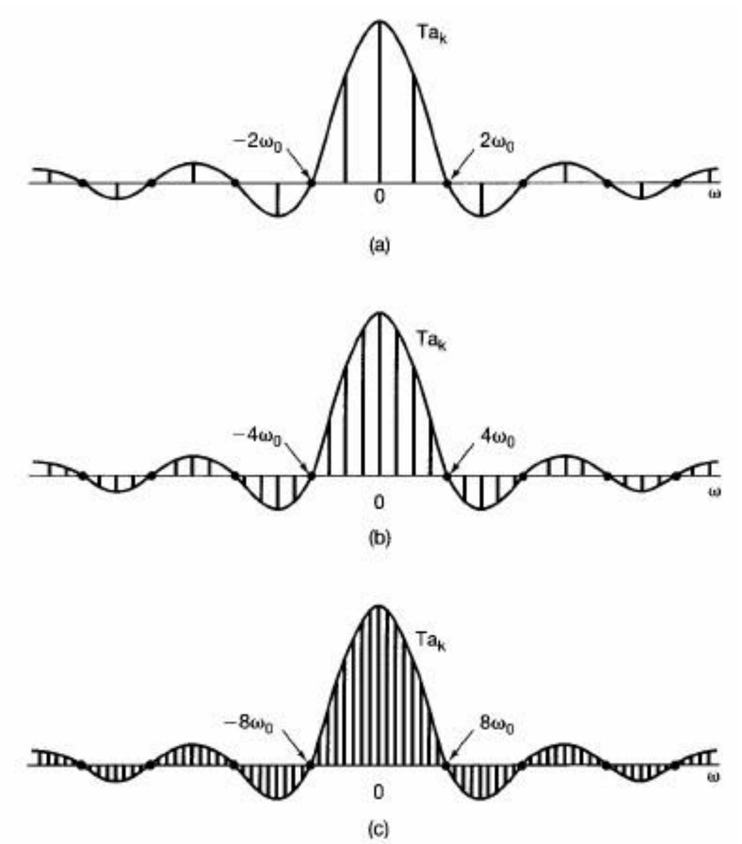
or alternatively 2.3

$$Ta_k = \frac{2\sin(\omega T_1)}{\omega}\bigg|_{\omega = k\omega_0}$$

where 2 sin(wT1) / w represent the envelope of Ta_k

When T increases or the fundamental frequency w $_0$ 2p / T decreases, the envelope is sampled with a closer and closer spacing. As T becomes arbitrarily large, the original periodic square wave approaches a rectangular pulse.

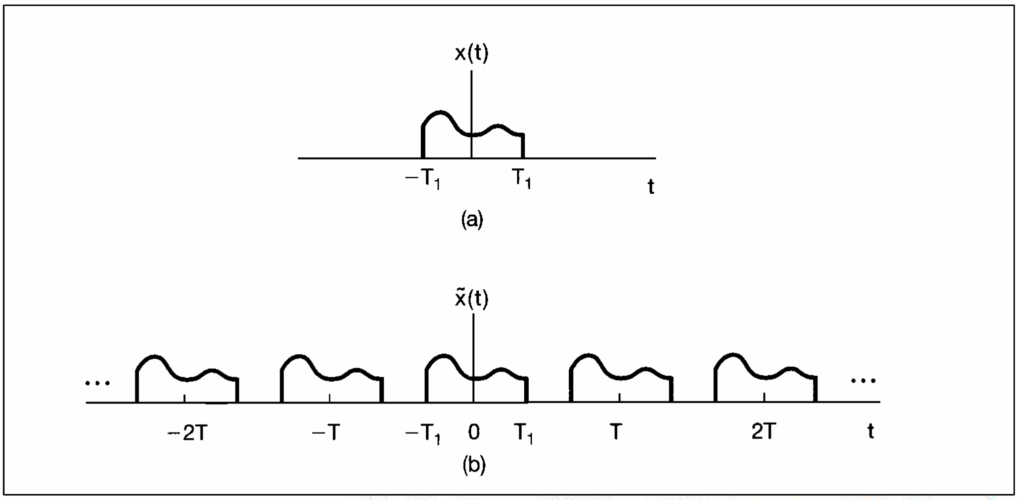
Ta_k becomes more and more closely spaced samples of the envelope, as T →∞, the Fourier series coefficients approaches the envelope function.



ThisexampleillustratesthebasicideabehindFourier’sdevelopmentofarepresentationfor aperiodic signals.

Based on this idea, we can derive the Fourier transform for aperiodic signals.

From this aperiodic signal, we construct a periodic signal (t) , shown in the figure below



As $T \to \infty$, $\tilde{x}(t) = x(t)$, for any infinite value of t.

The Fourier series representation of $\tilde{x}(t)$ is

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} , \quad 2.4$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt . \quad 2.5$$

Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since $x(t) = 0$ outside this interval, so we have

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt . \quad 2.6$$

Define the envelope X (jw) of Ta_k as,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt . \quad 2.7$$

we have for the coefficients a_k,

$$a_k = \frac{1}{T} X(jk\omega_0)$$

Then $\tilde{x}(t)$ can be expressed in terms of $X(j\omega)$, that is

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 . \qquad 2.8$$

As $T \to \infty$, $\tilde{x}(t) = x(t)$ and consequently,

Equation 2.8 becomes representation of *x(t).* In addition the right hand side of equation becomes an integral.

This results in the following Fourier Transform.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \textit{Inverse Fourier Transform} \qquad 2.9$$

and

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \textit{Fourier Transform} \qquad 2.10$$

Convergence of Fourier Transform

If the signal x(t) has finite energy, that is, it is square integrable,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty ,$$

Then we guaranteed that X(jw) is finite or equation 2.10 converges. If $e(t) = \tilde{x}(t) - x(t)$,

We have

$$\int_{-\infty}^{\infty} |e(t)|^2 dt = 0 .$$

An alterative set of conditions that are sufficient to ensure the convergence:

Contition1: Over any period, x(t) must be absolutely integrable, that is

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty ,$$

Condition 2: In any finite interval of time, x(t) have a finite number of maxima and minima.
Condition 3: In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Examples of Continuous-Time Fourier Transform

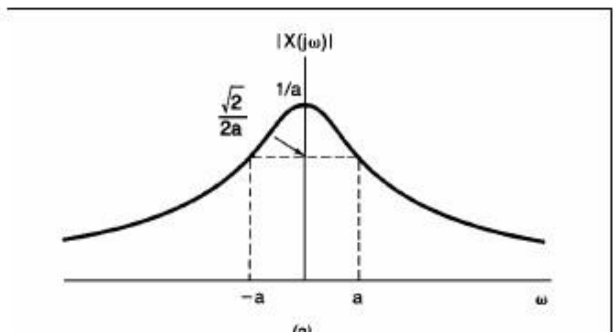
consider signal $x(t) = e^{-at}u(t)$, $a > 0$.

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = -\frac{1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty} = \frac{1}{a+j\omega}, \qquad a > 0$$

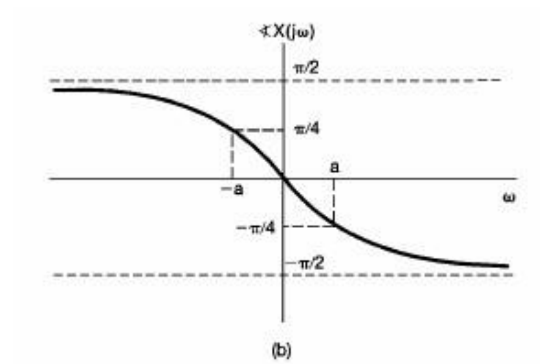
If a is complex rather than real, we get the same result if Re{a}>0

The Fourier transform can be plotted in terms of the magnitude and phase, as shown in the figure below.

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \qquad \angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$



(a)

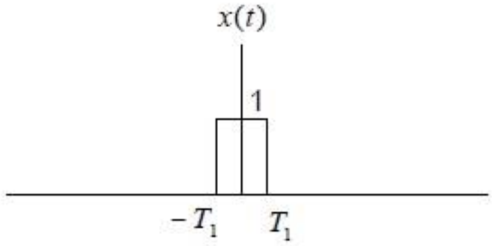


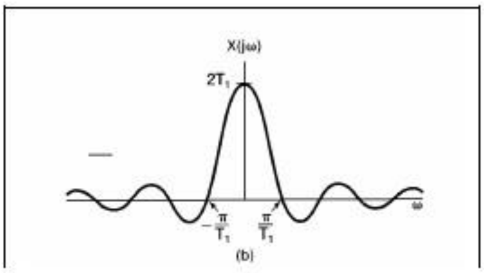
(b)

Example

Calculate the Fourier transform of the rectangular pulse signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}.$$





$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-T_1}^{T_1} 1e^{-j\omega t}dt = 2\frac{\sin \omega T_1}{\omega}.$$

The inverse Fourier Transform of the sinc function is

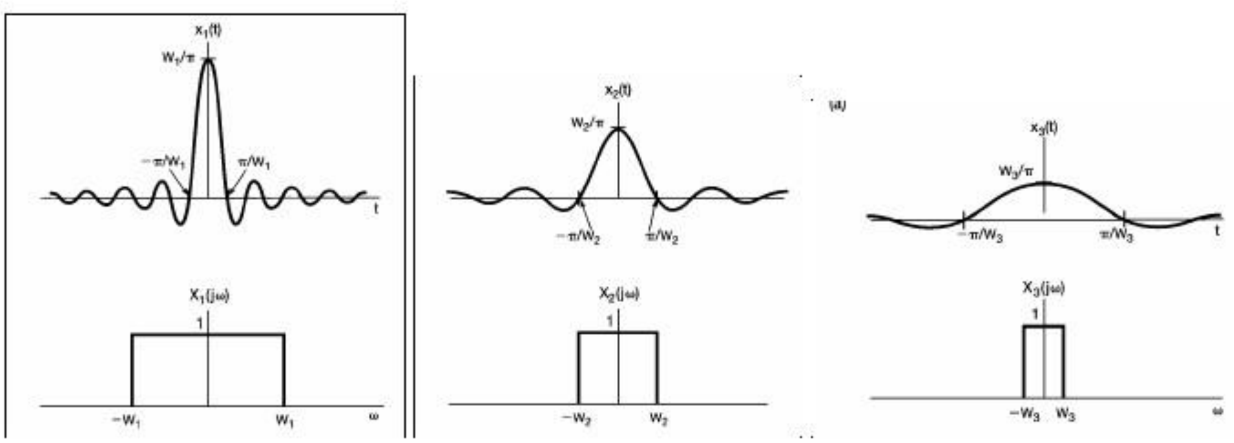
$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega = \frac{\sin Wt}{\pi t}$$

Comparing the results we have,

$$\text{Square wave} \; \underset{FT^{-1}}{\overset{FT}{\rightleftarrows}} \; \text{Sinc Function}$$

This means a square wave in the time domain, its Fourier transform is a sinc function. However, if the signal in the time domain is a sinc function, then its Fourier transform is a square wave. This property is referred to as **Duality Property.**

We also note that when the width of $X(jw)$ increases, its inverse Fourier transform x(t) will be compressed. When W → ∞, $X(jw)$ converges to an impulse. The transform pair with several different values of W is shown in the figure below.



The Fourier Transform for Periodic Signals

The Fourier series representation of the signal $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} .$$

It's Fourier transform is

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) .$$

Properties of Fourier Transform

1. **Linearity**

   If $x(t) \xrightarrow{F} X(jw)$ $y(t) \xrightarrow{F} Y(jw)$

   then

   $$ax(t) + by(t) \xrightarrow{F} aX(jw) + bY(jw)$$

2. **Time Shifting**

   If $x(t) \xrightarrow{F} X(jw)$

Then

$x(t-t_0) \overset{F}{\longleftrightarrow}$

3. Conjugation and Conjugate Symmetry

If $x(t) \overset{F}{\longleftrightarrow} X(jw)$
Then

$x^*(t) \overset{F}{\longleftrightarrow} X^*(-jw)$

4. Differentiation and Integration

If $x(t) \overset{F}{\longleftrightarrow} X(jw)$ then $\overset{F}{\longrightarrow} \omega(\omega)$

$$\tau \ \tau \overset{F}{} \frac{1}{\omega} + \pi \ 0 \ \delta \ \omega$$
$$-\infty$$

5. Time and Frequency Scaling

$x(t) \overset{F}{} (\omega)$

then,

$x(at) \overset{F}{} \frac{1}{} \frac{\omega}{}$

From the equation we see that the signal is compressed in the time domain, the spectrum will be extended in the frequency domain. Conversely, if the signal is extended, the corresponding spectrum will be compressed.

If a = -1, we get from the above equation,

$X(-t) \overset{F}{} (\omega)$

That is reversing a signal in time also reverses its Fourier transform.

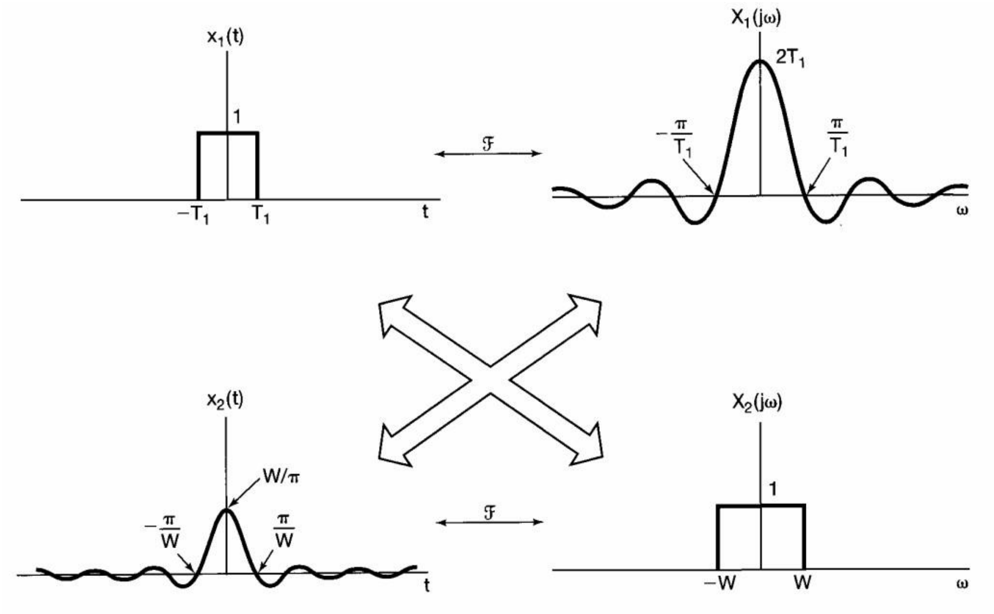
6. Duality

The duality of the Fourier Transform can be demonstrated using the following example.

$1, \quad < 1 \overset{F}{} \quad (\omega) \quad \frac{2 \sin \omega 1}{}$

$x_1(t)=$ 0, $>_1$ $_1$ = ω

$x_2(t) = \frac{n\ 1}{\pi}\ \overset{F}{}\ \ _2\ \omega\ = \begin{cases} 1, & \omega < \\ 0, & \omega > \end{cases}$



For any transform pair, there is a dual pair with the time and frequency variables interchanged.

$$-jtx(t) \xleftrightarrow{F} \frac{dX(j\omega)}{d\omega}$$

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X(j(\omega-\omega_0))$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xleftrightarrow{F} \int_{-\infty}^{\omega} x(\eta)d\eta$$

Parseval's Relation

If *x(t)* $\overset{F}{}$ (ω),

We have,

$$\int_{-\infty}^{\infty} ()^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} ()^2 \omega$$

Parseval‟s relation states that the total energy may be determined either by computing the energy per unit time $()^2$ and integrating over all time or by

computing the energy per unit frequency $(\omega)^2 / 2\pi$ and integrating over all frequencies. For this reason, $(\omega)^2$ is often referred toas the energy density spectrum.

The Parseval's theorem states that the inner product between signals is preserved in going from time to the frequency domain. This is interpreted physically as "Energy calculated in the time domain is sameas the energy calculated in the frequency domain"

The convolution properties

$$= h \quad * \boldsymbol{x} \qquad \overset{\boldsymbol{F}}{} \qquad \omega \;= H \;\omega\; (\omega)$$

The equation shows that the Fourier transform maps the convolution of two signals into product of their Fourier transforms.

H(jω), the transform of the impulse response is the frequency response of the LTI system, which also completely characterizes an LTI system.

Example

The frequency response of a differentiator.

y(t)=$\frac{()}{}$

From the differentiation property,

Y(jω)= *jω* ω

The frequency response of the differentiator is,

H(jω)=$\frac{(\omega)}{(\omega)}$=*jω*

The Multiplication Property

$$r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta$$

Multiplication of one signal by another can be thought of as one signal to scale or modulate the amplitude of the other, and consequently, the multiplication of two signals is often referred to as amplitude modulation.

Laplace Transform

The Laplace Transform is the more generalized representation of CT complex exponential signals. The Laplace transform provide solutions to most of the signals and systems, which are not possible with Fourier method. The Laplace transform can be used to analyze most of the signals which are not absolutely integrable such as the impulse response of an unstable system. Laplace Transform is a powerful tool for analysis and design of Continuous Time signals and systems. The Laplace Transform differs from Fourier Transform because it covers a broader class of CT signals and systems which may or may not be stable.

Till now, we have seen the importance of Fourier analysis in solving many problems involving signals. Now, we shall deal with signals which do not have a Fourier transform. We note that the Fourier Transform only exists for signals which can absolutely integrated and have a finite energy. This observation leads to generalization of continuous-time Fourier transform by considering a broader class of signals using the powerful tool of "Laplace transform". With this introduction let us go on to formally defining both Laplace transform.

Definition of Laplace Transform

The Laplace transform of a function x(t) can be shown to be,

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt$$

This equation is called the Bilateral or double sided Laplace transform equation.

$$x(t) = \int_{-\infty}^{\infty} X(s)\, e^{st} ds$$

This equation is called the Inverse Laplace Transform equation, x(t) being called the Inverse Laplace transform of X(s).

The relationship between x(t) and X(s) is
$x(t) \overset{LT}{\longleftrightarrow} X(s)$

Region of Convergence (ROC):

The range of values for which the expression described above is finite is called as the Region of Convergence (ROC).

Convergence of the Laplace transform

The bilateral Laplace Transform of a signal x(t) exists if

X(s)= $_{-\infty}^{\infty}()^{-}$

Substitute s=$\sigma + \omega$
X(s) = $_{-\infty}^{\infty}()^{-\sigma - \omega}$

Relationship between Laplace Transform and Fourier Transform

The Fourier Transform for Continuous Time signals is in fact a special case of Laplace Transform. This fact and subsequent relation between LT and FT are explained below.

Now we know that Laplace Transform of a signal 'x'(t)' is given by:

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

The s-complex variable is given by $s = \sigma + j\Omega$

But we consider $\sigma = 0$ and therefore „s" becomes completely imaginary. Thus we have s=jΩ. This means that we are only considering the vertical strip at $\sigma = 0$.

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt$$

From the above discussion it is clear that the LT reduces to FT when the complex variable only consists of the imaginary part. Thus LT reduces to FT along the jΩ axis. (imaginary axis)

Fourier Transform of x(t) = Laplace Transform of $()_{=\Omega}$

Laplace transform becomes Fourier transform

if σ=0 and s=jω.

$$X(s)\big|_{s=j\omega} = FT\{x(t)\}$$

Example of Laplace Transform

(1) Find the Laplace transform and ROC of x(t) = $^{-}$ ()

we notice that by multiplying by the term u(t) we are effectively considering the unilateral Laplace Transform whereby the limits tend from 0 to +∞

Consider the Laplace transform of x(t) as shown below

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t} dt$$

$$= \frac{1}{s+a} ;\ \text{for } (s+a) > 0$$

(2) Find the Laplace transform and ROC of x(t)=-$^{-}$ (−)

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} - e^{-at}\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-(s+a)t} dt$$

$$= \frac{1}{s+a} ;\ \text{for } (s+a) < 0$$

If we consider the signals $e^{-at}u(t)$ and $-e^{-at}u(-t)$, we note that although the signals are differing, their Laplace Transforms are identical which is 1/(s+a). Thus we conclude that to distinguish L.T's uniquely their ROC's must be specified.

Properties of Laplace Transform

1. Linearity

If $x_1(t) \overset{L}{\longleftrightarrow} {}_1()$ with ROC R1 and $x_2(t) \overset{L}{\longleftrightarrow} {}_2()$ with ROC R2, then

$a_1 \quad +b_2 \quad \overset{L}{\longleftrightarrow} \quad a_1 \quad +b_2$ with ROC containing ${}_1 \cap {}_2$

The ROC of X(s) is at least the intersection of R_1 and R_2, which could be empty, in which case x(t) has no Laplace Transform.

2. Differentiation in the time domain

If *x(t)* $\overset{L}{\longleftrightarrow}$ with ROC = R then $\frac{()}{} \overset{L}{\longleftrightarrow}$ () with ROC = R.

This property follows by integration by parts.

Hence, $\frac{}{} \overset{L}{\longleftrightarrow}$ () The ROC of *sX(s)* includes the ROC of X(*s*) and may be larger.

3. Time Shift

If x(t) $\overset{L}{\longleftrightarrow}$ with ROC = R then

x(t-t₀) $\overset{L}{\longleftrightarrow}$ ${}^{-}{}_0$ () with ROC = R

4. Time Scaling

If *x(t)* $\overset{L}{\longleftrightarrow}$ () with ROC = R, then

x(at) $\overset{L}{\longleftrightarrow} \frac{1}{}$; ROC = ie., $-\epsilon$

–

5. Multiplication

x(t) X y(t) $\xleftrightarrow{L} \frac{1}{2\pi} \quad * ()$

6. Time Reversal

When the signal x(t) is time reversed(180° Phase shift)

X(-t) $\xleftrightarrow{L} (-)$

7. Frequency Shifting

$$e^{0}x \xleftrightarrow{L} X(-{}_{0})$$

8. Conjugation symmetry

$x^{*}(t) \xleftrightarrow{L} {}^{*}(-)$

9. Parseval's Relation of Continuous Signal

It states that the total average power in a periodic signal x(t) equals the sum of the average in individual harmonic components, which in turn equals to the squared magnitude of X(s) Laplace Transform.

$$\int_{0}^{\infty} ()^{2} = \frac{1}{2\pi} \int_{\sigma-\infty}^{\sigma+\infty} ()^{2}$$

10.Differentiation in Frequency

When x(t) is differentiated with respect to frequency then,

-t x(t) $\xleftrightarrow{L} \frac{()}{}$

11.Integration Property

When a periodic signal x(t) is integrated, then the Laplace Transform becomes,

$$\int_{-\infty} \xleftrightarrow{L} \frac{1}{} + \frac{\int_{-\infty}^{\infty} \tau \, \tau}{}$$

12.Convolution Property

x(t) * y(t) $\overset{L}{\longleftrightarrow}$. ()

13.Initial Value Theorem

The initial value theorem is used to calculate initial value x(0^+) of the given sequence x(t) directly from the Laplace transform X(S). The initial value theorem does not apply to rational functions X(S) whose numerator polynomial order is greater than the denominator polynomial orders.

The initial value theorem states that,

$$\lim_{\rightarrow\infty} \quad = (0^{+})$$

14.Final Value Theorem

It states that,

$$\lim_{\rightarrow\infty} \quad = (\infty)$$