SMTX 1011 APPLIED NUMERICAL METHODS

COMMON TO ALL ENGINEERINGS EXCEPT BIO MED AND BIO INFO

III YEAR V SEMESTER (BATCH 2010 ONWARDS)

COURSE MATERIAL

COURSE OBJECTIVE: The ability to identify, reflect upon, evaluate and apply different types of information and knowledge to form independent judgments. Analytical, logical thinking and conclusions based on quantitative information will be the main objective of learning this subject.

UNIT I- CURVE FITTING

Curve Fitting –Method of group averages-Principle of least squares- Method of moments –Finite Difference – Operators E &D – Relationship between Operators.

APPLIED NUMERICAL METHODS

Subject code : SMTX1011

UNIT - I - Curve Fitting

Concepts:

The Method of Group Averages is based on the assumption that the sum of the residuals is zero.

The Method of Least Square is based on the principle that the sum of the squares of the residuals is Minimum.

The Normal equations to fit a Straight line of the form $y = ax+b$ are

$$\Sigma y = a\Sigma x + nb$$
$$\Sigma xy = a\Sigma x^2 + b\Sigma x$$

The normal equations to fit a parabola of the form $y = ax^2+bx+c$ are

$$\Sigma y = a\Sigma x^2 + b\Sigma x + nc$$
$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$$
$$\Sigma x^2y = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x^2$$

The Method of Moments is based on the Principle that the Calculated moments (μ_i) and the expected Moments (γ_i) are equal for all i.

Operators

Forward difference operator (Δ), Backward difference operator (∇), Central difference operator (δ) Shifting operator (E), Averaging operator (μ) and the Differential operator (D) are all satify the linear law.

$$\Delta(af(x)+bg(x)) = a\Delta f(x) + b\Delta g(x) \text{ where}$$

a and b are constants.

$$\Delta[f(x)g(x)] = f(x+h)\Delta g(x) + g(x)\Delta f(x)$$

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}$$

If $y = f(x)$ is a polynomial of degree n then its n^{th} differences are constants.

(ie) $\Delta^n[f(x)]$ is a constant

$\Delta^{n+1}[f(x)] = 0$.

Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots\cdots + a_n$

then $\Delta^n f(x) = a_0 n!\, h^n$

Solved Problems

* Fit a straight line of the form $y = ax+b$ by the method of Group Averages for the following data

x:	0	5	10	15	20	25	30
y:	10	14	19	25	31	36	39

Solution: The required straight line is $y = ax+b$ —— (1)

Let the data is divided into 2 groups and the averages of both the groups are substituted in (1).

Group I		Group II	
x_1	y_1	x_2	y_2
0	10	20	31
5	14	25	36
10	19	30	39
15	25		
30	68	75	106

Hence $\bar{x}_1 = \frac{30}{4} = 7.5$, $\bar{y}_1 = \frac{68}{4} = 17$

$\bar{x}_2 = \frac{75}{3} = 25$ $\quad \bar{y}_2 = \frac{106}{3} = 35.33$

Substituting in ①

$$7.5a + b = 17 \quad \text{—— ②}$$
$$25a + b = 35.33 \quad \text{—— ③}$$

③ − ② $\Rightarrow$ $17.5a = 18.33$

$\therefore a = \frac{18.33}{17.5} = 1.0474$

Substituting the value of a in ②

② $\Rightarrow b = 9.1445$

Hence the required Straight line is

$$\boxed{Y = 1.0474X + 9.1445}$$

* Fit a curve of the form $y = ax^b + c$ by the Method of Group Averages for the following data

x :	250	500	900	1200	1600	2000
y :	0.25	0.38	0.80	1.38	2.56	4.10

Solution: The value of c should be evaluated first. Choose 3 values of x from the given data which are in G.P

Choose $x_1 = 900$, $x_2 = 1200$ and $x_3 = 1600$.

Hence $y_1 = 0.80$, $y_2 = 1.38$ and $y_3 = 2.56$

$$\therefore \quad c = \frac{y_1 y_3 - y_2^2}{y_1 + y_3 - 2y_2}$$

$$= \frac{(0.80)(2.56) - (1.38)^2}{0.80 + 2.56 - 2(1.38)}$$

$$= 0.2393$$

Hence the required curve $y = ax^b + c$ can be written as $y - 0.2393 = ax^b$ which is not linear.

Taking $\log_{10}$ on both sides

$$\log_{10}(y - 0.2393) = \log_{10} a + b \log_{10} x \quad \text{—} ②$$

Let $X = \log_{10} x$ and $Y = \log_{10}(y - 0.2393)$

$$② \Rightarrow Y = A + bX \; \text{—} ③ \quad \text{where } A = \log_{10} a \; \text{—} ④$$

By dividing the data into 2 Groups we can fit the above linear form which can be used to evaluate a, b

Group I

x_1	X_1	y_1	Y_1
250	2.3979	0.25	−1.9706
500	2.6990	0.38	−0.8517
900	2.9542	0.80	−0.2513
	8.0511		−3.0736

Group II

x_2	X_2	y_2	Y_2
1200	3.0792		0.0572
1600	3.2041		0.3656
2000	3.3010		0.5867
	9.5843		1.0095

$$\therefore \bar{X_1} = \frac{8.0511}{3} = 2.6837 \qquad \bar{X_2} = \frac{9.5843}{3} = 3.1948$$

$$\bar{Y_1} = \frac{-3.0736}{3} = -1.0245 \qquad \bar{Y_2} = \frac{1.0095}{3} = 0.3365$$

Substituting in ③

$A + 2.6837b = -1.0245$ — (5)

$A + 3.1948b = 0.3365$ — (6)

(6) − (5) ⇒ $0.5111b = 1.3610$

$\Rightarrow b = 2.6629$

Substituting b in (5) we have

$A = -8.1709$

∴ (4) ⇒ $a = \text{Antilog of}_{10} -8.1709$

$= 10^{-8.1709} = 6 \times 10^{-9}$

Hence the required relation is

$y = (6 \times 10^{-9})\, x^{2.6629} + 0.2393$

* Fit a curve of the form $y = ab^x$ by the method of Group Averages for the following data

x :	2	3	4	5	6
y :	144	172.8	207.4	248.8	298.5

Solution: Given equation is $y = ab^x$ — (1)

Taking $\log_{10}$ on both sides

$\log_{10} y = \log_{10} a + x \log_{10} b$

$Y = A + xB$ — (2)

Group I

x_1	y_1	Y_1
2	144	2.1584
3	172.8	2.2375
4	207.4	2.3168
9		6.7127

Group II

x_2	y_2	Y_2
5	248.8	2.3959
6	298.5	2.4749
11		4.8708

$\bar{x}_1 = \frac{9}{3} = 3,\quad \bar{Y}_1 = \frac{6.7127}{3} = 2.2376$

$\bar{x}_2 = \frac{11}{2} = 5.5 \quad \bar{Y}_2 = \frac{4.8708}{2} = 2.4354$

Substituting in ②

$$2.2376 = A + 3B \quad \text{—③}$$
$$2.4354 = A + 5.5B \quad \text{—④}$$

④ − ③ ⇒ $2.5B = 0.1978$

$$B = \frac{0.1978}{2.5} = 0.07912$$

$$b = \text{Antilog}_{10}\, 0.07912 = 10^{0.07912} = 1.19$$

Substituting $b = 1.19$ in ① we get the required curve. Also substituting B in ③ we have

$$A = 2.2376 - 3(0.07912) = 2.00024$$

$$a = \text{Antilog}_{10}\, 2.00024 = 100$$

∴ The required curve $y = 100(1.19)^x$

* Fit a straight line of the form $y = ax + b$ by the method of least square for the following data

x :	0	1	2	3	4
y :	1	5	10	22	38

Solution: Let the required straight line be $y = ax + b$ —①

Let the Normal equations are

$$\Sigma y = a\Sigma x + nb \quad \text{—②}$$
$$\Sigma xy = a\Sigma x^2 + b\Sigma x \quad \text{—③}$$

Here $\Sigma x = 10$, $\Sigma y = 76$, $\Sigma x^2 = 30$, $\Sigma xy = 243$

$\Sigma x^2 = 220$, $\Sigma y^2 = 12733$, $\Sigma xy = 1616$

$\Sigma x^3 = 1800$, $\Sigma x^2 y = 14,120$, $\Sigma x^4 = 15,664$

Substituting the required values in ②, ③ & ④

② $\Rightarrow$ $197 = 220a + 30b + 6c$ — ⑤ $(n=6)$

③ $\Rightarrow$ $1616 = 1800a + 220b + 30c$ — ⑥

④ $\Rightarrow$ $14,120 = 15,664a + 1800b + 220c$ — ⑦

Solving ⑤, ⑥ and ⑨ for the values of a, b and c we have

$a = \cdot 86$, $b = -\cdot 81$ & $c = \cdot 98$

Hence ① $\Rightarrow$ $y = \cdot 86x^2 - \cdot 81x + \cdot 98$

(Note: Since the values of Σx^2, Σxy etc are huge values the origin can be shifted by some transformations. Let $u = \frac{x-2}{2}$. Using this we can evaluate the values of a, b and c using $y = au^2 + bu + c$. After calculating a, b, c and replacing u interms of x, the required parabola can be evaluated)

* Fit a straight line to the following data by the method of moments:

x:	0	1	2	3	4	5	6
y:	0.4	0.7	1.1	1.6	1.9	2.3	2.6

Solution :- Let $y = ax+b$ be the straight line $\rightarrow (1)$

The values of x are equally spaced with $h=1$

$x_1 = 0, \; x_n = 6.$

The moment equations are

$[\alpha = x_1 - \frac{h}{2}, \; x_n + \frac{h}{2} = \beta]$

(i) $\frac{a}{2}(\beta^2-\alpha^2) + b(\beta-\alpha) = h\Sigma y \longrightarrow (2)$

(ii) $\frac{a}{3}(\beta^3-\alpha^3) + \frac{b}{2}(\beta^2-\alpha^2) = h\Sigma xy \longrightarrow (3)$

Now $\alpha = 0 - \frac{1}{2} = -0.5, \; \beta = 6 + \frac{1}{2} = 6.5$

$\beta - \alpha = 6.5 - (-0.5) = 7$

$\beta^2 - \alpha^2 = (6.5)^2 - (-0.5)^2 = (6.5)^2 - (0.5)^2 = 42$

$\beta^3 - \alpha^3 = (6.5)^3 - (-0.5)^3 = (6.5)^3 + (0.5)^3 = 274$

$\Sigma y = 10.6; \; \Sigma xy = 42.4$

$(2) \Rightarrow$

$\frac{42}{2}a + 7b = 10.6$

$\Rightarrow 21a + 7b = 10.6 \longrightarrow (4)$

$(3) \Rightarrow \frac{274}{3}a + \frac{42}{2}b = 42.4$

$\Rightarrow 91.58a + 21b = 42.4 \longrightarrow (5)$

solving (4) and (5) $a = 0.3786, b = 0.385$

$\therefore$ Best fit of the straight line is

$y = 0.3786x + 0.3785$

* Find the missing values in the following table

x :	2	4	6	8	10
y :	5.6	8.6	13.9	-	35.6

solution

we know that if (n+1) values of y are given y may be a polynomial of degree n.

Here 4 values are known.

$\therefore$ The polynomial is of degree 3

(ie) $\Delta^4 y_0 = 0.$

$$(E-1)^4 y_0 = 0$$

$$\Rightarrow (E^4 - 4C_1 E^3 + 4C_2 E^2 - 4C_3 E + 4C_4 E^0) y_0 = 0$$

$$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$$

$$\Rightarrow E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0$$

$$\Rightarrow y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

(ie) $35.6 - 4y_3 + 6(13.9) - 4(8.6) + 5.6 = 0$

$$\Rightarrow 4y_3 = 35.6 + 6(13.9) - 4(8.6) + 5.6$$

$$\Rightarrow y_3 = \frac{1}{4}[35.6 + 6(13.9) - 4(8.6) + 5.6]$$

$$= \frac{1}{4}[35.6 + 83.4 - 34.4 + 5.6]$$

$$y_3 = 22.55.$$

* Prove that $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}.$

solution:-

$$\text{R.H.S } \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}} = \frac{\delta^2}{2} + \delta\sqrt{\frac{4+\delta^2}{4}}$$

$$= \frac{\delta}{2}\left[\delta + \sqrt{4+\delta^2}\right]$$

C

$$= \frac{1}{2}\delta\left[\left(E^{1/2} - E^{-1/2}\right) + \sqrt{4 + \left(E^{1/2} - E^{-1/2}\right)^2}\right]$$

$$= \frac{1}{2}\delta\left[E^{1/2} - E^{-1/2} + \sqrt{4 + E + E^{-1} - 2E^{1/2}\cdot E^{-1/2}}\right]$$

$$= \frac{1}{2}\delta\left[E^{1/2} - E^{-1/2} + \sqrt{E + E^{-1} + 2}\right]$$

$$= \frac{1}{2}\delta\left[E^{1/2} - E^{-1/2} + \sqrt{\left(E^{1/2} + E^{-1/2}\right)^2}\right]$$

$$= \frac{1}{2}\delta\left[E^{1/2} - \cancel{E^{-1/2}} + E^{1/2} + \cancel{E^{-1/2}}\right]$$

$$= \frac{1}{\cancel{2}}\delta\cdot\cancel{2}E^{1/2} = \left(E^{1/2} - E^{-1/2}\right)E^{1/2}$$

$$= E - 1 = \Delta = L.H.S$$

* Prove that $\left(\frac{\Delta^2}{E}\right)y_x \neq \frac{\Delta^2 y_x}{E y_x}$.

Solution:-

L.H.S $\left(\frac{\Delta^2}{E}\right)y_x = \left(\Delta^2 E^{-1}\right)y_x$

$$= \Delta^2\left(E^{-1}y_x\right) = \Delta^2\left(y_{x-h}\right)$$

$$= \Delta\left(\Delta y_{x-h}\right)$$

$$= \Delta\left[y_x - y_{x-h}\right]$$

$$= \Delta y_x - \Delta y_{x-h}$$

$$= y_{x+h} - y_x - \left[y_x - y_{x-h}\right]$$

$$= y_{x+h} - y_x - y_x + y_{x-h}.$$

$$= y_{x+h} - 2y_x + y_{x-h} \rightarrow (1)$$

Now R.H.S $\frac{\Delta^2 y_x}{E y_x} = \frac{(E-1)^2 y_x}{y_{x+h}} = \frac{\left(E^2 - 2E + 1\right)y_x}{y_{x+h}}$

$$= \frac{E^2(y_x) - 2E(y_x) + y_x}{y_{x+h}}$$

$$= \frac{y_{x+2h} - 2y_{x+h} + y_x}{y_{x+h}} \longrightarrow (2).$$

From (1) and (2)

$$\left(\frac{\Delta^2}{E}\right) y_x \neq \frac{\Delta^2 y_x}{E(y_x)}$$

* Express $2x^3+x^2+3x+4$ interms of factorial polynomials, taking $h=3$ and hence find its forward differences.

Solution

$f(x)$ can be divided successively by $x, x-3, x-6$, the successive remainders are the coefficients in the factorial polynomial expression of $f(x)$ in the reverse order.

0	2	1	3	4
	0	0	0	0
3	2	1	3	4
	0	6	21	
6	2	7	24	
	0	12		
	2	19		

$\therefore f(x) = 2x^{(3)} + 19x^{(2)} + 24x^{(1)} + 4$

$\Delta f(x) = 2\cdot 3\cdot h\, x^{(2)} + 19\cdot 2\cdot h\, x^{(1)} + 24\cdot 1\cdot h\, x^{(0)}$

$= 6hx^{(2)} + 38hx^{(1)} + 24h \qquad \because \Delta x^{(n)} = nhx^{(n-1)}$

$= 6\times 3\, x^{(2)} + 38\times 3\, x^{(1)} + 24\times 3$

$= 18x^{(2)} + 114x^{(1)} + 72$

$\Delta^2 f(x) = 36h\, x^{(1)} + 114\, h\, x^{(0)} = 108x^{(1)} + 342$

$\Delta^3 f(x) = 108\, h\, x^{(0)} = 324.$

Assignment Problems

* The following data satisfies the law $y = a + bx^2$

x:	20	30	35	40	45	50
y:	10	11	11.8	12.4	13.5	14.4

Using the Method of Group Averages find the best value of a and b

(Ans: $y = 9.15 + 0.002x^2$)

* Fit a straight line to the following data by the method of least square.

x:	0	5	10	15	20
y:	7	11	16	20	26

(Ans: a = 0.94, b = 6.6)

* Fit a parabola to the data given below:

x:	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y:	1.1	1.3	1.6	2.0	2.7	3.4	4.1

(Ans: $y = 1.0 - .2x + .24x^2$)

* Fit a law of the type $y = ae^{bx}$ to the following data

x:	0	1	2	3
y:	1.05	2.10	3.85	8.30

(Ans: $y = 0.7386e^{0.6808x}$)

* Fit a straight line by the method of moments to the data

x:	1	2	3	4
y:	16	19	23	26

(Ans: $y = 3.188x + 13.03$)

* By the Method of moments fit a Second degree parabola to the data

x:	1	2	3	4
y:	1.7	1.8	2.3	3.2

(Ans: $y = 0.01x^2 + .45x + 1.09$)

* Find the Missing terms in the following table.

x:	10	15	20	25	30	35	40
y:	270	–	222	200	–	164	148

(Ans: 246 and 180.8)

* Prove that $\Delta = \mu\delta + \frac{1}{2}\delta^2$

* Prove that $\mu = \frac{2+\Delta}{2\sqrt{1+\Delta}} + \sqrt{1+\frac{\delta^2}{4}}$

* Prove that $1+\delta^2\mu^2 = \left(1+\frac{\delta^2}{2}\right)^2$

* Show that $\frac{\Delta^2 x^2}{E(x+\log x)} = \frac{2}{(x+1)+\log(x+1)}$

* Evaluate $\Delta(e^{3x}\log 2x)$

* Express x^3+x^2+x+1 in factorial polynomial and get its successive forward differences taking $h=1$.