SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS
COURSE MATERIAL

COURSE NAME: ENGINEERING MATHEMATICS IV **COURSE CODE: SMT1204**

UNIT I

INTRODUCTION

The Fourier series is named in honour of Jean-Baptiste Joseph Fourier (1768–1830), who made important contributions to the study of trigonometric series, after preliminary investigations by Leonhard Euler, Jean le Rond d'Alembert, and Daniel Bernoulli.Fourier introduced the series for the purpose of solving the heat equation in a metal plate. Although the original motivation was to solve the heat equation, it later became obvious that the same techniques could be applied to a wide array of mathematical and physical problems, and especially those involving linear differential equations with constant coefficients, for which the eigen solutions are sinusoids.Fourier series has many such applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, quantum mechanics, econometrics.

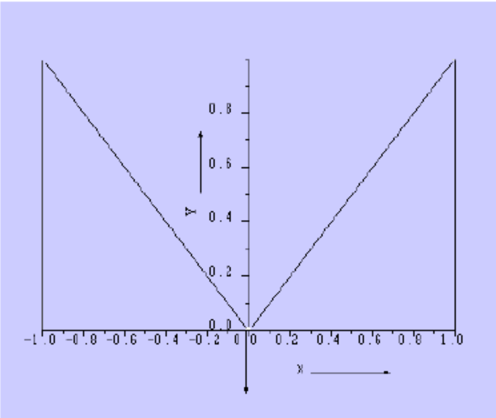
PRELIMINARIES

Definitions :

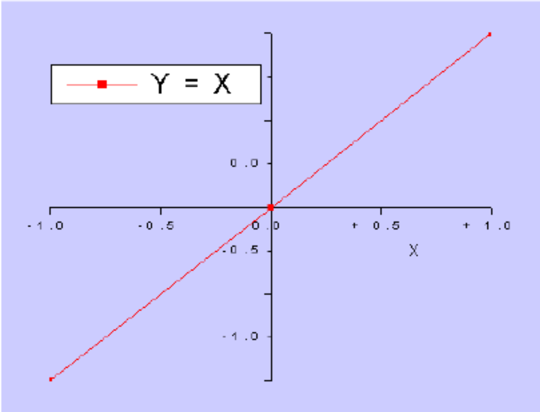
A function y = f(x) is said to be even, if f(-x) = f(x). The graph of the even function is always symmetrical about the y-axis.

A function y=f(x) is said to be odd, if f(-x) = - f(x). The graph of the odd function is always symmetrical about the origin.

For example, the function f(x) = x in [-1,1] is even as f(-x) = $-x = x$ = f(x) and the function f(x) = x in [-1,1] is odd as f(-x) = -x = -f(x). The graphs of these functions are shown below.



Graph of f(x) = *x*



Graph of f(x) = x

Note that the graph of f(x) = *x* is symmetrical about the y-axis and the graph of f(x) = x is symmetrical about the origin.

1. If f(x) is even and g(x) is odd, then
   - h(x) = f(x) . g(x) is odd
   - h(x) = f(x) . f(x) is even
   - h(x) = g(x) .g(x) is even

For example,

1. h(x) = x^2 cosx is even, since both x^2 and cosx are even functions
2. h(x) = xsinx is even, since x and sinx are odd functions
3. h(x) = x^2 sinx is odd, since x^2 is even and sinx is odd.

2. If f(x) is even, then $$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

3. If f(x) is odd, then $$\int_{-a}^{a} f(x)dx = 0$$

PERIODIC FUNCTIONS

A periodic function has a basic shape which is repeated over and over again. The fundamental range is the time (or sometimes distance) over which the basic shape is defined. The length of the fundamental range is called the period.

A general periodic function f(x) of period T satisfies the condition f(x+T) = f(x)

Here f(x) is a real-valued function and T is a positive real number.

As a consequence, it follows that

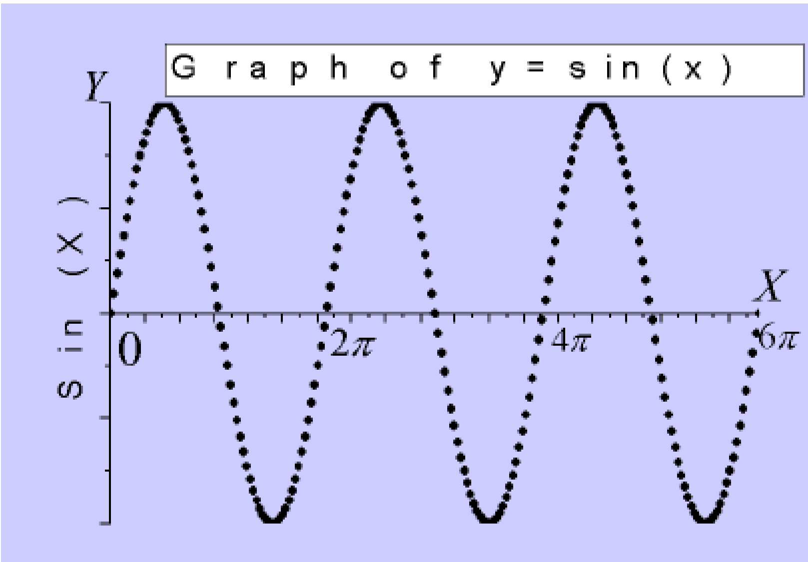
f(x) = f(x+T) = f(x+2T) = f(x+3T) = >.. = f(x+nT)

Thus, f(x) = f(x+nT), n=1,2,3,>..

The function f(x) = sinx is periodic of period 2π since

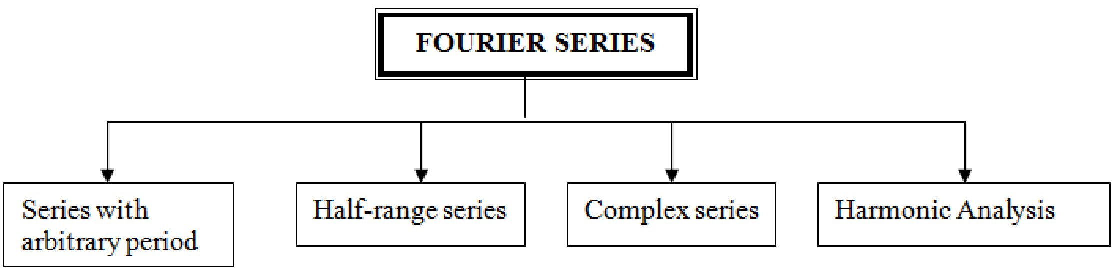
Sin(x+2nπ) = sinx, n=1,2,3,...

The graph of the function is shown below :



FOURIER SERIES

A Fourier series of a periodic function consists of a sum of sine and cosine terms. Sines and cosines are the most fundamental periodic functions. The Fourier series is named after the French Mathematician and Physicist Jacques Fourier (1768 – 1830). Fourier series has its application in problems pertaining to Heat conduction, acoustics, etc. The subject matter may be divided into the following sub topics.



FORMULA FOR FOURIER SERIES

Dirichlet conditions

Dirichlet conditions are sufficient conditions for a real-valued, periodic function *f*(*x*) to be equal to the sum of its Fourier series at each point where *f* is continuous. Moreover, the behaviour of the Fourier series at points of discontinuity is determined as well (it is the midpoint of the values of the discontinuity). These conditions are named after Peter Gustav Lejeune Dirichlet.

The conditions are:

• *f*(*x*) must be absolutely integrable over a period.

• *f*(*x*) must have a finite number of extrema in any given bounded interval, i.e. there must be a finite number of maxima and minima in the interval.

• *f*(*x*) must have a finite number of discontinuities in any given bounded interval, however the discontinuity cannot be infinite.

Let

$$a_0 = \frac{1}{l}\int_a^{a+2l} f(x)dx \qquad (1)$$

$$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx, \qquad n = 1,2,3,..... \quad (2)$$

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx, \qquad n = 1,2,3,...... \quad (3)$$

Then, the infinite series

$$f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x \qquad (4)$$

is called the Fourier series of f(x) in the interval (a,a+2*l*). Also, the real numbers a_0, a_1, a_2,…

.a_n, and b_1, b_2 , …,.b_n are called the Fourier coefficients of f(x). The formulae (1), (2) and (3) are called Euler's formulae. It can be proved that the sum of the series (4) is f(x) if f(x) is continuous at x. Thus we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}\right)x + b_n \sin\left(\frac{n\pi}{l}\right)x \ldots\ldots \qquad (5)$$

Suppose f(x) is discontinuous at x, then the sum of the series (4) would be

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

where $f(x^+)$ and $f(x^-)$ are the values of f(x) immediately to the right and to the left of f(x) respectively.

Some useful Results

1. The following rule called Bernoulli's generalized rule of integration by parts is useful in evaluating the Fourier coefficients.

$\int uv dx = uv_1 - u' v_2 + u'' v_3 + \ldots\ldots$ Here u', u'' ,>.. are the successive derivatives of u and $v_1 = \int v dx, v_2 = \int v_1 dx, \ldots\ldots$

We illustrate the rule, through the following examples :

$$\int x^2 \sin nx dx = x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)$$

$$\int x^3 e^{2x} dx = x^3\left(\frac{e^{2x}}{2}\right) - 3x^2\left(\frac{e^{2x}}{4}\right) + 6x\left(\frac{e^{2x}}{8}\right) - 6\left(\frac{e^{2x}}{16}\right)$$

2. The following integrals are also useful :

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2}\left[a \cos bx + b \sin bx\right]$$

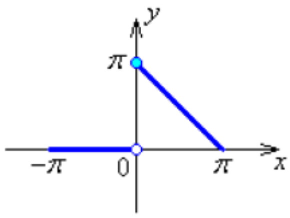
$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2}\left[a \sin bx - b \cos bx\right]$$

3. If 'n' is integer, then sin nπ = 0, cosnπ = $(-1)^n$, sin2nπ = 0, cos2nπ =1

Examples for the interval (0,2π) and (-π, π)

1. Expand the following in a Fourier series

$$f(x) = \begin{cases} 0 & (-\pi < x < 0) \\ \pi - x & (0 \le x < +\pi) \end{cases}$$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{(\pi - x)^2}{-2}\right]_0^{\pi} = \frac{\pi}{2}$$

D		I
$\pi - x$		$\cos nx$
	+	
-1		$\frac{1}{n}\sin nx$
	−	
0		$-\frac{1}{n^2}\cos nx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\sin nx - \cos nx}{n^2}\right]_0^{\pi} = \frac{1 - (-1)^n}{n^2\pi}$$

D		I
$\pi - x$		nx
	+	
-1		$-\frac{1}{n}\cos nx$
	−	
0		$-\frac{1}{n^2}\sin nx$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\cos nx + \sin nx}{-n^2}\right]_0^{\pi} = \frac{1}{n}$$

Therefore the Fourier series for $f(x)$ is

$$\boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right) \qquad (-\pi < x < +\pi)}$$

2. Obtain the Fourier expansion of f(x) = ½ (π-x) in -π<x<π.

   We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{-\pi}^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\cos nx dx$$

Here we use integration by parts, so that

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}[0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\sin nx dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{-\cos nx}{n} - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{(-1)^n}{n}$$

Using the values of a0 , an and bn in the Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

We get

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

This is the required Fourier expansion of the given function.

3. Obtain the Fourier expansion of f(x) =x^2 in (-π, π). Deduce that

$$\frac{\pi^2}{6} = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \infty$$

Solution

The function f(x) is even, Hence

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx, \quad \text{since f(x)cosnx is even}$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx dx$$

Integrating by parts, we get

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4(-1)^n}{n^2}$$

Also, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$ since f(x).sinnx is odd.

Thus

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

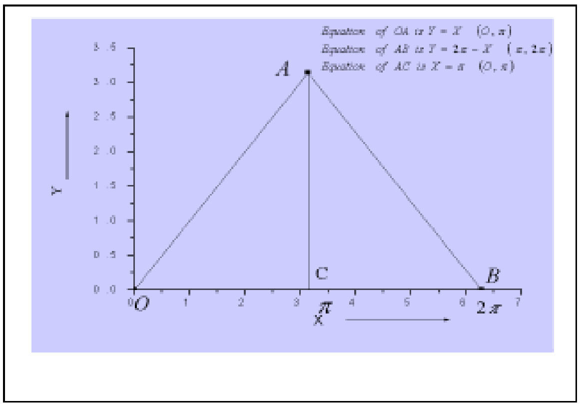
Hence, $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ...$

4. Obtain the Fourier expansion of $f(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x \le 2\pi \end{cases}$

   Deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

   Solution

   The graph of f(x) is shown below.



Here OA represents the line f(x)=x, AB represents the line f(x)=(2π-x) and AC represents the line x=π. Note that the graph is symmetrical about the line AC, which in turn is parallel to y-axis. Hence the function f(x) is an even function. Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$

$$= \frac{2}{\pi}\int_{0}^{\pi} x dx = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx$$ since f(x)cosnx is even.

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0,$$ since f(x)sinnx is odd

Thus the Fourier series of f(x) is

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx$$

For x=π , we get

$$f(\pi) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos n\pi$$

or $$\pi = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-2\cos(2n-1)\pi}{(2n-1)^2}$$

Thus, $$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

or $$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

5. Find the Fourier series expansion for the standard square wave,

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 \le x < 1 \end{cases}$$

Solution $l = 1$.

The function is odd ($f(-x) = -f(x)$ for all x).

Therefore $a_n = 0$ for all n. We will have a Fourier sine series only.

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0} -\sin n\pi x\,dx + \int_{0}^{1} \sin n\pi x\,dx$$

$$= \left[\frac{\cos n\pi x}{n\pi}\right]_{-1}^{0} + \left[\frac{-\cos n\pi x}{n\pi}\right]_{0}^{1} = \frac{2\left(1-(-1)^n\right)}{n\pi}$$

HALF-RANGE FOURIER SERIES

The Fourier expansion of the periodic function f(x) of period $2l$ may contain both sine and cosine terms. Many a time it is required to obtain the Fourier expansion of f(x) in the interval $(0,l)$ which is regarded as half interval. The definition can be extended to the other half in such a manner that the function becomes even or odd. This will result in cosine series or sine series only.

Half Range Sine series

Suppose f(x) is given in the interval $(0,l)$. Then Half range sine series of f(x) over $(0,l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

The half-range sine series of f(x) over $(0,\pi)$ given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Half Range cosine series :

The half-range cosine series of f(x) over (0, l) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where,

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

The half-range cosine series over (0, π) I given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \quad \text{n} = 1,2,3,\ldots..$$

6. The Fourier series of $f(x) = |x|$ in [-1, 1] .

   Solution

$$a_n = \frac{2}{1}\int_0^1 x\cos\left(\frac{n\pi x}{1}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

$$\Rightarrow \quad a_n = 2\left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)\right]_0^1$$

$$= \frac{2\left((-1)^n - 1\right)}{(n\pi)^2}$$

and $a_0 = \frac{2}{1}\int_0^1 x\,dx = \left[x^2\right]_0^1 = 1$

Evaluating the first few terms,

$$a_0 = 1, \quad a_1 = \frac{-4}{\pi^2}, \quad a_2 = 0, \quad a_3 = \frac{-4}{9\pi^2}, \quad a_4 = 0, \quad a_5 = \frac{-4}{25\pi^2}, \quad a_6 = 0, \ldots$$

or $a_n = \begin{cases} 1 & (n=0) \\ \frac{-4}{(n\pi)^2} & (n = 1,3,5,\ldots) \\ 0 & (n = 2,4,6,\ldots) \end{cases}$

Therefore the Fourier cosine series for $f(x) = x$ on [0, 1] (which is also the Fourier series for $f(x) = |x|$ on [–1, 1]) is

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos((2k-1)\pi x)}{(2k-1)^2}$$

or

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos\pi x + \frac{\cos 3\pi x}{9} + \frac{\cos 5\pi x}{25} + \frac{\cos 7\pi x}{49} + \ldots\right)$$

7. Expand f(x) = x(π-x) as half-range sine series over the interval (0,π).

Solution

We have

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\,dx$$

Integrating by parts, we get

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{4}{n^3\pi}\left[1-(-1)^n\right]$$

The sine series of f(x) is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[1-(-1)^n\right]\sin nx$$

8. Expand $f(x) = \cos x$, $0 < x < \pi$ in a Fourier sine series.

Solution

Fourier sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi \cos x\,\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^\pi 2\sin nx\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^\pi [\sin(n+1)x + \sin(n-1)x]\,dx, \quad n \neq 1$$

$$= \frac{1}{\pi}\left[\left(\frac{-\cos(n+1)x}{n+1}\right) + \left(\frac{-\cos(n-1)x}{n-1}\right)\right]_0^\pi$$

$$= -\frac{1}{\pi}\left[\left\{\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right\} - \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$ 2SinACosB = Sin(A+B) + Sin(A–B)

$$= -\frac{1}{\pi}\left[(-1)^n\left\{\frac{-1}{n+1} + \frac{-1}{n-1}\right\} - \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$\cos(n+1)\pi = (-1)^{n+1}$$
$$\cos(n-1)\pi = (-1)^{n-1}$$

$$= \frac{1}{\pi}\left[(-1)^n\left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} + \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left\{\frac{2n}{n^2-1}\right\} + \left\{\frac{2n}{n^2-1}\right\}\right]$$

$$b_n = \frac{2n}{\pi(n^2-1)}\left[(-1)^n + 1\right], \quad n \neq 1$$

When n = 1, we have

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\,dx = \frac{2}{\pi}\int_0^{\pi}\cos x\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = -\frac{1}{2\pi}(1-1) = 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = b_1\sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= 0 + \sum_{n=2}^{\infty}\frac{2n[(-1)^n+1]}{\pi(n^2-1)}\sin nx$$

Interval	Fourier series of f(x)=	a_0	a_n	b_n
$(0,2l)$	$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{l}\right)x + b_n\sin\left(\frac{n\pi}{l}\right)x$	$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$	$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx,$	$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx.$
$(-l, l)$		$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$	$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx$	$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx.$
$(0,2\pi)$	$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$	$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$	$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx$	$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx$
$(-\pi, \pi)$		$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$	$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$	$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$
	Half Range sine series			
$(0,l)$	$f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$	-	-	$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$
$(0,\pi)$	$f(x) = \sum_{n=1}^{\infty} b_n\sin(nx)$	-	-	$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$
	Half Range cosine series			
$(0,l)$	$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right)$	$a_0 = \frac{2}{l}\int_0^{l} f(x)dx$	$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$	-
$(0,\pi)$	$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(nx)$	$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$	$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx$	

Root Mean Square Value(RMS value)

The RMS value of a function *f(x)* in (a,b) is defined by

$$\bar{y} = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}$$

$$\bar{y}^2 = \frac{1}{b-a}\int_a^b [f(x)]^2\,dx$$

Parseval's Identity For Fourier Series

The Parseval's identity for Fourier series in the interval (c, c + 2*l*) is

$$\frac{1}{l}\int_c^{c+2l} [f(x)]^2\,dx = \frac{a_0^{\,2}}{2} + \sum_{n=1}^{\infty}(a_n^{\,2} + b_n^{\,2})$$

The Parseval's identity for Fourier series in the interval (c, c + 2π) is

$$\frac{1}{\pi}\int_c^{c+2\pi} [f(x)]^2\,dx = \frac{a_0^{\,2}}{2} + \sum_{n=1}^{\infty}(a_n^{\,2} + b_n^{\,2})$$

9. Expand $f(x) = x - x^2$ as a Fourier series in $-l < x < l$ and using this series find the root square mean value of $f(x)$ in the interval.

   Solution

   Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{l}(x - x^2)\,dx$$

$$= \frac{1}{l}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{\frac{l^2}{2} - \frac{l^3}{3}\right\} - \left\{\frac{l^2}{2} + \frac{l^3}{3}\right\}\right]$$

$$= \frac{1}{l}\left(\frac{-2l^3}{3}\right) = \frac{-2l^2}{3}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{1}{l}\int_{-l}^{l}(x-x^2)\cos\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[(x-x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{0 + (1-2l)\left(\frac{(-1)^n l^2}{n^2\pi^2}\right) + 0\right\} - \left\{0 + (1+2l)\left(\frac{(-1)^n l^2}{n^2\pi^2}\right) + 0\right\}\right]$$

$$= \frac{(-1)^n l^2}{l\, n^2\pi^2}\left[1 - 2l - 1 - 2l\right]$$

$$= \frac{(-1)^n l}{n^2\pi^2}\left[-4l\right] = \frac{4\, l^2\,(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{l}\int_{-l}^{l}(x-x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[(x-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{-(l-l^2)\left(\frac{(-1)^n l}{n\pi}\right)+0-\frac{2(-1)^n l^3}{n^3\pi^3}\right\}-\left\{-(-l-l^2)\left(\frac{(-1)^n l}{n\pi}\right)+0-\frac{2(-1)^n l^3}{n^3\pi^3}\right\}\right]$$

$$= \frac{-(-1)^n l}{l\,n\pi}\left[l-l^2+l+l^2\right]$$

$$= \frac{(-1)^{n+1}}{n\pi}[2l] = \frac{2l\,(-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$= \frac{1}{2}\left(\frac{-2l^2}{3}\right)+\sum_{n=1}^{\infty}\left(\frac{4l^2(-1)^{n+1}}{n^2\pi^2}\cos\frac{n\pi x}{l}+\frac{2l(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{l}\right)$$

$$(i.e.)\ f(x) = \frac{-l^2}{3}+\frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l}-\frac{1}{2^2}\cos\frac{2\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}-\frac{1}{4^2}\cos\frac{4\pi x}{l}+\dots\dots\right]$$
$$+\frac{2l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l}-\frac{1}{2}\sin\frac{2\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}-\frac{1}{4}\sin\frac{4\pi x}{l}+\dots\dots\right]$$

RMS value of $f(x)$ in $(-l, l)$ is

$$\bar{y}^2 = \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$= \frac{1}{4}\left(\frac{-2l^2}{3}\right)^2+\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{16l^4(-1)^{2n+2}}{n^4\pi^4}+\frac{4l^2(-1)^{2n+2}}{n^2\pi^2}\right]$$

$$(i.e.)\ \bar{y}^2 = \frac{l^4}{9}+\sum_{n=1}^{\infty}\left[\frac{8l^4}{n^4\pi^4}+\frac{2l^2}{n^2\pi^2}\right]$$

10. Find the half range cosine series for $f(x) = x(\pi - x)$ in $0 < x < \pi$.

Deduce that $\frac{\pi^2}{90} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$

Solution

Half range fourier cosine series is

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \;= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) - (0-0)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{6}\right]$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \;= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 + \frac{(-\pi)(-1)^n}{n^2} + 0\right\} - \left\{0 + \frac{(\pi)(1)}{n^2} + 0\right\}\right]$$

$$= \frac{2\pi}{\pi n^2}\left[-(-1)^n - 1\right]$$

$$= -\frac{2}{n^2}\left[(-1)^n + 1\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{1}{2}\left(\frac{\pi^2}{3}\right) + \sum_{n=1}^{\infty} -\frac{2}{n^2}\left[(-1)^n + 1\right]\cos nx$$

$$= \frac{\pi^2}{6} - 2\left[0 + \frac{2\cos 2x}{2^2} + 0 + \frac{2\cos 4x}{4^2} + 0 + \frac{2\cos 6x}{6^2} + 0 + \ldots\ldots\ldots\ldots\right]$$

$$= \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \ldots\ldots\ldots\ldots\right]$$

Parseval's identity for half range fourier cosine series is

$$\frac{2}{\pi}\int_0^{\pi}[f(x)]^2\,dx = \frac{a_0^{\,2}}{2} + \sum_{n=1}^{\infty} a_n^{\,2}$$

$$\frac{2}{\pi}\int_0^{\pi}[\pi x - x^2]^2\,dx = \frac{1}{2}\left(\frac{\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty}\frac{4}{n^4}[(-1)^n+1]^2$$

$$\frac{2}{\pi}\int_0^{\pi}(\pi^2x^2 + x^4 - 2\pi x^3)dx \quad = \frac{\pi^4}{18} + 4\left[0+\frac{4}{2^4}+0+\frac{4}{4^4}+0+\frac{4}{6^4}+0+\ldots\ldots\ldots\ldots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^2x^3}{3}+\frac{x^5}{5}-\frac{2\pi x^4}{4}\right]_0^{\pi} = \frac{\pi^4}{18}+\frac{16}{2^4}\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots\ldots\ldots\right]$$

$$\frac{2}{\pi}\left[\left(\frac{\pi^5}{3}+\frac{\pi^5}{5}-\frac{\pi^5}{2}\right)-0\right] = \frac{\pi^4}{18}+\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots\ldots\ldots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^5}{30}\right]-\frac{\pi^4}{18} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\pi^4}{15}-\frac{\pi^4}{18} \quad = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots\ldots\ldots$$

$$(i.e.)\quad \frac{\pi^4}{90} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots\ldots\ldots$$

11. Find the half range cosine series for the function *f(x)* = *x* in 0 < x < *l*.

    Hence deduce the value of the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$

    Solution

    Half range Fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx \quad = \frac{2}{l}\int_0^{l} x\,dx = \frac{2}{l}\left[\frac{x^2}{2}\right]_0^{l} = \frac{2}{l}\left[\frac{l^2}{2}-0\right] = l$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{0 + \frac{(-1)^n l^2}{n^2\pi^2}\right\} - \left\{0 + \frac{l^2}{n^2\pi^2}\right\}\right]$$

$$= \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= \frac{l}{2} + \sum_{n=1}^{\infty}\frac{2l[(-1)^n - 1]}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l}{2} + \frac{2l}{\pi^2}\left[-\frac{2}{1^2}\cos\frac{\pi x}{l} + 0 - \frac{2}{3^2}\cos\frac{3\pi x}{l} + 0 - \frac{2}{5^2}\cos\frac{5\pi x}{l} + 0 - \dots\dots\dots\right]$$

$$(i.e.)\ f(x) = \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \dots\dots\dots\right]$$

Using Parseval's identity for half range Fourier cosine series we have

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = \frac{a_0^{\ 2}}{2} + \sum_{n=1}^{\infty} a_n^{\ 2}$$

$$\frac{2}{l}\int_0^l (x)^2\,dx = \frac{l^2}{2} + \sum_{n=1}^{\infty}\left[\frac{4l^2\{(-1)^n - 1\}^2}{n^4\pi^4}\right]$$

$$\frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{l^2}{2} + \frac{4l^2}{\pi^4}\left[\frac{4}{1^4} + 0 + \frac{4}{3^4} + 0 + \frac{4}{5^4} + 0 + \dots\dots\dots\right]$$

$$\frac{2}{l}\left[\frac{l^3}{3} - 0\right] = \frac{l^2}{2} + \frac{16l^2}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\dots\dots\right]$$

$$\frac{2l^2}{3} - \frac{l^2}{2} = \frac{16l^2}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\dots\dots\right]$$

$$\frac{l^2}{6} = \frac{16l^2}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\dots\dots\right]$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\dots\dots \quad (i.e.)\ \frac{\pi^4}{96} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

COMPLEX FORM OF FOURIER SERIES

Complex form of Fourier Series of f(x) is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

Where $$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L} dx$$

12. Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 < x < 1$.

Solution

The complex form of Fourier series of *f(x)* is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$$

$$C_n = \frac{1}{2(1)}\int_{-1}^{1} f(x) e^{-in\pi x} dx \qquad 2l = 2,\ l = 1$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x} dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x} dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1}$$

$$= \frac{-1}{2(1+in\pi)}\left[e^{-(1+in\pi)} - e^{(1+in\pi)}\right]$$

$$= \frac{-(1-in\pi)}{2(1+n^2\pi^2)}\left[e^{-1}e^{-in\pi} - e^{1}e^{in\pi}\right]$$

$$= \frac{-(1-in\pi)}{2(1+n^2\pi^2)}\left[e^{-1}(\cos n\pi - i\sin n\pi) - e^{1}(\cos n\pi - i\sin n\pi)\right]$$

$$C_n = \frac{-(1-in\pi)}{2(1+n^2\pi^2)}\left[e^{-1}(-1)^n - e^{1}(-1)^n\right]$$

$$= \frac{(1-in\pi)(-1)^n}{2(1+n^2\pi^2)}\left[e^1 - e^{-1}\right]$$

$$= \frac{(1-in\pi)(-1)^n}{2(1+n^2\pi^2)} 2\sinh 1$$

$$C_n = \frac{(-1)^n \sinh 1(1-in\pi)}{1+n^2\pi^2}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh 1(1-in\pi)}{1+n^2\pi^2}\, e^{in\pi x}$$

HARMONIC ANALYSIS

The Fourier series of a known function f(x) in a given interval may be found by finding the Fourier coefficients. The method described cannot be employed when f(x) is not known explicitly, but defined through the values of the function at some equidistant points. In such a case, the integrals in Euler's formulae cannot be evaluated. Harmonic analysis is the process of finding the Fourier coefficients numerically.

To derive the relevant formulae for Fourier coefficients in Harmonic analysis, we employ the following result :

The mean value of a continuous function f(x) over the interval (a,b) denoted by [f(x)] is defined as $\left[f(x)\right] = \frac{1}{b-a}\int_a^b f(x)dx$.

The Fourier coefficients defined through Euler's formulae, (1), (2), (3) may be redefined as

$$a_0 = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)dx\right] = 2[f(x)]$$

$$a_n = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right] = 2\left[f(x)\cos\left(\frac{n\pi x}{l}\right)\right]$$

$$b_n = 2\left[\frac{1}{2l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right] = 2\left[f(x)\sin\left(\frac{n\pi x}{l}\right)\right]$$

Using these in (5), we obtain the Fourier series of f(x). The term a1cosx+b1sinx is called the first harmonic or fundamental harmonic, the term a2cos2x+b2sin2x is called the second harmonic and so on. The amplitude of the first harmonic is $\sqrt{a_1{}^2 + b_1{}^2}$ and that of second harmonic is $\sqrt{a_2{}^2 + b_2{}^2}$ and so on.

13. Find the first two harmonics of the Fourier series of f(x) given the following table

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
f(x)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Note that the values of y = f(x) are spread over the interval $0 \le x \le 2\pi$ and f(0) = f(2π) = 1.0. Hence the function is periodic and so we omit the last value f(2π) = 0. We prepare the following table to compute the first two harmonics.

Solution

x^0	y = f(x)	cosx	cos2x	sinx	sin2x	ycosx	ycos2x	ysinx	ysin2x
0	1.0	1	1	0	0	1	1	0	0
60	1.4	0.5	-0.5	0.866	0.866	0.7	-0.7	1.2124	1.2124
120	1.9	-0.5	-0.5	0.866	-0.866	-0.95	-0.95	1.6454	-1.6454
180	1.7	-1	1	0	0	-1.7	1.7	0	0
240	1.5	-0.5	-0.5	-0.866	0.866	-0.75	-0.75	1.299	1.299
300	1.2	0.5	-0.5	-0.866	-0.866	0.6	-0.6	-1.0392	-1.0392
Total						-1.1	-0.3	3.1176	-0.1732

We have

$$a_n = 2\left[f(x)\cos\left(\frac{n\pi x}{l}\right)\right] = 2[y\cos nx]$$

as the length of interval= 2*l* = 2π or *l*=π

$$b_n = 2\left[f(x)\sin\left(\frac{n\pi x}{l}\right)\right] = 2[y\sin nx]$$

Putting, n=1,2, we get

$$a_1 = 2[y\cos x] = \frac{2\sum y\cos x}{6} = \frac{2(1.1)}{6} == -0.367$$

$$a_2 = 2[y\cos 2x] = \frac{2\sum y\cos 2x}{6} = \frac{2(-0.3)}{6} = -0.1$$

$$b_1 = [y\sin x] = \frac{2\sum y\sin x}{6} = 1.0392$$

$$b_2 = [y\sin 2x] = \frac{2\sum y\sin 2x}{6} = -0.0577$$

The first two harmonics are a1cosx+b1sinx and a2cos2x+b2sin2x. That is (-0.367cosx + 1.0392sinx) and (-0.1cos2x – 0.0577sin2x).

14. Express y as a Fourier series upto the third harmonic given the following values :

x	0	1	2	3	4	5
y	4	8	15	7	6	2

The values of y at x=0,1,2,3,4,5 are given and hence the interval of x should be $0 \le x < 6$. The length of the interval = 6-0 = 6, so that 2*l* = 6 or *l* = 3.

Solution

The Fourier series upto the third harmonic is

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right) + \left(a_3 \cos\frac{3\pi x}{l} + b_3 \sin\frac{3\pi x}{l}\right)$$

or

$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}\right) + \left(a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3}\right) + \left(a_3 \cos\frac{3\pi x}{3} + b_3 \sin\frac{3\pi x}{3}\right)$$

Put $\theta = \frac{\pi x}{3}$, then

$$y = \frac{a_0}{2} + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta) + (a_3 \cos 3\theta + b_3 \sin 3\theta) \qquad (1)$$

We prepare the following table using the given values :

x	$\theta = \frac{\pi x}{3}$	y	ycosθ	ycos2θ	ycos3θ	ysinθ	ysin2θ	ysin3θ
0	0	04	4	4	4	0	0	0
1	60^0	08	4	-4	-8	6.928	6.928	0
2	120^0	15	-7.5	-7.5	15	12.99	-12.99	0
3	180^0	07	-7	7	-7	0	0	0
4	240^0	06	-3	-3	6	-5.196	5.196	0
5	300^0	02	1	-1	-2	-1.732	-1.732	0
Total		42	-8.5	-4.5	8	12.99	-2.598	0

$$a_0 = 2[f(x)] = 2[y] = \frac{2\sum y}{6} = \frac{1}{3}(42) = 14$$

$$a_1 = 2[y\cos\theta] = \frac{2}{6}(-8.5) = -2.833$$

$$b_1 = 2[y\sin\theta] = \frac{2}{6}(12.99) = 4.33$$

$$a_2 = 2[y\cos 2\theta] = \frac{2}{6}(-4.5) = -1.5$$

$$b_2 = 2[y\sin 2\theta] = \frac{2}{6}(-2.598) = -0.866$$

$$a_3 = 2[y\cos 3\theta] = \frac{2}{6}(8) = 2.667$$

$$b_3 = 2[y\sin 3\theta] = 0$$

Using these in (1), we get

$$y = 7 - 2{,}833\cos\left(\frac{\pi x}{3}\right) + (4.33)\sin\left(\frac{\pi x}{3}\right) - 1.5\cos\left(\frac{2\pi x}{3}\right) - 0.866\sin\left(\frac{2\pi x}{3}\right) + 2.667\cos \pi x$$

This is the required Fourier series upto the third harmonic.

15. The following table gives the variations of a periodic current A over a period T :

t(secs)	0	T/6	T/3	T/2	2T/3	5T/6	T
A (amp)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a constant part of 0.75amp. in the current A and obtain the amplitude of the first harmonic.

Note that the values of A at t=0 and t=T are the same. Hence A(t) is a periodic function of period T. Let us denote $\theta = \left(\frac{2\pi}{T}\right) t$

We have

$$a_0 = 2[A]$$
$$a_1 = 2\left[A\cos\left(\frac{2\pi}{T}\right)t\right] = 2[A\cos\theta] \qquad (1)$$
$$b_1 = 2\left[A\sin\left(\frac{2\pi}{T}\right)t\right] = 2[A\sin\theta]$$

Using the values of the table in (1), we get

$$a_0 = \frac{2\sum A}{6} = \frac{4.5}{3} = 1.5$$
$$a_1 = \frac{2\sum A\cos\theta}{6} = \frac{1.12}{3} = 0.3733$$
$$b_1 = \frac{2\sum A\sin\theta}{6} = \frac{3.0137}{3} = 1.0046$$

t	$\theta = \frac{2\pi t}{T}$	A	cosθ	sinθ	Acosθ	Asinθ
0	0	1.98	1	0	1.98	0
T/6	60^0	1.30	0.5	0.866	0.65	1.1258
T/3	120^0	1.05	-0.5	0.866	-0.525	0.9093
T/2	180^0	1.30	-1	0	-1.30	0
2T/3	240^0	-0.88	-0.5	-0.866	0.44	0.7621
5T/6	300^0	-0.25	0.5	-0.866	-0.125	0.2165
Total		4.5			1.12	3.0137

The Fourier expansion upto the first harmonic is

$$A = \frac{a_0}{2} + a_1 \cos\left(\frac{2\pi t}{T}\right) + b_1 \sin\left(\frac{2\pi t}{T}\right)$$
$$= 0.75 + 0.3733 \cos\left(\frac{2\pi t}{T}\right) + 1.0046 \sin\left(\frac{2\pi t}{T}\right)$$

The expression shows that A has a constant part 0.75 in it. Also the amplitude of the first harmonic is $\sqrt{{a_1}^2 + {b_1}^2} = 1.0717$.

SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS
COURSE MATERIAL
COURSE NAME: ENGINEERING MATHEMATICS IV **COURSE CODE: SMT1204**

UNIT II

INTRODUCTION

The problems related to fluid mechanics, solid mechanics, heat transfer, wave equation and other areas of physics are designed as Initial Boundary Value Problems consisting of partial differential equations and initial conditions. These problems can be solved by "Method of separation of variables," in this unit we derive and solve one dimensional heat equation, wave equation, Laplace's equation in two dimensions etc. by separation of variables method. The general solution of partial differential equation consists arbitrary functions which can be obtained by Fourier Series.

METHOD OF SEPARATION OF VARIABLES

In this method, we assume that the required solution is the product of two functions *i.e.*,

$$u(x, y) = X(x)Y(y) \qquad \ldots(i)$$

Then we substitute the value of $u(x, y)$ from (i) and its derivatives reduces the $P.D.E.$ to the form

$$f_1(X, X', \cdots\cdots) = f_2(Y, Y', \cdots\cdots) \qquad \ldots(ii)$$

which is separable in X and Y. Since $f_2(Y, Y', \cdots)$ is function Y only and $f_1(X, X', \cdots)$ is function of X only, then equation (ii) must be equal to a common constant say k. Thus (ii) reduces to

$$f_1(X, X', \cdots) = f_2(X, X', \cdots) = k.$$

ONE DIMENSIONAL WAVE EQUATION

The one dimensional wave equation arises in the study of transverse vibrations of an elastic string. Consider an elastic string, stretched to its length 'l' between two points O and A fixed. Let the function $y(x, t)$ denote the displacement of string at any point x and at any time $t > 0$ from the equilibrium position (x-axis). When the string released after stretching then it vibrates and therefore the transverse vibrations formed a one dimensional wave equation.

Let the string is perfectly flexible and does not offer resistance to bending. Let T_1 and T_2 betensions at the end points P and Q of the portion of the string. Since there is no motion in the horizontal direction. Thus the sum of the forces in the horizontal direction must be zero *i.e.,*

$$-T_1 \cos\alpha + T_2 \cos\beta = 0$$

$$\Rightarrow \quad T_1 \cos\alpha = T_2 \cos\beta = T \quad \text{(constant)} \qquad \ldots(i)$$

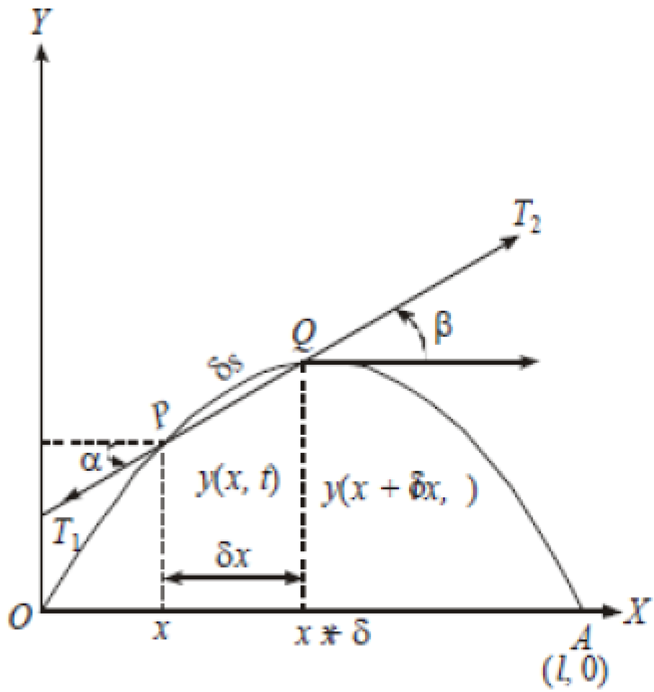
Let m be the mass of the string per unit length then the mass of portion $PQ = m\delta s$.

Now by Newton's second law of motion, the equation of motion in the vertical direction is

mass × acceleration = resultant of forces

$$m\delta s \frac{\partial^2 y}{\partial t^2} = T_2 \sin\beta - T_1 \sin\alpha \qquad \ldots(ii)$$

Dividing (ii) by (i), we have



$$\Rightarrow \qquad \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}$$

Thus $$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.}$$

Where $c^2 = \frac{T}{m}$. Equation (iii) is known as one dimensional wave equation.

Solution of One dimensional wave equation

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \ldots(i)$$

Let $y = X(x)\, T(t)$. $\qquad \ldots(ii)$

where X is the function of x only and T is the function of t only.

Then $\frac{\partial^2 y}{\partial t^2} = X\frac{d^2 T}{dt^2}$ and $\frac{\partial^2 y}{\partial x^2} = T\frac{d^2 X}{dx^2}$

Putting these values in equation (i) we get

$$X\frac{d^2 T}{dt^2} = c^2 T\frac{d^2 X}{dx^2}$$

$$\Rightarrow \quad \frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{c^2 T}\frac{d^2 T}{dt^2} = k \qquad \ldots(iii)$$

Now $\frac{1}{X}\frac{d^2 X}{dx^2} = k$

$\Rightarrow \quad (D^2 - k)X = 0; \quad \frac{d}{dx} = D$

The A.E. is $m^2 - k = 0$

$$m = \pm\sqrt{k}$$

$$\therefore \quad \boxed{X = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}}$$

and, again from (iii), we get

$$\frac{d^2 T}{dt^2} = kc^2 T \quad \Rightarrow \quad (D^2 - kc^2)T = 0\,; \quad D = \frac{d}{dt}$$

$\Rightarrow$ The A.E. is $m^2 - kc^2 = 0 \quad \Rightarrow \quad m = \pm c\sqrt{k}$

$$\therefore \quad \boxed{T = c_3 e^{c\sqrt{k}t} + c_4 e^{-c\sqrt{k}t}}.$$

Thus, from equation (ii), we get

$$y = (c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x})(c_3 e^{c\sqrt{k}t} + c_4 e^{-c\sqrt{k}t})$$

There are arise following cases:

Case I: If $k > 0$ let $k = p^2$

then $$\boxed{y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt}).} \quad ...(A)$$

Case II: If $k < 0$, let $k = -p^2$

then $m^2 = -p^2 \Rightarrow m = \pm pi$

$\therefore$ $X = c_1 \cos px + c_2 \sin px$

and $m^2 = -p^2 c^2 \Rightarrow m = \pm ipc$

$\therefore$ $T = c_3 \cos cpt + c_4 \sin cpt$

then $$\boxed{y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt.} \quad ...(B)$$

Case III: If $k = 0$

then $D^2 X = 0 \Rightarrow m = 0, 0$

$\therefore$ $X = (c_1 + c_2 x)$

And $D^2 T = 0 \Rightarrow m = 0, 0$

$\therefore$ $T = (c_3 + c_4 t)$

Then $$\boxed{y = (c_1 + c_2 x)(c_3 + c_4 t).} \quad ...(C)$$

Of these three solutions, we have choose the solution which is consistent with the physical nature of the problem.

Since the physical nature of the one dimensional wave equation is periodic so we consider the solution which is periodic in nature.

Here the solution (B) is periodic (as both sine and cosine are periodic)

Thus the desired solution for one dimensional periodic equation is

$$\boxed{y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt).} \qquad \ldots(iv)$$

Now using the boundary conditions

At $x = 0$ (origin), $y = 0$ from (iv), we get

$$0 = c_1(c_3 \cos cpt + c_4 \sin cpt) \quad \Rightarrow \quad c_1 = 0$$

using the value of c_1 in (iv), we get

$$y(x,t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt) \qquad \ldots(v)$$

At $x = l$ (at A), $y = 0$ from (v), we get

$$0 = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt)$$

$$\Rightarrow \qquad \sin pl = 0 = \sin n\pi \quad \Rightarrow \quad pl = n\pi$$

$$\Rightarrow \qquad \boxed{p = \frac{n\pi}{l}}, \quad \text{where } n = 1, 2, 3, \ldots\ldots$$

Hence, the solution of wave equation satisfying the boundary conditions is, from (v)

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right) \qquad \left|\, p = \frac{n\pi}{l}\right.$$

$$= \left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \qquad \left|\, \begin{matrix} c_2c_3 = a_n \\ c_2c_4 = b_n \end{matrix}\right.$$

$\therefore$ The general solution of wave equation is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}} \qquad \ldots(vi)$$

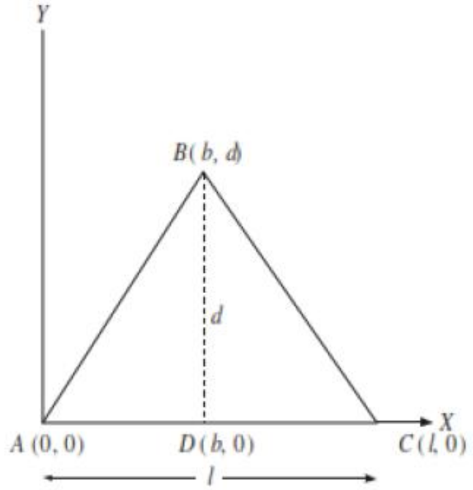
Remark: We can apply initial conditions on above equation (vi) in time domain *i.e.*, at $t = 0$.

Example 1: A string of length l is fastened at both ends A and C. At a distance 'b' from the end A, the string is transversely displaced to a distance 'd' and is released from rest when it is in this position. Find the equation of the subsequent motion.

Solution: Let $y(x, t)$ is the displacement of the string.

Now, by the one dimensional wave equation, we have

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$



$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad ...(2)$$

Now using the boundary conditions as follows:

The boundary conditions are

(*i*) At $x = 0$(at A), $y = 0 \Rightarrow y(0, t) = 0$

and (*ii*) At $x = l$ (at C), $y = 0 \Rightarrow y(l, t) = 0$

From (2), we have

$$0 = c_1(c_3 \cos cpt + c_4 \sin cpt) \Rightarrow c_1 = 0$$

Using $c_1 = 0$, in equation (2), we get

$$y(x,t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt) \quad ...(3)$$

Using (*ii*) boundary condition, from (3), we have

$$0 = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt)$$

$$\Rightarrow \quad \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}.}$$

Using the value of p in (3), we obtain

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right) \quad ...(4)$$

Next, the initial conditions are as follows:

(*iii*) velocity $\frac{\partial y}{\partial t} = 0$ at $t = 0$

and displacement at $t = 0$ is

$$(iv)\ y(x, 0) = \begin{cases} \dfrac{d \cdot x}{b}, & 0 \le x \le b \\ \dfrac{d(x-l)}{(b-l)}, & b \le x \le l \end{cases}$$

Equation of AB is $y = \frac{d \cdot x}{b}$ and equation of BC is $y = \frac{d(x-l)}{(b-l)}$

From (4),

$$\frac{\partial y}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(-\frac{n\pi c c_3}{l}\sin\frac{n\pi ct}{l} + \frac{n\pi c c_4}{l}\cos\frac{n\pi ct}{l}\right)$$

Using (*iii*) in above equation, we get

$$0 = c_2 c_4 \frac{n\pi c}{l} \cdot \sin\frac{n\pi x}{l} \Rightarrow c_4 = 0$$

Using $c_4 = 0$ in equation (4), we get

$$y(x,t) = c_2 c_3 \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi ct}{l}$$

$\therefore$ The general solution of given problem is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi ct}{l} \quad ...(5)$$

Using initial condition (*iv*) in equation (5), we get

$$y(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

which is half range Fourier sine series, so we have

$$b_n = \frac{2}{l}\int_0^l y(x,0)\cdot \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^b \frac{d}{b}x\cdot\sin\left(\frac{n\pi x}{l}\right)dx + \frac{2}{l}\frac{d}{(b-l)}\int_b^l (x-l)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2d}{bl}\left[x\left(\frac{-l}{n\pi}\right)\cos\frac{n\pi x}{l} - \left(\frac{-l^2}{n^2\pi^2}\right)\sin\frac{n\pi x}{l}\right]_0^b$$

$$+ \frac{2d}{l(b-l)}\left[(x-l)\left(\frac{-l}{n\pi}\right)\cos\frac{n\pi x}{l} - \left(\frac{-l^2}{n^2\pi^2}\right)\cdot\sin\frac{n\pi x}{l}\right]_b^l$$

$$\Rightarrow \quad b_n = -\frac{2d}{n\pi}\cos\frac{n\pi b}{l} + \frac{2dl^2}{bl n^2\pi^2}\cdot\sin\frac{n\pi b}{l} + \frac{2d}{n\pi}\cdot\cos\frac{n\pi b}{l}$$

$$- \frac{2dl^2}{l(b-l)n^2\pi^2}\sin\frac{n\pi b}{l}$$

$$\Rightarrow \quad b_n = \frac{2dl^2}{b(l-b)n^2\pi^2}\sin\frac{n\pi b}{l}$$

$\therefore$ From (5), we get

$$y(x,t) = \frac{2dl^2}{b(l-b)\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi b}{l}\times\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi ct}{l}.$$

Example 2: A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin \frac{\pi x}{l}$ from which it is released at a time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by

$$y(x,t) = a \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right).$$

Solution: Let $y(x, t)$ be the displacement at any point $P(x, y)$ at any time.

Then by the wave equation, we have

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The solution of equation (1) is of the form

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \qquad ...(2)$$

Now using the boundary conditions

(*i*) At $x = 0$, the displacement $y = 0 \Rightarrow y(0, t) = 0$

(*ii*) At $x = l$, the displacement $y = 0 \Rightarrow y(l, t) = 0$

Using (*i*) boundary condition in (2), we get

$$y(0,t) = 0 = c_1(c_3 \cos cpt + c_4 \sin cpt) \Rightarrow \boxed{c_1 = 0.}$$

$\therefore$ From (2), we get

$$y(x,t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt) \qquad ...(3)$$

Using (*ii*) boundary condition in equation (3), we get

$$y(l,t) = 0 = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt)$$

$$y(l,t) = 0 = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt)$$

$$\Rightarrow \qquad \sin pl = 0 = \sin n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}.}$$

Now using the initial conditions

(*iii*) At $t = 0$, the velocity $\frac{\partial y}{\partial t} = 0 \Rightarrow \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

(*iv*) At $t = 0$, the displacement $y = a\sin\frac{\pi x}{l} \Rightarrow y(x,0) = a\sin\frac{\pi x}{l}$

$\therefore$ From (3), we have

$$\frac{\partial y}{\partial t} = c_2 \sin px[c_3(-cp)\sin cpt + c_4(cp)\cos cpt]$$

Using (*iii*) initial condition in above equation, we get

$$0 = c_2 c_4 cp \sin px \Rightarrow c_2 c_4 cp = 0$$

$$\Rightarrow \boxed{c_4 = 0.}$$

$c_2 \neq 0$, otherwise there is trivial solution

Using $p = \frac{n\pi}{l}$ and $c_4 = 0$, in equation (3), we get

$$y(x,t) = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

$\therefore$ The general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad (b_n = c_2 c_3) \qquad \ldots(4)$$

Finally using (*iv*) initial condition in equation (4), we get

$$y(x,0) = a\sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

or

$$a\sin\frac{\pi x}{l} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + \cdots\cdots$$

Equating the coefficient of $\sin\frac{\pi x}{l}$, we get

$$b_1 = a, \quad b_2 = b_3 = \cdots = 0$$

Hence the required solution of given problem is

$$y(x,t) = a\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}; \quad n = 1.$$ **Proved.**

Example 3: Find the displacement of a string stretched between two fixed points at a distance $2l$ apart when the string is initially at rest in equilibrium position and points of the string are given initial velocity v where

$$v = \begin{cases} \dfrac{x}{l}, & \text{when } 0 < x < l \\ \dfrac{2l - x}{l}, & \text{when } l < x < 2l \end{cases}$$

x being the distance measured from one end.

Solution: The displacement $y(x, t)$ is given by wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \ldots(1)$$

The solution of equation is given by

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \qquad \ldots(2)$$

Now, the boundry conditions are

(i) At $x = 0, \quad y = 0 \Rightarrow y(0, t) = 0$

(ii) At $x = 2l, \quad y = 0 \Rightarrow y(2l, t) = 0$

Using (i) boundary condition in (2), we get

$$0 = c_1(c_3 \cos cpt + c_4 \sin cpt) \quad \Rightarrow \quad c_1 = 0$$

$\therefore$ From (2), we get

$$y(x, t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt) \qquad \ldots(3)$$

Using (*ii*) condition in (3), we get

$$0 = c_2 \sin 2pl(c_3 \cos cpt + c_4 \sin cpt)$$

$$\Rightarrow \quad \sin 2pl = 0 = \sin n\pi \quad \Rightarrow \quad \boxed{p = \frac{n\pi}{2l}}.$$

Now, the initial conditions are

(*iii*) At $t = 0$ the displacement $y(x, 0) = 0$.

(*iv*) At $t = 0$, $\frac{\partial y}{\partial t} = v$.

Making use of initial condition (*iii*) in (3), we get

$$y(x, 0) = 0 = c_2 \sin px(c_3) \Rightarrow c_3 = 0$$

$\therefore$ From (3), we get

$$y(x, t) = c_2 c_4 \sin \frac{n\pi x}{2l} \sin \frac{n\pi c}{2l} t \qquad \Big| \, p = \frac{n\pi}{2l}$$

The general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2l} \cdot \sin \frac{n\pi c}{2l} t \qquad \ldots(4)$$

$$\Rightarrow \quad \frac{\partial y}{\partial t} = \frac{\pi c}{2l} \sum_{n=1}^{\infty} n b_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi c}{2l} t$$

Using initial condition (*iv*) in above equation, we get

$$v = \frac{\pi c}{2l} \sum_{n=1}^{\infty} n b_n \sin \frac{n\pi x}{2l}$$

which represents half range Fourier sine series

$$\therefore \quad \frac{\pi c}{2l} n b_n = \frac{2}{2l} \int_0^{2l} v \sin \frac{n\pi x}{2l} dx$$

$$= \frac{1}{l} \int_0^{l} \frac{x}{l} \sin \frac{n\pi x}{2l} dx + \frac{1}{l} \int_l^{2l} \left(\frac{2l - x}{l} \right) \sin \frac{n\pi x}{2l} dx$$

$$= \frac{1}{l} \left[\frac{x}{l} (-1) \left(\frac{2l}{n\pi} \right) \cos \frac{n\pi x}{2l} - \frac{1}{l} (-1) \frac{4l^2}{n^2 \pi^2} \sin \frac{n\pi x}{2l} \right]_0^l$$

$$+ \frac{1}{l} \left[\left(\frac{2l - x}{l} \right) \left(\frac{-2l}{n\pi} \right) \cdot \cos \frac{n\pi x}{2l} - \left(\frac{-1}{l} \right) \left(\frac{-4l^2}{n^2 \pi^2} \right) \sin \frac{n\pi x}{2l} \right]_l^{2l}$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\Rightarrow \qquad b_n = \frac{2l}{n\pi c}\left[\frac{8}{n^2\pi^2}\cdot\sin\frac{n\pi}{2}\right]$$

$$\Rightarrow \qquad b_n = \frac{16l}{n^3\pi^3 c}\cdot\sin\frac{n\pi}{2}$$

Hence the displacement function is given by, from equation (4), we get

$$y(x, t) = \frac{16l}{\pi^3 c}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\times\sin\frac{n\pi x}{2l}\cdot\sin\frac{n\pi ct}{2l} \qquad \textbf{Ans.}$$

Example 4: A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of x from one end at time t.

Solution: Let the displacement $y(x, t)$ given by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

$$\therefore \qquad y(x,t) = (c_1\cos px + c_2\sin px)(c_3\cos cpt + c_4\sin cpt) \qquad ...(2)$$

Using the boundary conditions (*i*) $y(0, t) = 0$ (*ii*) $y(l, t) = 0$

Using (*i*) in equation (2), we get

$$y = c_2\sin px\,(c_3\cos cpt + c_4\sin cpt) \qquad ...(3)$$

Using (*ii*), in equation (3), we get

$$0 = c_2\sin pl(c_3\cos cpt + c_4\sin cpt) \quad\Rightarrow\quad \sin pl = 0$$

$$\Rightarrow \qquad \sin pl = \sin n\pi \quad\Rightarrow\quad \boxed{p = \frac{n\pi}{l}}.$$

and the initial conditions are

$$(iii)\ \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad (iv)\ y(x, 0) = k(lx - x^2)$$

$\therefore$ From (3), we get

$$\frac{\partial y}{\partial t} = c_2 \sin px\{-c_3 c \sin cpt + c_4 c \cos cpt\}$$

Using (*iii*) in above relation, we get

$$0 = c_2(c_4 c)\sin px \quad \Rightarrow \quad \boxed{c_4 = 0.}$$

Using the values of c and p in equation (3), we get

$$y = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

$\therefore$ The general solution is

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad \text{...(4)}$$

Making use of (*iv*) in (4), we get

$$k(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

which represents half range Fourier sine series

$$\therefore \qquad b_n = \frac{2k}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$\Rightarrow \qquad b_n = \frac{2k}{l}\left[(lx - x^2)\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{\pi x} - (l - 2x)\left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right.$$

$$\left. + (-2)\left(\cos\frac{n\pi x}{l}\right)\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$\Rightarrow \qquad b_n = \frac{2k}{l}\left[(-1)^{n+1}\frac{2l^3}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right] = \frac{8kl^2}{n^3\pi^3} \text{ when } n \text{ is odd.}$$

$$= 0 \quad \text{when } n \text{ is even.}$$

Hence the required displacement is, from (4), we get

$$y = \sum_{n=1}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l} \cos\frac{n\pi c}{l} t, \quad \text{when } n \text{ is odd.} \quad \textbf{Ans.}$$

Example 5: A string of length l is fastened at both ends A and C. At a distance 'b' from the end A, the string is transversely displaced to a distance 'd' and is released from rest when it is in this position. Find the equation of the subsequent motion.

Solution: We know that the solution of one dimensional wave equation with boundary conditions

$y(0, t) = y(l, t) = 0$ is

$$y(x, t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) \quad \ldots(1)$$

where $p = \frac{n\pi}{l}$.

Now the initial conditions are

(a) $y(x,0) = 0$ $\quad$ (b) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l-x)$

Making use of (a) in (1), we get

$$0 = c_3 c_2 \sin px \quad \Rightarrow \quad \boxed{c_3 = 0.}$$

$\therefore$ From (1), we get

$$y(x,t) = c_2 c_4 \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}$$

The general solution of wave equation is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cdot \sin\frac{n\pi ct}{l} \quad \ldots(2)$$

From (2), $\quad \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} (n\pi c/l) \cdot b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$

$$\Rightarrow \quad \left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l-x) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{l}\right) b_n \sin\frac{n\pi x}{l}$$

$$\therefore \quad \frac{n\pi c}{l} b_n = \frac{2}{l}\int_0^l \lambda x(l-x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2\lambda}{l}\left[x(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\right.$$

$$\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)+(-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\Bigg]_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3}(1-\cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\Rightarrow \qquad b_n = \begin{cases} \dfrac{8\lambda l^3}{cn^4\pi^4}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$\therefore$ From (2) the required solution is

$$y(x,t) = \frac{8\lambda l^3}{c\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}, \quad n \text{ is odd.} \quad \textbf{Ans.}$$

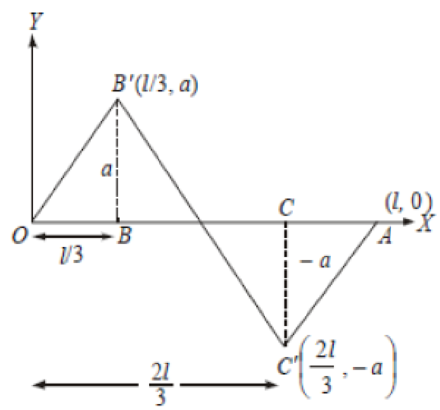
Example 6

The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.

Solution: Let the string OA be trisected at B and C.

Let the equation of vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \ldots(1)$$



$\therefore$ The solution of equation (1) is

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad ...(2)$$

Now using the boundary conditions

(*i*) $y(0, t) = 0$ (*ii*) $y(l, t) = 0$

Making use of (*i*) and (*ii*) in equation (2), we get

$$y(x, t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt) \quad ...(3)$$

where $p = \frac{n\pi}{l}$.

Next equation of OB' is $y = \frac{ax}{l/3} \Rightarrow y = \frac{3a}{l}x$

Equation of $B'C'$ is $y - a = \dfrac{a+a}{\frac{l}{3} - \frac{2l}{3}}\left(x - \frac{l}{3}\right)$ $\quad \left| y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\right.$

$$\Rightarrow \quad y = \frac{3a}{l}(l - 2x)$$

and equation of $C'A$ is $y - 0 = \dfrac{-a - 0}{\frac{2l}{3} - l}(x - l)$

$$\Rightarrow \quad y = \frac{3a}{l}(x - l)$$

$\therefore$ The initial conditions of given problem are

(*iii*) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

$$(iv)\ y(x, 0) = \begin{cases} \frac{3a}{l}x, & 0 \le x \le l/3 \\ \frac{3a}{l}(l - 2x), & \frac{l}{3} \le x \le \frac{2l}{3} \\ \frac{3a}{l}(x - l), & \frac{2l}{3} \le x \le l \end{cases}$$

From equation (3), we get

$$\frac{\partial y}{\partial t} = c_2 \sin px(-c_3 cp \sin cpt + c_4 cp \cos cpt)$$

Using (*iii*) initial condition in above, we obtain

$$0 = c_2 \sin px(c_4 cp) \Rightarrow \boxed{c_4 = 0.}$$

Again from (3), we have

$$y(x,\,t) \;=\; c_2 c_3 \sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi ct}{l}$$

$\therefore$ The general solution of equation (1) is

$$y(x,\,t) \;=\; \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad \ldots(4)$$

Using (*iv*) condition in equation (4), we get

$$y(x,\,0) \;=\; \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\therefore \qquad b_n \;=\; \frac{2}{l}\int_0^l y(x,0)\cdot\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{3}} \frac{3ax}{l}\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{3}}^{\frac{2l}{3}} \frac{3a}{l}(l-2x)\sin\frac{n\pi x}{l}\,dx + \int_{\frac{2l}{3}}^{l} \frac{3a}{l}(x-l)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6a}{l^2}\left[x\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)-(1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{\frac{l}{3}} + \frac{6a}{l^2}\left[(l-2x)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right)\right.$$

$$\left.-(-2)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{\frac{l}{3}}^{\frac{2l}{3}} + \frac{6a}{l^2}\left[(x-l)\left(\frac{-l^2}{n\pi}\cos\frac{n\pi x}{l}\right)-(1)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{\frac{2l}{3}}^{l}$$

$$= \frac{6a}{l^2}\left[\left(-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{l^2}{n^2\pi^2}\sin n\pi\right)+\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}-\frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right.$$

$$\left.+\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3}-\left(\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}+\frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right)\right]$$

$$= \frac{6a}{l^2}\cdot\frac{3l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3}-\sin\frac{2n\pi}{3}\right)$$

$$= \frac{18a}{n^2\pi^2}\sin\frac{n\pi}{3}[1+(-1)^n] \qquad \sin\frac{2n\pi}{3} = \sin\left(n\pi-\frac{n\pi}{3}\right) = -(-1)^n\sin\frac{n\pi}{3}$$

$$\Rightarrow \qquad b_n = \begin{cases} 0, \text{ when } n \text{ is odd} \\ \dfrac{36a}{n^2\pi^2}\sin\dfrac{n\pi}{3}, \text{ when } n \text{ is even} \end{cases}$$

Putting the value of b_n in equation (4), we get

$$y(x, t) = \sum_{n=2(\text{even})}^{\infty} \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}. \quad \textbf{Ans.}$$

Example 7

If the string of length l is initially at rest in equilibrium position and each of its points is given the velocity.

$v_0 \sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{2\pi x}{l}\right)$ where $0 < x < l$ at $t = 0$ determine the displacement function $y(x, t)$.

Solution: The displacement $y(x, t)$ given by wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

we know that the solution of (1) is

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \qquad ...(2)$$

Using the boundary conditions

(*i*) $y(0, t) = 0$ (*ii*) $y(l, t) = 0$

We get from (2)

$$y(x, t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt) \qquad ...(3)$$

where $p = \frac{n\pi}{l}$.

and the initial conditions are

(*iii*) $y(x, 0) = 0$ (*iv*) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{2\pi x}{l}\right)$

Using (*iii*) in equation (3), we get

$$0 = c_2 c_3 \sin px \cos cpt \Rightarrow \boxed{c_3 = 0.}$$

Making use $c_3 = 0$ in equation (3), we get

$$y(x, t) = c_2 c_4 \sin px \sin cpt$$

where $p = \frac{n\pi}{l}$.

or the general form of solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l} \qquad \ldots(4)$$

Differentiating partially w.r.t. 't' equation (4), we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi c}{l}\right) \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi ct}{l}$$

$$\therefore \qquad \left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin\frac{3\pi x}{l} \cos\frac{2\pi x}{l} = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi c}{l}\right) \sin\frac{n\pi x}{l}$$

$$\Rightarrow \qquad \frac{v_0}{2}\left[\sin\frac{5\pi x}{l} + \sin\frac{\pi x}{l}\right] = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi c}{l}\right) \sin\frac{n\pi x}{l}$$

Equating the coefficient of like terms, we have

$$\frac{v_0}{2} = b_1\left(\frac{\pi c}{l}\right) \quad \Rightarrow \quad b_1 = \frac{l v_0}{2c\pi}$$

$$\frac{v_0}{2} = b_5\left(\frac{5\pi c}{l}\right) \quad \Rightarrow \quad b_5 = \frac{l v_0}{5c\pi}$$

and $\qquad b_2 = b_3 = b_4 = b_5 = b_6 = \cdots = 0$

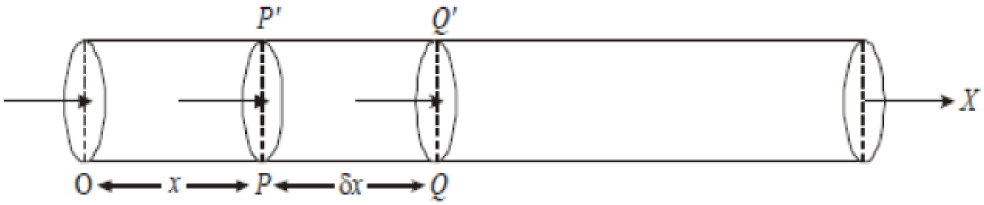
Using these values in equation (4), we get the required solution

$$y(x,t) = \left(\frac{l v_0}{2c\pi}\right) \sin\left(\frac{\pi x}{l}\right) \sin\frac{\pi ct}{l} + \left(\frac{l v_0}{5c\pi}\right) \sin\left(\frac{5\pi x}{l}\right) \sin\frac{5\pi ct}{l}.$$

ONE DIMENSIONAL FLOW

In this section we set up the mathematical model for one dimensional heat flow and derive the corresponding partial differential equation.

Consider a bar or a rod of equal thickness at every point.



Let the area of cross-sectional = A cm^2.

and density of material of rod = ρ gr/cm^3

Here we consider a small element PQ of length δx.

$\therefore$ The mass of the element PQ = $A\rho\delta x$

Let $u(x, t)$ is the temperature of the rod at a distance x at time t.

We know that the amount of heat in a body is always proportional to the mass of the body and to the temperature change.

Thus the rate of increase of heat in element

$$= sA\rho\delta x \frac{\partial u}{\partial t} \qquad (s \text{ is specific heat}) \qquad \ldots(i)$$

Since the direction of heat flow in a body becomes always toward decreasing temperature. Physical experiment shows that the rate of flow is proportional to the area and to the temperature gradient normal to the area. If we suppose Q_1 and Q_2 are the quantities of heat flowing at the points P and Q respectively,

then $$Q_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x \text{ per second}$$

and $$Q_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \text{ per second}$$

The negative sign shows the direction of heat flow towards lower temperature

where k is a constant known as thermal conductivity.

$$\therefore \text{ Total amount of heat in the element} = Q_1 - Q_2 = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \text{ per second} \qquad \ldots(ii)$$

where k is a constant known as thermal conductivity.

$\therefore$ Total amount of heat in the element $= Q_1 - Q_2 = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$ per second ...(*ii*)

From (*i*) and (*ii*)

$$sA\rho\,\delta x\,\frac{\partial u}{\partial t} = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = \frac{k}{\rho s}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

Taking the limit as $\delta x \to 0$ *i.e.*, when $x + \delta x \to x$

$$\therefore \quad \frac{\partial u}{\partial t} = \frac{k}{\rho s}\lim_{\delta x \to 0}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

$$= \frac{k}{\rho s}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{k}{\rho s}\frac{\partial^2 u}{\partial x^2}$$

Let $\frac{k}{\rho s} = c^2$ is called diffusivity of the substance

Thus $$\boxed{\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}.}$$

SOLUTION OF ONE DIMENSIONAL HEAT EQUATION

We know that the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Let $\quad u(x, t) = X(x)\, T(t) \qquad ...(2)$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = T\frac{dX}{dx} \text{ or } \frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2}$$

and
$$\frac{\partial u}{\partial t} = X\frac{dT}{dt}$$

Using these values in equation (1), we get

$$X\frac{dT}{dt} = c^2 T\frac{d^2X}{dx^2}$$

$$\Rightarrow \quad \frac{1}{c^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = k \qquad ...(3)$$

Taking 2nd and 3rd terms

$$\therefore \quad \frac{1}{X}\frac{d^2X}{dx^2} = k \quad \Rightarrow \quad \frac{d^2X}{dx^2} - kX = 0$$

$$\Rightarrow \quad (D^2 - k)X = 0$$

Hence
$$X = c_1e^{\sqrt{k}x} + c_2e^{-\sqrt{k}x}$$

and
$$\frac{1}{c^2T}\frac{dT}{dt} = k \quad \Rightarrow \quad \frac{dT}{T} = kc^2\,dt$$

On integrating, $\quad log_e T = kc^2t + \log c_3$

$$\Rightarrow \quad T = c_3e^{kc^2t}$$

$\therefore$ From (2), we get

$$\boxed{u(x,t) = (c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}) c_3 e^{kc^2 t}}$$

There are arise following cases:

Case I: If $k > 0$, let $k = p^2$

then $$\boxed{u(x,t) = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{p^2 c^2 t}.} \qquad ...(A)$$

Case II: If $k < 0$, Let $k = -p^2$

then A.E. is $\quad m^2 = -p^2 \quad \Rightarrow \quad m = \pm\, pi$

$\therefore \quad X = c_1 \cos px + c_2 \sin px$

then $$\boxed{u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-p^2 c^2 t}.} \qquad ...(B)$$

Case III: If $k = 0$, then

$$\boxed{u(x,t) = (c_1 + c_2 x) c_3.} \qquad ...(C)$$

Since the physical nature of the problem is periodic so the suitable solution of the heat equation is

$$\boxed{u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-p^2 c^2 t}.} \qquad ...(4)$$

The boundary conditions are

(*i*) $u(0, t) = 0$ and (*ii*) $u(l, t) = 0$

and initial condition is (*iii*) $u(x, 0) = f(x)$.

Using (*i*) boundary condition in (4), we get

$$0 = c_1 c_3 e^{-p^2c^2t} \Rightarrow c_1 = 0$$

$\therefore$ From (4), we get

$$u(x, t) = c_2 c_3 \sin px \cdot e^{-p^2c^2t} \qquad \ldots(5)$$

Using (*ii*) boundary condition in (5), we get

$$0 = c_2 c_3 \sin pl \Rightarrow \sin pl = 0 = \sin n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

$$\therefore \qquad u(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} e^{\frac{n^2\pi^2c^2t}{l^2}}$$

The general form of above solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-n^2\pi^2c^2t/l} \qquad (b_n = c_2 c_3) \qquad \ldots(6)$$

Again using initial condition (*iii*) in equation (6), we get

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which represents Fourier half range sine series so, we have

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

Thus the required solution is

$$\boxed{\begin{array}{l} u(x, t) = \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2c^2t}{l}} \\ \text{where } \; b_n = \displaystyle\frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx. \end{array}}$$

Remark: In steady state $\frac{\partial u}{\partial t} = 0$, so $\boxed{\frac{\partial^2 u}{\partial x^2} = 0.}$

Example 8

Determine the solution of one dimension heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Under the conditions $u(0, t) = u(l, t) = 0$ and $u(x,0) = \begin{cases} x & \text{if } 0 \le x \le l/2 \\ l - x & \text{if } \frac{l}{2} \le x \le l. \end{cases}$

Solution: We have

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

$$u(x, t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-p^2 c^2 t} \qquad ...(2)$$

$$u(x, t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-p^2 c^2 t} \qquad ...(2)$$

At $x = 0$, we get

$$0 = c_1 c_3 e^{-p^2 c^2 t} \quad \Rightarrow \quad c_1 = 0.$$

$\therefore$ From equation (2), we get

$$u(x,t) = c_2 c_3 \sin px \cdot e^{-p^2 c^2 t} \qquad ...(3)$$

Again at $x = l$ from (3), we get

$$0 = c_2 c_3 \sin pl \cdot e^{-p^2c^2t} \Rightarrow \sin pl = 0 = \sin n\pi$$

$$\Rightarrow \quad p = \frac{n\pi}{l}$$

From (3), we get

$$u(x, t) = c_2 c_3 \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2c^2}{l^2}t}$$

$\Rightarrow$ Therefore the general form of solution can be written as

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2c^2t}{l^2}} \qquad \ldots(4)$$

At $t = 0$, from equation (4), we get

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\therefore \quad b_n = \frac{2}{l}\int_0^l u(x,0)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2}{l}\left\{\left[-x\frac{l}{n\pi}\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^{l/2} + \left[-(l-x)\frac{l}{n\pi}\cos\frac{n\pi x}{l} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_{l/2}^{l}\right\}$$

$$= \frac{2}{l}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4l}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).$$

$\therefore$ From equation (4), we get

$$u(x, t) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cdot\sin\frac{n\pi x}{l}\cdot e^{-\frac{n^2\pi^2c^2}{l^2}t}. \quad \textbf{Ans.}$$

Example 9

An insulated rod of length l has its ends A and B maintained 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C find the temperature at a distance x from A at time t.

Solution: From one dimensional and equation, we have

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

The boundary conditions are

(*i*) $u(0, t) = 0$°C and (*ii*) $u(l, t) = 100$°C

In steady state condition $\frac{\partial u}{\partial t} = 0$ here from (1), we get

$$\frac{\partial^2 u}{\partial x^2} = 0$$

On integrating, we get $u(x) = c_1x + c_2$ ...(2)

where c_1 and c_2 are constants to be determined

At $x = 0$, from equation (2), we have

$$\boxed{0 = c_2.}$$

and at $x = l$, $100 = c_1 l + 0 \Rightarrow \boxed{c_1 = \frac{100}{l}}$.

$\therefore$ From (2)

$$u(x) = \frac{100}{l}x \quad \text{...(3)}$$

Now the temperature at B is suddenly changed we have again transient state. If $u(x, t)$ is the subsequent temperature function, the boundary conditions are

(*iii*) $u(0, t) = 0°C$, (*iv*) $u(l, t) = 0°C$ and the initial condition (*v*) $u(x, 0) = \frac{100}{l}x$

Since the subsequent steady state function $u_s(x)$ satisfies the equation

$$\frac{\partial^2 u_s}{\partial x^2} = 0$$

or $$\frac{d^2 u_s}{dx^2} = 0 \Rightarrow u_s(x) = c_3 x + c_4$$

at $x = 0$, we get $0 = c_4$

and at $x = l$, we get $0 = c_3 l + 0 \Rightarrow c_3 = 0$

Thus $$u_s(x) = 0 \quad \text{...(4)}$$

If $u_T(x, t)$ is the temperature in transient state then the temperature distribution in the rod $u(x, t)$ can be expressed in the form

$$u(x, t) = u_s(x) + u_T(x, t)$$

$$\Rightarrow \quad u(x, t) = u_T(x, t) \quad \Big| \text{As } u_s(x) = 0 \quad \text{...(5)}$$

Again from heat equation, we have

$$\frac{\partial u_T}{\partial t} = c^2 \frac{\partial^2 u_T}{\partial x^2} \quad \text{...(6)}$$

The solution of equation (6) is

$$u_T(x, t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t}$$

$\Rightarrow$
$$u(x, t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t} \qquad ...(7)$$

At $\quad x = 0,\ u(0, t) = 0$

$\Rightarrow \quad 0 = c_1 e^{-c^2 p^2 t} \Rightarrow c_1 = 0.$

From (7), we get

$$u(x, t) = c_2 \sin px \cdot c_3 e^{-c^2 p^2 t} \qquad ...(8)$$

Again at $\quad x = l, \quad u(l, t) = 0$

$\Rightarrow \quad 0 = c_2 c_3 \sin pl \cdot e^{-c^2 p^2 t} \Rightarrow \sin pl = 0 = \sin n\pi$

$\Rightarrow \quad p = \dfrac{n\pi}{l}$

From (8), we get

$$u(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2 c^2 \pi^2 t}{l^2}}$$

$\Rightarrow$
$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2 \pi^2 c^2 t}{l^2}} \qquad ...(9)$$

Using initial condition *i.e.*, at $t = 0,\ u = \dfrac{100}{l} x$, we get

$$u(x, 0) = \frac{100}{l} x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$\therefore$
$$b_n = \frac{2}{l} \int_0^l \frac{100x}{l} \cdot \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{200}{l^2} \int_0^l x \cdot \sin \frac{n\pi x}{l} \, dx = \frac{200}{l^2} \left[-\frac{xl}{n\pi} \cos \frac{n\pi x}{l} + \frac{l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} \right]_0^l$$

$\Rightarrow$
$$b_n = \frac{200}{l^2} \left[\frac{-l^2}{n\pi} \cos n\pi \right] = \frac{200}{n\pi} (-1)^{n+1}$$

Hence from equation (9), we get

$$u(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}e^{-\frac{n^2\pi^2c^2}{l^2}t}.$$

Example 10

The temperature of a bar 50 cm long with insulated sides is kept at 0°C at one end and 100°C at the other end until steady conditions prevail. The two ends are then suddenly insulated so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.

Solution: The temperature function $u(x, t)$ is the solution of the one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

When the steady state condition prevails $\frac{\partial u}{\partial t} = 0$ and hence from (1), we get

$$\frac{\partial^2 u}{\partial x^2} = 0$$

On integrating, we get

$$u(x) = c_1x + c_2 \qquad ...(2)$$

At $\quad x = 0, \quad u = 0$

$\therefore \quad 0 = c_2$

and at $x = 50$, $u = 100$, from (2), we get

$$100 = 50\,c_1 + 0 \quad \Rightarrow \quad c_1 = 2$$

Hence $\quad u(x) = 2x \quad \Rightarrow \quad u(x, 0) = 2x \qquad ...(3)$

and the subsequent temperature function $u_1(x, t)$ satisfy the boundary conditions

$$u_1(0, t) = 0, \quad u_1(50, t) = 0$$

Under these conditions, we find the steady state function $u_s(x)$ vanishes *i.e.,* $u_s(x) = 0 \quad \quad |u_1 \to u_s$

$\Rightarrow \quad u(x, t) = u_s(x) + u_T(x, t) = 0 + u_T(x, t)$

$\Rightarrow \quad u(x, t) = u_T(x, t) \quad \ldots(4)$

where $u_T(x, t)$ is the temperature in transient state which satisfied the boundary conditions

$$u_T(0, t) = 0 = u_T(50, t)$$

$\therefore$ The temperature $u_T(x, t)$ can be obtained by the solution of one dimensional heat equation

$$\Rightarrow \quad u_T(x, t) = (c_1 \cos px + c_2 \sin px)c_3 e^{-c^2p^2t} \quad \ldots(5)$$

At $\quad x = 0, \quad u_T = 0$

$\Rightarrow \quad 0 = c_1c_3e^{-c^2p^2t} \quad \Rightarrow \quad c_1 = 0$ (otherwise $u_T(x, t) = 0$)

From (5), we get

$$u_T(x, t) = c_2c_3 \sin px \cdot e^{-c^2p^2t} \quad \ldots(6)$$

And at $\quad x = 50, \quad u_T = 0$

$\Rightarrow \quad 0 = c_2c_3 \sin 50p \cdot e^{-c^2p^2t}$

$\Rightarrow \quad \sin 50p = 0 = \sin n\pi \quad \Rightarrow \quad p = \frac{n\pi}{50}$

$\therefore$ From (6), we get

$$u_T(x, t) = c_2c_3 \sin\frac{n\pi x}{50} e^{-\frac{n^2\pi^2c^2t}{2500}}$$

The general form of solution is

$$u_T(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{50} e^{-\frac{n^2\pi^2c^2t}{2500}} \quad \ldots(7)$$

At $\quad t = 0, u_T = 2x$ (from equation 3)

$$\Rightarrow \quad 2x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{50}$$

$$\therefore \quad b_n = \frac{2}{50}\int_0^{50} 2x \sin\frac{n\pi x}{50}dx = \frac{2}{25}\left[-\frac{50}{n\pi}x\cos\frac{n\pi x}{50} + \frac{2500}{n^2\pi^2}\sin\frac{n\pi x}{50}\right]_0^{50}$$

$$b_n = \frac{2}{25}\left[-\frac{2500}{n\pi}\cos n\pi + \frac{2500}{n^2\pi^2}\sin n\pi + 0\right] = \frac{200}{n\pi}(-1)^{n+1}$$

Putting the value of b_n in equation (7), we get

$$u_T(x, t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}(-1)^{n+1} \cdot \sin\frac{n\pi x}{50} \cdot e^{-\frac{n^2\pi^2c^2t}{2500}} \quad \ldots(8)$$

Hence from (4) and (8), we get

$$u(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{50} \cdot e^{-\frac{n^2\pi^2c^2t}{2500}}. \quad \textbf{Ans.}$$

Example 11

Two ends A and B of a rod 20 cm long have the temperature at 30°C and 80°C respectively until steady state prevails. The temperature at the end are changed to 40°C and 60°C respectively find the temperature distribution in the rod.

Solution: The heat equation in one dimensional is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

The boundary conditions are

(i) $u(0, t) = 30°C$ (ii) $u(20, t) = 80°C$

In steady condition $\frac{\partial u}{\partial t} = 0$

$\therefore$ From (1), we get $\frac{\partial^2 u}{\partial x^2} = 0$, on integrating, we get

$$u(x) = c_1 x + c_2 \qquad ...(2)$$

at $x = 0$, $u = 30$, so $30 = 0 + c_2 \Rightarrow c_2 = 30$

and at $x = 20$ $u = 80$ so $80 = c_1 \times 20 + 30 \Rightarrow c_1 = \frac{5}{2}$

From equation (2), we get

$$u(x) = \frac{5x}{2} + 30$$

Now the temperatures at A and B are suddenly changed we have again gain transient state.

If $u_1(x, t)$ is subsequent temperature function then the boundary conditions are

$$u_1(0, t) = 40°C \quad \text{and} \quad u_1(20, t) = 60°C$$

and the initial condition *i.e.*, at $t = 0$, is given by equation (3)

Since the subsequent steady state function $u_s(x)$ satisfies the equation

$$\frac{\partial^2 u_s}{\partial x^2} = 0 \quad \text{or} \quad \frac{d^2 u_s}{dx^2} = 0$$

The solution of above equation is

$$u_s(x) = c_3 x + c_4 \qquad ...(4)$$

At $x = 0$, $u_s = 40 \Rightarrow 40 = 0 + c_4 \Rightarrow c_4 = 40$ | $u_s(0) = 40°C$

and at $x = 20$, $u_s = 60 \Rightarrow 60 = 20\,c_3 + 40 \Rightarrow c_3 = 1$ | $u_s(20) = 60°C$

$\therefore$ From (4), we get

$$u_s(x) = x + 40 \qquad ...(5)$$

Thus the temperature distribution in the rod at time t is given by

$$u(x, t) = u_s(x) + u_T(x, t)$$

$$\Rightarrow \quad u(x, t) = (x + 40) + u_T(x, t) \qquad ...(6)$$

where $u_T(x, t)$ is the transient state function which satisfying the conditions

and
$$\left.\begin{array}{l} u_T(0, t) = u_1(0, t) - u_s(0) = 40 - 40 = 0 \\ u_T(20, t) = u_1(20, t) - u_s(20) = 60 - 60 = 0 \\ u_T(x, 0) = u_1(x, 0) - u_s(x) = \frac{5x}{2} + 30 - x - 40 = \frac{3x}{2} - 10 \end{array}\right\}$$

The general solution for $u_T(x, t)$ is given by

$$u_T(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20} e^{-\frac{n^2\pi^2c^2t}{400}} \qquad ...(7)$$

At $t = 0$, from (7), we get

$$\frac{3x}{2} - 10 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20}$$

$$\therefore \qquad b_n = \frac{2}{20}\int_0^{20}\left(\frac{3x}{2} - 10\right)\sin\frac{n\pi x}{20}dx$$

$$= \frac{1}{10}\left[\left(\frac{3x}{2} - 10\right)\left(-\frac{20}{n\pi}\cos\frac{n\pi x}{20}\right) - \frac{3}{2}\left(-\frac{400}{n^2\pi^2}\sin\frac{n\pi x}{20}\right)\right]_0^{20}$$

$$= \frac{1}{10}\left[-20\left(\frac{20}{n\pi}\right)(-1)^n - (-10)\left(\frac{20}{n\pi}\right)\right] = -\frac{20}{n\pi}\left[2(-1)^n + 1\right]$$

Putting the value of b_n in equation (7), we get

$$u_T(x, t) = -\frac{20}{\pi}\sum_{n=1}^{\infty}\frac{2(-1)^n + 1}{n}\sin\frac{n\pi x}{20} \cdot e^{-\frac{n^2\pi^2c^2t}{400}} \qquad ...(8)$$

From (6) and (8), we get

$$u(x, t) = (x + 40) - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{2(-1)^n + 1}{n} \cdot \sin\frac{n\pi x}{20} \cdot e^{-\frac{n^2\pi^2c^2t}{400}}. \qquad \textbf{Ans.}$$

TWO DIMENSIONAL HEAT EQUATIONS

We have

$$\frac{\partial u}{\partial t} = c^2(\nabla^2 u) \qquad \ldots(i)$$

where $c^2 = \frac{k}{s\rho}$, k is the thermal conductivity of the body, s is the specific heat of the material of the body and ρ is the density.

In case of two dimensional, we may suppose that z-coordinate is constant.

$$\therefore \qquad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 0$$

As $u(z) = \text{constant}$

$$\therefore \frac{\partial^2 u}{\partial z^2} = 0$$

$$\Rightarrow \qquad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \ldots(ii)$$

From (i) and (ii), we get

$$\boxed{\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).} \qquad \ldots(A)$$

In steady state u always independent of t so that $\frac{\partial u}{\partial t} = 0$.

Hence from equation (A), we get

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.} \qquad \ldots(B)$$

The equation (B) is known as Laplace's equation.

Solution of Two dimensional Heat equation

We have

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \ldots(i)$$

Let $\qquad u(x, y, t) = XYT \qquad \ldots(ii)$

Putting the value of $u(x, y, t)$ from (ii) in equation (i), we get

$$XY\frac{dT}{dt} = Tc^2\left(Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}\right)$$

$$\Rightarrow \quad \frac{1}{C^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}$$

There are three possibilities

(a) $\frac{1}{X}\frac{d^2X}{dx^2}=0,\ \frac{1}{Y}\frac{d^2y}{dy^2}=0,\ \frac{1}{c^2T}\frac{dT}{dt}=0$

(b) $\frac{1}{X}\frac{d^2X}{dx^2}=p_1^2,\ \frac{1}{Y}\frac{d^2Y}{dy^2}=p_2^2,\ \frac{1}{c^2T}\frac{dT}{dt}=p^2.$

(c) $\frac{1}{X}\frac{d^2X}{dx^2}=-p_1^2,\ \frac{1}{Y}\frac{d^2Y}{dy^2}=-p_2^2,\ \frac{1}{c^2T}\frac{dT}{dt}=-p^2.$

where $p^2=p_1^2+p_2^2$.

Out of these three possibilities, we have to select that solution which suits the physical nature of the problem and the given boundary conditions.

Here $u(x,y,t) = (c_1x+c_2)(c_3y+c_4)c_5$ (For a)

$u(x,y,t) = (c_1e^{p_1x}+c_2e^{-p_1x})(c_3e^{p_2y}+c_4e^{-p_2y})c_5e^{p^2c^2t}$ (For b)

and $u(x,y,t) = (c_1\cos p_1x+c_2\sin p_1x)(c_3\cos p_2y+c_4\sin p_2y)c_5e^{-p^2c^2t}$ (For c)

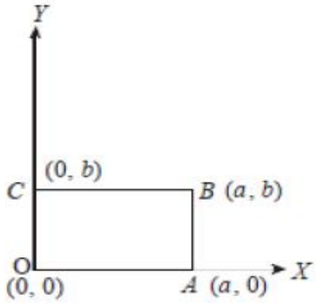
Example 12

A thin rectangular plate whose surface is impervious to heat flow has $t=0$ an arbitrary distribution of temperature $f(x,y)$. Its four edges $x=0$, $x=a$, $y=0$ and $y=b$ are kept at zero temperature. Determine the temperature at a point of the plate as t increases.

Solution: The heat equation in two dimensional is

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right) \quad \ldots(1)$$

Let $u = XYT$ ...(2)



From (1) $XY\frac{dT}{dt} = Tc^2\left(Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}\right)$

$$\Rightarrow \quad \frac{1}{c^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2} \quad \ldots(3)$$

Since the physical nature of problem is periodic so we choosen the constant as follows:

From (3), we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = -p_1^2 \Rightarrow \frac{d^2X}{dx^2}+p_1^2X=0 \Rightarrow (D^2+p_1^2)X=0$$

$\therefore$ The A.E. is $m^2+p_1^2=0 \Rightarrow m^2=i^2p_1^2 \Rightarrow m=\pm ip_1$

$\Rightarrow$ $X = c_1\cos p_1x + c_2\sin p_1x$

Again from (3), we get

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -p_2^2 \Rightarrow (D^2+p_2^2)X=0$$

so $Y = (c_3\cos p_2y + c_4\sin p_2y)$

and $\frac{1}{c^2T}\frac{dT}{dt} = -p^2$, where $p^2 = p_1^2+p_2^2$.

$\Rightarrow$ $\frac{dT}{T} = -p^2c^2dt \Rightarrow \log_e T = -p^2c^2t+\log_e c_5$

$\Rightarrow$ $T = c_5e^{-p^2c^2t}$

Hence from equation (2), we get

$$u = (c_1\cos p_1x + c_2\sin p_1x)(c_3\cos p_2y + c_4\sin p_2y)c_5e^{-p^2c^2t} \quad \ldots(4)$$

Now the boundary conditions are:

(*i*) $u(0, y, t) = 0$, (*ii*) $u(a, y, t) = 0$, (*iii*) $u(x, 0, t) = 0$; and (*iv*) $u(x, b, t) = 0$

Using first boundary condition in (4), we get

$$0 = c_1(c_3\cos p_2y + c_4\sin p_2y)c_5e^{-p^2c^2t}$$

$\Rightarrow$ $c_1 = 0$ (otherwise $u(x, y, t) = 0$)

From (4), we get

$$u = c_2c_5\sin p_1x(c_3\cos p_2y + c_4\sin p_2y)e^{-p^2c^2t} \quad \ldots(5)$$

Using second boundary condition in (5), we get

$$0 = c_2c_5\sin p_1a(c_3\cos p_2y + c_4\sin p_2y)e^{-p^2c^2t}$$

$\Rightarrow$ $\sin p_1a = 0 = \sin m\pi \Rightarrow \boxed{p_1 = \frac{m\pi}{a}}$

Next using 3rd boundary condition in (5), we get

$$0 = c_3 c_2 c_5 \sin p_1 x \cdot e^{-p^2 c^2 t} \Rightarrow c_3 = 0 \quad (\text{otherwise } u(x, y, t) = 0)$$

From (5), we get

$$u = c_2 c_4 c_5 \sin p_1 x \cdot \sin p_2 y \cdot e^{-p^2 c^2 t} \qquad \ldots(6)$$

And using 4th boundary condition in (6), we get

$$0 = c_2 c_4 c_5 \sin p_1 x \cdot \sin p_2 b \cdot e^{-p^2 c^2 t}$$

$$\Rightarrow \qquad \sin p_2 b = 0 = \sin n\pi \Rightarrow \boxed{p_2 = \frac{n\pi}{b}}.$$

Since $\qquad p^2 = p_1^2 + p_2^2$

so $\qquad p^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$

Putting the values of p_1, p_2 and p in equation (6), we get

$$u(x, y, t) = c_2 c_4 c_5 \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b} e^{-\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)c^2 t}$$

The general form of solution is

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b} e^{-\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)c^2 t} \qquad \ldots(7)$$

and the initial condition is

$$u(x, y, 0) = f(x, y)$$

From (7), we get

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \cdot \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi x}{b}$$

which is the double Fourier since series of $f(x, y)$.

$$\therefore \qquad A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_{x=0}^{a} \int_{y=0}^{b} f(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi x}{b}\, dx dy \qquad \ldots(8)$$

Hence the equation (7) is required temperature distribution with the equation (8). **Ans.**

Example 13

Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ which satisfies the conditions

$$u(0, y) = u(l, y) = u(x, 0) = 0 \text{ and } u(x, a) = \sin\left(\frac{n\pi x}{l}\right).$$

Solution: We have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(1)$$

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad \ldots(2)$$

At $x = 0, \quad u = 0$

$$0 = c_1(c_3 e^{py} + c_4 e^{-py}) \Rightarrow c_1 = 0 \quad [\text{otherwise } u(x, y) = 0]$$

Putting the value of c_1 in equation (2), we get

$$u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \qquad \ldots(3)$$

at $x = l, \quad u = 0$

$$0 = c_2 \sin pl (c_3 e^{py} + c_4 e^{-py})$$

$$\Rightarrow \qquad \sin pl = 0 = \sin n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}.}$$

From equation (3), we get

$$u(x, y) = c_2 \sin\frac{n\pi x}{l}\left(c_3 e^{\frac{n\pi y}{l}} + c_4 e^{-\frac{n\pi y}{l}}\right) \qquad \ldots(4)$$

At $y = 0, \quad u = 0,$ we get

$$0 = 2c_2 \sin\frac{n\pi x}{l}(c_3 + c_4) \Rightarrow c_3 + c_4 = 0 \Rightarrow \boxed{c_4 = -c_3}$$

Putting the value of c_4 in equation (4), we get

$$u(x, y) = c_2 c_3 \sin\frac{n\pi x}{l}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right) = 2c_2 c_3 \sin\frac{n\pi x}{l} \cdot \sinh\frac{n\pi y}{l} \quad \left|\ \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}\right.$$

The general form of above solution is given as

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \sin h\frac{n\pi y}{l} \qquad (2c_1c_2 = b_n) \qquad \ldots(5)$$

Putting $y = a$ and $u = \sin\frac{n\pi x}{l}$ in equation (5), we get

$$\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \sin h\frac{n\pi a}{l}$$

Equating the coefficient of $\sin\frac{n\pi x}{l}$ on both sides, we get

$$1 = b_n \sinh\frac{n\pi a}{l} \quad \Rightarrow \quad b_n = \frac{1}{\sin h\frac{n\pi a}{l}}$$

and $$b_1 = b_2 = b_3 = \ldots\ldots = b_{n-1} = 0$$

Hence, from equation (5), we get

$$u(x, y) = \frac{\sin(n\pi x/l)}{\sin h\left(\frac{n\pi a}{l}\right)} \cdot \sin h\frac{n\pi y}{l}. \quad \textbf{Ans.}$$

SATHYABAMA
INSTITUTE OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS
COURSE MATERIAL

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UNIT III

INTRODUCTION

Solution of Algebraic and Transcendental Equations

A polynomial equation of the form

$$f(x) = p_n(x) = a_0 x^{n-1} + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$$

is called an Algebraic equation. For example,

$x^4 - 4x^2 + 5 = 0,\ 4x^2 - 5x + 7 = 0;\ 2x^3 - 5x^2 + 7x + 5 = 0$ are algebraic equations.

An equation which contains polynomials, trigonometric functions, logarithmic functions, exponential functions etc., is called a Transcendental equation. For example,

$$\tan x - e^x = 0;\quad \sin x - xe^{2x} = 0; \qquad x\, e^x = \cos x$$

are transcendental equations.

Finding the roots or zeros of an equation of the form $f(x) = 0$ is an important problem in science and engineering. We assume that $f(x)$ is continuous in the required interval. A root of an equation $f(x) = 0$ is the value of x, say $x = \alpha$ for which $f(\alpha) = 0$. Geometrically, a root of an equation $f(x) = 0$ is the value of x at which the graph of the equation $y = f(x)$ intersects the x – axis (see Fig. 1)

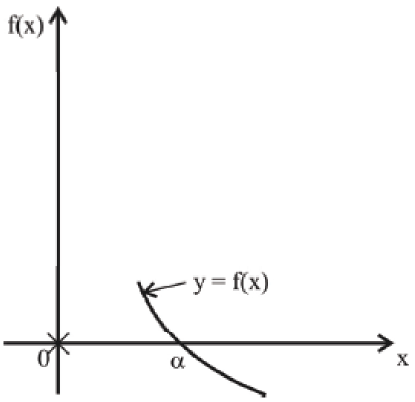


Fig. 1 Geometrical Interpretation of a root of $f(x) = 0$

A number α is a simple root of $f(x) = 0$; if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Then, we can write

$f(x)$ as, $f(x) = (x - \alpha)\, g(x)$, $g(\alpha) \neq 0$.

A number α is a multiple root of multiplicity m of $f(x) = 0$,

and $$f^{m}(\alpha) = 0.$$

Then, $f(x)$ can be writhen as,

$$f(x) = (x - \alpha)^{m}\, g(x),\ g(\alpha) \neq 0$$

A polynomial equation of degree n will have exactly n roots, real or complex, simple or multiple. A transcendental equation may have one root or no root or infinite number of roots depending on the form of $f(x)$.

The methods of finding the roots of $f(x) = 0$ are classified as,

1. Direct Methods
2. Numerical Methods.

Direct methods give the exact values of all the roots in a finite number of steps. Numerical methods are based on the idea of successive approximations. In these methods, we start with one or two initial approximations to the root and obtain a sequence of approximations x_0, x_1, … x_k which in the limit as $k \to \infty$ converge to the exact root $x = a$. There are no direct methods for solving higher degree algebraic equations or transcendental equations. Such equations can be solved by Numerical methods. In these methods, we first find an interval in which the root lies. If a and b are two numbers such that $f(a)$ and $f(b)$ have opposite signs, then a root of $f(x) = 0$ lies in between a and b. We take a or b or any valve in between a or b as first approximation x_1. This is further improved by numerical methods. Here we discuss few important Numerical methods to find a root of $f(x) = 0$.

REGULA FALSI METHOD

This is another method to find the roots of $f(x) = 0$. This method is also known as Regular False Method. In this method, we choose two points a and b such that $f(a)$ and $f(b)$ are of opposite signs. Hence a root lies in between these points. The equation of the chord joining the two points.

$(a, f(a))$ and $(b, f(b))$ is given by

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a} \qquad .....(5)$$

We replace the part of the curve between the points $[a, f(a)]$ and $[b, f(b)]$ by means of the chord joining these points and we take the point of intersection of the chord with the x axis as an approximation to the root (see Fig.3). The point of intersection is obtained by putting y = 0 in (5), as

$$x = x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} \qquad .....(6)$$

x_1 is the first approximation to the root of $f(x) = 0$.

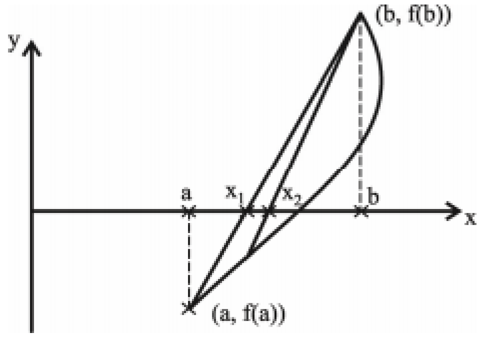


Fig. 3 Method of False Position

If f (x_1) and f (a) are of opposite signs, then the root lies between a and x_1 and we replace b by x_1 in (6) and obtain the next approximation x_2. Otherwise, we replace a by x_1 and generate the next approximation. The procedure is repeated till the root is obtained to the desired accuracy. This method is also called linear interpolation method or chord method.

1. Find the root of the equation $2x - \log x = 7$ which lies between 3.5 and 4 by Regula–False method. (JNTU 2006)

Solution

Given $f(x) = 2x - \log x_{10} = 7$ .....(1)

Take $x_0 = 3.5$, $x_1 = 4$

Using Regula Falsi method

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x)} . f(x_0)$$

$$x_2 = 3.5 - \frac{4 - 3.5}{(0.3979 + 0.5441)} (-0.5441)$$

$$x_2 = 3.7888$$

Now taking $x_0 = 3.7888$ and $x_1 = 4$

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} . f(x_0)$$

$$x_3 = 3.7888 - \frac{4 - 3.7888}{0.3988} (-0.0009)$$

$$x_3 = 3.7893$$

The required root is = 3.789

2. Find a real root of $xe^x = 3$ using Regula-Falsi method.

Solution

Given $f(x) = x\,e^x - 3 = 0$

$f(1) = e - 3 = -0.2817 < 0$

$f(2) = 2e^2 - 3 = 11.778 > 0$

$\therefore$ One root lies between 1 and 2

Now taking $x_0 = 1, x_1 = 2$

Using Regula – Falsi method

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$\therefore \quad x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{1(11.778) - 2(-0.2817)}{11.778 + 0.2817}$$

$$x_2 = 1.329$$

Now $f(x_2) = f(1.329) = 1.329\, e^{1.329} - 3 = 2.0199 > 0$

$f(1) = -0.2817 < 0$

$\therefore$ The root lies between 1 and 1.329 taking $x_0 = 1$ and $x_2 = 1.329$

$\therefore$ Taking $x_0 = 1$ and $x_2 = 1.329$

$$\therefore \quad x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)}$$

$$= \frac{1(2.0199) + (1.329)(0.2817)}{(2.0199) + (0.2817)}$$

$$= \frac{2.3942}{2.3016} = 1.04$$

Now $f(x^3) = 1.04\, e^{1.04} - 3 = -0.05 < 0$

The root lies between x^2 and x^3

i.e., 1.04 and 1.329 $[\because f(x_2) > 0 \text{ and } f(x_3) < 0]$

$$\therefore \quad x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{(1.04)(-0.05) - (1.329)(2.0199)}{(-0.05) - (2.0199)}$$

$x_4 = 1.08$ is the approximate root

3. Find a real root of $e^x \sin x = 1$ using Regula – Falsi method

Solution

Given $f(x) = e^x \sin x - 1 = 0$

Consider $x_0 = 2$

$f(x_0) = f(2) = e^2 \sin 2 - 1 = -0.7421 < 0$

$f(x_1) = f(3) = e^3 \sin 3 - 1 = 0.511 > 0$

$\therefore$ The root lies between 2 and 3

Using Regula – Falsi method

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{2(0.511) + 3(0.7421)}{0.511 + 0.7421}$$

$x_2 = 2.93557$

$f(x_2) = e^{2.93557} \sin(2.93557) - 1$

$f(x_2) = -0.35538 < 0$

$\therefore$ Root lies between x_2 and x_1

i.e., lies between 2.93557 and 3

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$= \frac{(2.93557)(0.511) - 3(-35538)}{0.511 + 0.35538}$$

$x_3 = 2.96199$

$f(x_3) = e^{2.90199} \sin(2.96199) - 1 = -0.000819 < 0$

$\therefore$ root lies between x_3 and x_1

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$x_4 = \frac{2.96199(0.511) + 3(0.000819)}{0.511 + 0.000819} = 2.9625898$$

$f(x^4) = e^{2.9625898} \sin(2.9625898) - 1$

$f(x^4) = -0.0001898 < 0$

$\therefore$ The root lies between x_4 and x_1

$$x_5 = \frac{x_4 f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)}$$

$$= \frac{2.9625898(0.511)+3(0.0001898)}{0.511+(0.0001898)}$$

$$x_5 = 2.9626$$

we have

$$x_4 = 2.9625$$

$$x_5 = 2.9626$$

$\therefore \quad x_5 = x_4 = 2.962$

$\therefore$ The root lies between 2 and 3 is 2.962

4. Find a real root of $x\,e^x = 2$ using Regula – Falsi method (J

Solution

$f(x) = x\,e^x - 2 = 0$

$f(0) = -2 < 0, \quad f(1) = \text{i.e., } -2 = (2.7183)-2$

$f(1) = 0.7183 > 0$

$\therefore$ The root lies between 0 and 1

Considering $x_0 = 0, x_1 = 1$

$f(0) = f(x_0) = -2; \quad f(1) = f(x_1) = 0.7183$

By Regula – Falsi method

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{0(0.7183)-1(-2)}{0.7183-(-2)} = \frac{2}{2.7183}$$

$$x_2 = 0.73575$$

Now $f(x^2) = f(0.73575) = 0.73575\,e^{0.73575} - 2$

$f(x_2) = -0.46445 < 0$

and $f(x_1) = 0.7183 > 0$

$\therefore$ The root x_3 lies between x_1 and x_2

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$x_3 = \frac{(0.73575)(0.7183)}{0.7183 + 0.46445}$$

$$x_3 = \frac{0.52848 + 0.46445}{1.18275}$$

$$x_3 = \frac{0.992939}{1.18275}$$

$$x_3 = 0.83951 \quad f(x^3) = \frac{(0.83951)}{(0.83951)e^{-2}}$$

$f(x_3) = (0.83951)\, e^{0.83951} - 2$

$f(x_3) = -0.056339 < 0$

$\therefore$ One root lies between x_1 and x_3

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)} = \frac{(0.83951)(0.7183) - 1(-0.056339)}{0.7183 + 0.056339}$$

$$x_4 = \frac{0.65935}{0.774639} = 0.851171$$

$f(x_4) = 0.851171\ e0.851171 - 2 = -0.006227 < 0$

Now x_5 lies between x_1 and x_4

$$x_5 = \frac{x_4 f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)}$$

$$x_5 = \frac{(0.851171)(0.7183) + (.006227)}{0.7183 + 0.006227}$$

$$x_5 = \frac{0.617623}{0.724527} = 0.85245$$

Now $f(x_5) = 0.85245\ e^{0.85245}\ e^{0.85245} - 2 = -0.0006756 < 0$

$\therefore$ One root lies between x_1 and x_5, (i.e., x_6 lies between x_1 and x_5)

Using Regula – Falsi method

$$x_6 = \frac{(0.85245)(0.7183) + 0.0006756}{0.7183 + 0.0006756}$$

$$x_6 = 0.85260$$

Now $f(x_6) = -0.00006736 < 0$

$\therefore$ One root x_7 lies between x_1 and x_6

By Regula – Falsi method

$$x_7 = \frac{x_6 f(x_1) - x_1 f(x_6)}{f(x_1) - f(x_6)}$$

$$x_7 = \frac{(0.85260)(0.7183) + 0.0006736}{0.7183 + 0.0006736}$$

$$x_7 = 0.85260$$

From $x^6 = 0.85260$ and $x_7 = 0.85260$

$\therefore$ A real root of the given equation is 0.85260

NEWTON RAPHSON METHOD

This is another important method. Let x_0 be approximation for the root of $f(x) = 0$. Let $x_1 = x_0 + h$ be the correct root so that $f(x_1) = 0$. Expanding $f(x_1) = f(x_0 + h)$ by Taylor series, we get

$$f(x_1) = f(x_1 + h) = f(x_0) + h\ f'(x_0) + \frac{h^2}{2!}\ f''(x_0) + \ldots\ldots = 0 \qquad \ldots..(1)$$

For small valves of h, neglecting the terms with h^2, h^3 ….. etc,. We get

$$\therefore \qquad f(x_0) + h\ f'(x_0) = 0 \qquad \ldots..(2)$$

and $$h = -\frac{f(x_0)}{f^1(x_0)}$$

$$\therefore \qquad x_1 = x_0 + h$$
$$= x_0 - \frac{f(x_0)}{f'(x_0)}$$

Proceeding like this, successive approximation $x_2, x_3, \ldots x_{n+1}$ are given by,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \ldots..(3)$$

For n = 0, 1, 2, ……

Note:

(i) The approximation x_{n+1} given by (3) converges, provided that the initial approximation x_0 is chosen sufficiently close to root of $f(x) = 0$.

(ii) Convergence of Newton-Raphson method: Newton-Raphson method is similar to iteration method

$$\phi(x) = x - \frac{f(x)}{f'(x)} \qquad \ldots..(1)$$

differentiating (1) w.r.t to 'x' and using condition for convergence of iteration method i.e.

$$|\phi'(x)| < 1,$$

We get

$$\left|1-\frac{f'(x).f'(x)-f(x)f''(x)}{[f'(x)]^2}\right|<1$$

Simplifying we get condition for convergence of Newton-Raphson method is

$$|f(x).f''(x)|<[f(x)]^2$$

Example 1

Using Newton-Raphson method (a) Find square root of a number (b) Find a reciprocal of a number.

Solution

(a) Let n be the number and $x = \sqrt{n}$ $x^2 = n$

If $f(x) = x^2 - n = 0$ …..(1)

Then the solution to $f(x) = x^2 - n = 0$ is $x = \sqrt{n}$

$f^1(x) = 2x$

by Newton Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} = x_i - \left(\frac{x_i^2 - n}{2x_i}\right)$$

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{x}{x_i}\right)$$

using the above formula the square root of any number 'n' can be found to required accuracy.

(b) To find the reciprocal of a number 'n'

$f(x) = \frac{1}{x} - n = 0$ …..(1)

$\therefore$ solution of (1) is $x = \frac{1}{n}$

$f^1(x) = -\frac{1}{x^2}$

Now by Newton-Raphson method,

$$x_{i+1} = x_i - \left(\frac{f(x_i)}{f^1(x_i)} \right)$$

$$x_{i+1} = x_i - \left(\frac{\frac{1}{x_i} - N}{-\frac{1}{x_1^2}} \right)$$

$$x_{i+1} = x_i\ (2 - x_i\ n)$$

using the above formula the reciprocal of a number can be found to required accuracy.

Example 2

Find the reciprocal of 18 using Newton–Raphson method

Solution

The Newton-Raphson method

$x_{i+1} = x_i\ (2 - x_i\ n)$ …..(1)

considering the initial approximate value of x as $x_0 = 0.055$ and given $n = 18$

$\therefore\ x_1 = 0.055\ [2 - (0.055)\ (18)]$

$\therefore\ x_1 = 0.0555$

$x_2 = 0.0555\ [2 - 0.0555 \times 18]$

$x_2 = (0.0555)\ (1.001)$

$x_2 = 0.0555$

Hence $x_1 = x_2 = 0.0555$

$\therefore$ The reciprocal of 18 is 0.0555.

Example 3

Find a real root for $x \tan x + 1 = 0$ using Newton–Raphson method

Solution

Given $f(x) = x \tan x + 1 = 0$

$f^1(x) = x \sec 2x + \tan x$

$f(2) = 2 \tan 2 + 1 = -3.370079 < 0$

$f(3) = 2 \tan 3 + 1 = -0.572370 > 0$

$\therefore$ The root lies between 2 and 3

Take $x_0 = \frac{2+3}{2} = 2.5$ (average of 2 and 3), By Newton-Raphson method

$$x_{i+1} = x_i - \left(\frac{f(x_i)}{f^1(x_i)}\right)$$

$$x_1 = x_0 - \left(\frac{f(x_0)}{f^1(x_0)}\right)$$

$$x_1 = 2.5 - \frac{(-0.86755)}{3.14808}$$

$$x_1 = 2.77558$$

$$x_2 = x_1 - \frac{f(x_i)}{f^1(x_i)};$$

$f(x_1) = -0.06383,\quad f^1(x_1) = 2.80004$

$$x_2 = 2.77558 - \frac{(-0.06383)}{2.80004}$$

$$x_2 = 2.798$$

$f(x_2) = -0.001080,\quad f^1(x_2) = 2.7983$

$$x_3 = x_2 - \frac{f(x_2)}{f^1(x_2)} = 2.798 - \frac{[-0.001080]}{2.7983}$$

$$x_3 = 2.798.$$

$\therefore \quad x_2 = x_3$

$\therefore$ The real root of $x \tan x + 1 = 0$ is 2.798

Example 4

Find a root of $e^x \sin x = 1$ using Newton–Raphson method

Solution

Given $f(x) = e^x \sin x - 1 = 0$

$f^1(x) = e^x \sin x + e^x \cos x$

Take $x_1 = 0$, $x_2 = 1$

$f(0) = f(x_1) = e^0 \sin 0 - 1 = -1 < 0$

$f(1) = f(x_2) = e^1 \sin(1) - 1 = 1.287 > 0$

The root of the equation lies between 0 and 1.Using Newton Raphson Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$$

Now consider x_0 = average of 0 and 1

$$x_0 = \frac{1+0}{2} = 0.5$$

$$x_0 = 0.5$$

$$f(x_0) = e^{0.5} \sin(0.5) - 1$$

$$f^1(x_0) = e^{0.5} \sin(0.5) + e^{0.5} \cos(0.5) = 2.2373$$

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 0.5 - \frac{(-0.20956)}{2.2373}$$

$$x_1 = 0.5936$$

$$f(x_1) = e^{0.5936} \sin(0.5936) - 1 = 0.0128$$

$$f^1(x_1) = e^{0.5936} \sin(0.5936) + e^{0.5936} \cos(0.5936) = 2.5136$$

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)} = 0.5936 - \frac{(0.0128)}{2.5136}$$

$\therefore$ $x_2 = 0.58854$

similarly $x_3 = x_2 - \dfrac{f(x_1)}{f^1(x_1)}$

$$f(x_2) = e^{0.58854} \sin(0.58854) - 1 = 0.0000181$$

$$f^1(x_2) = e^{0.58854} \sin(0.58854) + e^{0.58854} \cos(0.58854)$$

$$f(x_2) = 2.4983$$

$\therefore$ $x_3 = 0.58854 - \dfrac{0.0000181}{2.4983}$

$$x_3 = 0.5885$$

$\therefore$ $x_2 - x_3 = 0.5885$

0.5885 is the root of the equation $e^x \sin x - 1 = 0$

GAUSS ELIMINATION METHOD

This is the elementary elimination method and it reduces the system of equations to an equivalent upper – triangular system, which can be solved by back substitution.

We consider the system of n linear equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

There are two steps in the solution viz., the elimination of unknowns and back substitution.

Example 1

Solve the following system of equations using Gaussian elimination.

$$x_1 + 3x_2 - 5x_3 = 2$$

$$3x_1 + 11x_2 - 9x_3 = 4$$

$$-x_1 + x_2 + 6x_3 = 5$$

Solution

An augmented matrix is given by

$$\left[\begin{array}{ccc|c} \boxed{1} & 3 & -5 & 2 \\ 3 & 11 & -9 & 4 \\ -1 & 1 & 6 & 5 \end{array}\right].$$

We use the boxed element to eliminate any non-zeros below it.

This involves the following row operations

$$\left[\begin{array}{ccc|c} \boxed{1} & 3 & -5 & 2 \\ 3 & 11 & -9 & 4 \\ -1 & 1 & 6 & 5 \end{array}\right] \begin{array}{l} R2 - 3 \times R1 \\ R3 + R1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} \boxed{1} & 3 & -5 & 2 \\ 0 & 2 & 6 & -2 \\ 0 & 4 & 1 & 7 \end{array}\right].$$

And the next step is to use the 2 to eliminate the non-zero below it. This requires the final row operation

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 0 & \boxed{2} & 6 & -2 \\ 0 & 4 & 1 & 7 \end{array}\right] \; R3 - 2 \times R2 \quad \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 0 & \boxed{2} & 6 & -2 \\ 0 & 0 & -11 & 11 \end{array}\right].$$

This is the augmented form for an upper triangular system, writing the system in extended form we

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 2 \\ 2x_2 + 6x_3 &= -2 \\ -11x_3 &= 11 \end{aligned}$$

This gives $x_3 = -1$; $x_2 = 2$; $x_1 = -9$.

Example 2

Solve the system of equation

2x + 4y +6z = 22

3x + 8y + 5x = 27

-x + y +2z = 2

Solution

$$\begin{bmatrix} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

$\boldsymbol{R_1}$ ‘$= \mathbf{1/2}\boldsymbol{R}_1$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

$\boldsymbol{R_2}$’ $= \boldsymbol{R_2} - \mathbf{3}\boldsymbol{R}_1$; $\boldsymbol{R_3}$’ $= \boldsymbol{R_3} + \boldsymbol{R}_1$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{bmatrix}$$

$\boldsymbol{R_2}$' = $\mathbf{1/2}\boldsymbol{R}_2$; $\boldsymbol{R_1}$' = $\boldsymbol{R_1} - \mathbf{2}\boldsymbol{R}_2$; $\boldsymbol{R_3}$' = $\boldsymbol{R_3} - \mathbf{3}\boldsymbol{R_2}$

$$\begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 11 & 22 \end{bmatrix}$$

$\boldsymbol{R_3}$' = $\mathbf{1/11}\boldsymbol{R}_1$; $\boldsymbol{R_1}$' = $\boldsymbol{R_1} - \mathbf{7}\boldsymbol{R}_3$; $\boldsymbol{R_1}$' = $\boldsymbol{R_1} - \mathbf{7}\boldsymbol{R}_3$; $\boldsymbol{R_2}$' = $\boldsymbol{R_2} + \mathbf{2}\boldsymbol{R}_3$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Thus the solution to the system is x = 3, y = 1, z = 2.

ITERATIVE METHODS FOR SOLVING LINEAR SYSTEMS

As a numerical technique, Gaussian elimination is rather unusual because it is direct. That is, a solution is obtained after a single application of Gaussian elimination. Once a “solution” has been obtained, Gaussian elimination offers no method of refinement. The lack of refinements can be a problem because, as the previous section shows, Gaussian elimination is sensitive to rounding error. Numerical techniques more commonly involve an iterative method. For example, in calculus you probably studied Newton’s iterative method for approximating the zeros of a differentiable function. In this section you will look at two iterative methods for approximating the solution of a system of n linear equations in n variables.

The Jacobi Method The first iterative technique is called the Jacobi method, after Carl Gustav Jacob Jacobi (1804–1851). This method makes two assumptions: (1) that the system given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

has a unique solution and (2) that the coefficient matrix A has no zeros on its main diagonal. If any of the diagonal entries are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal. A matrix A is diagonally dominated if, in each row, the absolute value of the entry on the diagonal is greater than the sum of the absolute values of the other entries. More compactly, A is diagonally dominated if

$$|A_{ii}| > \sum_{j, j\neq i} |A_{ij}| \quad \text{for all } i$$

To begin the Jacobi method, solve the first equation for the second equation for and so on, as follows

$$x_1 = 1/\, a_{11}[b_1 - a_{12}x_2 - \dots - a_{1n}x_n]$$

$$x_2 = 1/\, a_{22}[\, b_2 - a_{21}x_1 - \dots - a_{2n}x_n]$$

$$\vdots$$

$$x_n = 1/a_{nn}[\, b_n - a_{n1}x_1 - a_{n2}x_2 - \dots]$$

Then make an initial approximation of the solution, Initial approximation and substitute these values of into the right-hand side of the rewritten equations to obtain the first approximation. After this procedure has been completed, one iteration has been performed. In the same way, the second approximation is formed by substituting the first approximation's x-values into the right-hand side of the rewritten equations. By repeated iterations, you will form a sequence of approximations that often converges to the actual solution.

Example

Use the Jacobi method to approximate the solution of the following system of linear equations.

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Solution

To begin, write the system in the form

$$x_1 = -\tfrac{1}{5} + \tfrac{2}{5}x_2 - \tfrac{3}{5}x_3$$
$$x_2 = \tfrac{2}{9} + \tfrac{3}{9}x_1 - \tfrac{1}{9}x_3$$
$$x_3 = -\tfrac{3}{7} + \tfrac{2}{7}x_1 - \tfrac{1}{7}x_2.$$

Let $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

as a convenient initial approximation. So, the first approximation is

$$x_1 = -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200$$
$$x_2 = \tfrac{2}{9} + \tfrac{3}{9}(0) - \tfrac{1}{9}(0) \approx 0.222$$
$$x_3 = -\tfrac{3}{7} + \tfrac{2}{7}(0) - \tfrac{1}{7}(0) \approx -0.429.$$

Continuing this procedure, you obtain the sequence of approximations shown in Table

n	0	1	2	3	4	5	6	7
x_1	0.000	-0.200	0.146	0.192	0.181	0.185	0.186	0.186
x_2	0.000	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0.000	-0.429	-0.517	-0.416	-0.421	-0.424	-0.423	-0.423

Because the last two columns in the above table are identical, you can conclude that to three significant digits the solution is x_1= 0.186 , x_2 = 0.331 , x_3 = -0.423.

GAUSS SEIDEL METHOD

Intuitively, the Gauss-Seidel method seems more natural than the Jacobi method. If the solution is converging and updated information is available for some of the variables, surely it makes sense to use that information! From a programming point of view, the Gauss-Seidel method is definitely more convenient, since the old value of a variable can be overwritten as soon as a new value becomes available. With the Jacobi method, the values of all variables from the previous iteration need to be retained throughout the current iteration, which means that twice as much as storage is needed. On the other hand, the Jacobi method is perfectly suited to parallel computation, whereas the Gauss-Seidel method is not. Because the Jacobi method updates or ‘displaces’ all of the variables at the same time (at the end of each iteration) it is often called the method of simultaneous displacements. The Gauss-Seidel method updates the variables one by one (during each iteration) so its corresponding name is the method of successive displacements.

Example 1

Solve the following system of equations by Gauss – Seidel method

$28x + 4y - z = 32$

$x + 3y + 10z = 24$

$2x + 17y + 4z = 35$

Solution

Since the diagonal element in given system are not dominant, we rearrange the equation as follows

$28x + 4y - z = 32$

$2x + 17y + 4z = 35$

$x + 3y + 10z = 24$

Hence

$x = 1/28[32 - 4y + z]$

$y = 1/17[35 - 2x - 4z]$

$z = 1/10[24 - x - 3y]$

Setting y = 0 and z = 0, we get,

First iteration

$x^{(1)} = 1/28\ [\ 32 - 4(0) + (0)] = 1.1429$

$y^{(1)} = 1/17\ [\ 35 - 2(1.1429) - 4(0)] = 1.9244$

$z^{(1)} = 1/10\ [\ 24 - 1.1429 - 3(1.9244)] = 1.8084$

Second Iteration

$x^{(2)}$ = 1/28 [32- 4(1.9244) +(1.8084)] = 0.9325

$y^{(2)}$ = 1/17 [35 – 2(0.9325) -4(1.8084)] = 1.5236

$z^{(2)}$ = 1/10 [24 – 0.9325 – 3(1.5236)] = 1.8497

Third Iteration

$x^{(3)}$ = 1/28 [32- 4(1.5236) +(1.8497)] = 0.9913

$y^{(3)}$ = 1/17 [35 – 2(0.9913) -4(1.8497)] = 1.5070

$z^{(3)}$ = 1/10 [24 –0.9913– 3(1.5070)] = 1.8488

Fourth Iteration

$x^{(4)}$ = 1/28 [32- 4(1.5070) +(1.8488)] = 0.9936

$y^{(4)}$ = 1/17 [35 – 2(0.9936) -4(1.8488)] = 1.5069

$z^{(4)}$ = 1/10 [24 – 0.9936 – 3(1.5069)] = 1.8486

Fifth Iteration

$x^{(5)}$ = 1/28 [32- 4(1.5069) +(1.8486)] = 0.9936

$y^{(5)}$ = 1/17 [35 – 2(0.9936) -4(1.8486)] = 1.5069

$z^{(5)}$ = 1/10 [24 – 0.9936 – 3(1.5069)] =1.8486

Since the values of x, y, z are same in the 4^{th} and 5^{th} Iteration, we stop the procedure here.

Hence x = 0.9936, y = 1.5069, z = 1.8486.

UNIT IV

Interpolation

The process of computing intermediate values of (x_0, x_n) for a function $y(x)$ from a given set of values of a function

Gregory-Newton's forward interpolation formula

$$y(x) = y_0 + \frac{\Delta y_0}{1}u + \frac{\Delta^2 y_0}{2}u(u-1) + \frac{\Delta^3 y_0}{6}u(u-1)(u-2) + \frac{\Delta^4 y_0}{24}u(u-1)(u-2)(u-3) + ---(a)$$

$$where\ u = \frac{1}{h}(x - x_0)$$

Gregory-Newton's backward interpolation formula

$$y(x) = y_n + \frac{\nabla y_n}{1}v + \frac{\nabla^2 y_n}{2}v(v+1) + \frac{\nabla^3 y_n}{6}v(v+1)(v+2) + \frac{\nabla^4 y_n}{24}v(v+1)(v+2)(v+3) + ---(b)$$

$$where\ v = \frac{1}{h}(x - x_n)$$

Remark:

(i) The process of finding the values of $y(x_i)$ outside the interval (x_0, x_n) is called *extrapolation*

(ii) The *interpolating polynomial* is a function $p_n(x)$ through the data points $y_i = f(x_i) = P_n(x_i)$ i=0,12,..n

(iii) Gregory-Newton's forward interpolation formula (a) can be applicable if the interval difference h is constant and used to interpolate the value of $y(x_i)$ nearer to beginning value x_0 of the data set

(iv) If $y = f(x)$is the exact curve and $y = p_n(x)$ is the interpolating polynomial then the *Error in polynomial interpolation* is $y(x) - p_n(x)$ given by

$$Error = \frac{h^{n+1} y^{(n+1)}(c)}{(n+1)!}(x - x_0)(x - x_1) -- (x - x_n) : x_0 < x < x_n, x_0 < c < x_n ---(c)$$

(v) *Error in Newton's forward interpolation is*

$$Error = \frac{h^{n+1} y^{(n+1)}(c)}{(n+1)!}u(u-1)(u-2) -- (u-n) : x_0 < x < x_n, x_0 < c < x_n ----(d)$$

(vi) *Error in Newton's backward interpolation is*

$$Error = \frac{h^{n+1} y^{(n+1)}(c)}{(n+1)!} v(v+1)(v+2)--(v+n): x_0 < x < x_n, x_0 < c < x_n ----(e)$$

Problem1: Estimate θ at $x = 43$ & $x = 84$ from the following table .also find $y(x)$

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

Solution: Here all the intervals are equal with h=x_1-x_0=10 we apply Newton interpolation

Difference Table:

x	$\theta = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
40	$184 = y_0$	$y_1 - y_0 = 20 = \Delta y_0$				
50	$204 = y_1$	$y_2 - y_1 = 22 = \Delta y_1$	$2 = \Delta^2 y_0$	$0 = \Delta^3 y_0$		
60	$226 = y_2$	$y_3 - y_2 = 24 = \Delta y_2$	$2 = \Delta^2 y_1$	$0 = \Delta^3 y_1$	$0 = \Delta^4 y_0$	$0 = \nabla^5 y_n$
70	$250 = y_3$	$y_4 - y_3 = 26 = \Delta y_3$	$2 = \Delta^2 y_2$	$0 = \nabla^3 y_n$	$0 = \nabla^4 y_n$	
80	$276 = y_4$	$y_n - y_{n-1} = 20.18 = \nabla y_n$	$2 = \nabla^2 y_n$			
90	$304 = y_n$					

Case (i): to find the value of θ at $x = 43$

Since $x = 43$ is nearer to x_0 we apply Newton's forward Interpolation

$$y(x) = y_0 + \frac{\Delta y_0}{1} u + \frac{\Delta^2 y_0}{2} u(u-1) + \frac{\Delta^3 y_0}{6} u(u-1)(u-2) + \frac{\Delta^4 y_0}{24} u(u-1)(u-2)(u-3) + ---(1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{10}(43-40) = \frac{3}{10} = 0.3 \Rightarrow u-1 = -0.7, u-2 = -1.7, u-3 = -2.7 --- (2)$$

Substituting (2) in (1), we get $y(x=43) = 184 + \frac{20}{1}(\frac{3}{10}) + \frac{2}{2}(\frac{3}{10})(\frac{-7}{10}) + 0 = \frac{18979}{10} = 189.79$

Case (ii): to find the value of θ at $x = 84$

Since $x = 84$ is nearer to x_n we apply Newton's backward Interpolation

$$y(x) = y_n + \frac{\nabla y_n}{1} v + \frac{\nabla^2 y_n}{2} v(v+1) + \frac{\nabla^3 y_n}{6} v(v+1)(v+2) + \frac{\nabla^4 y_n}{24} v(v+1)(v+2)(v+3) + ---(3)$$

$$where\ v = \frac{1}{h}(x - x_n) = \frac{1}{10}(84-90) = \frac{-6}{10} \Rightarrow v+1 = \frac{4}{10}, v+2 = \frac{14}{10}, v+3 = \frac{24}{10} ---(4)$$

Substituting (4) in (3), we get $y(x=84) = 304 + \frac{28}{1}(\frac{-6}{10}) + \frac{2}{2}(\frac{-6}{10})(\frac{4}{10}) + 0 = \frac{7174}{25} = 286.96$

To find polynomial $y(x)$, from (1) we get

$$y(x) = y_0 + \frac{\Delta y_0}{1} u + \frac{\Delta^2 y_0}{2} u(u-1) + \frac{\Delta^3 y_0}{6} u(u-1)(u-2) + \frac{\Delta^4 y_0}{24} u(u-1)(u-2)(u-3) + ---(1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{10}(x-40) \Rightarrow u-1 = \frac{1}{10}(x-50), u-2 = \frac{1}{10}(x-60), u-3 = \frac{1}{10}(x-60) ---(2)^1$$

Substituting (4) in (3), we get

$$y(x) = 184 + \frac{20}{1}\frac{1}{10}(x-40) + \frac{2}{2}\frac{1}{10}(x-40)\frac{1}{10}(x-50) + 0 = 184 + 2x - 80 + \frac{1}{100}(x^2 - 90x + 2000)$$

$$\Rightarrow y(x) = \frac{1}{100}(x^2 + 110x + 12400) ----------(5)$$

To Estimate θ at $x = 43$ & $x = 84$, put $x = 43$ & $x = 84$ in (5), we get

$$y(43) = \frac{1}{100}(18979) = 189.79\ and\ \ y(84) = \frac{1}{100}(28696) = 286.96$$

Problem2: Estimate the number of students whose weight is between 60 lbs and 70 lbs from the following data

Weight(lbs)	0-40	40-60	60-80	80-100	100-120
No.Students	250	120	100	70	50

Solution: let x-Weight less than 40 lbs, y-Number of Students, $\Rightarrow x_0 = 40, x_1 = 60, x_2 = 80, x_3 = 100, x_n = 120$, Here all the intervals are equal with h=x_1-x_0=20 we apply Newton interpolation

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	$250 = y_0$	$y_1 - y_0 = 120 = \Delta y_0$			
60	$370 = y_1$	$y_2 - y_1 = 100 = \Delta y_1$	$-20 = \Delta^2 y_0$	$-10 = \Delta^3 y_0$	
80	$470 = y_2$	$y_3 - y_2 = 70 = \Delta y_2$	$-30 = \Delta^2 y_1$	$10 = \nabla^2 y_n$	$20 = \Delta^4 y_0 = \nabla^4 y_n$
100	$540 = y_3$	$y_n - y_{n-1} = 50 = \nabla y_n$	$-20 = \nabla^2 y_n$		
120	$590 = y_n$				

Case (i): to find the number of students y whose weight less than 60 lbs ($x = 60$)

From the difference table the number of students y whose weight less than 60 lbs ($x = 60) = 370$

Case (ii): to find the number of students y whose weight less than 70 lbs ($x = 70$)

Since $x = 70$ is nearer to x_0 we apply Newton's forward Interpolation

$$y(x) = y_0 + \frac{\Delta y_0}{1}u + \frac{\Delta^2 y_0}{2}u(u-1) + \frac{\Delta^3 y_0}{6}u(u-1)(u-2) + \frac{\Delta^4 y_0}{24}u(u-1)(u-2)(u-3) + ----(1)$$

$$where\ u = \frac{1}{h}(x - x_0) = \frac{1}{20}(70-40) = \frac{3}{2} \Rightarrow u-1 = \frac{3}{2}, u-2 = \frac{2}{2}, u-2 = \frac{-1}{2}, u-3 = \frac{-3}{2} -----(2)$$

Substituting (2) in (1), we get

$$y(x=70) = 250 + \frac{120}{1}(\frac{3}{2}) + \frac{-20}{2}(\frac{3}{2})(\frac{1}{2}) + \frac{-10}{6}(\frac{3}{2})(\frac{1}{2})(\frac{-1}{2}) + \frac{20}{24}(\frac{3}{2})(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2}) = 423.59$$

The number of students y whose weight less than 70 lbs ($x = 70$) =424

Number of students whose weight is between 60 lbs and 70 lbs $=$

$$\begin{Bmatrix} \text{The number of students } y \\ \text{whose weight less than 70 lbs} \end{Bmatrix} - \begin{Bmatrix} \text{The number of students } y \\ \text{whose weight less than 60 lbs} \end{Bmatrix} = 424\text{-}370 = 54$$

Lagrange's interpolation formula Unequal intervals

$$y(x)=\frac{(x-x_1)(x-x_2)--(x-x_n)}{(x_0-x_1)(x_0-x_2)--(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)--(x-x_n)}{(x_1-x_0)(x_1-x_2)--(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)--(x-x_n)}{(x_2-x_0)(x_2-x_1)--(x_2-x_n)}y_2+---+\frac{(x-x_0)(x-x_1)--(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)--(x_n-x_{n-1})}y_n$$

Problem 3: Determine the value of $y(1)$ from the following data using Lagrange's Interpolation

x	-1	0	2	3
y	-8	3	1	12

Solution: given

x	$x_0=-1$	$x_1=0$	$x_2=3$	$x_n=3$
y	$y_0=-8$	$y_1=3$	$y_2=1$	$y_n=12$

Since the intervals ere not uniform we cannot apply Newton's interpolation.

Hence by Lagrange's interpolation for unequal intervals

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_n)}y_2+\frac{(x-x_0)(x-x_1)(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_{n-1})}y_n$$

$$y(x)=\frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8)+\frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3)$$
$$+\frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1)+\frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(12)----(1)$$

To compute $y(1)$ put $x=1$ in (1), we get

$$y(x=1)=\frac{(1-0)(1-2)(1-3)}{(-1-0)(-1-2)(-1-3)}(-8)+\frac{(1+1)(1-2)(1-3)}{(0+1)(0-2)(0-3)}(3)$$
$$+\frac{(1+1)(1-0)(1-3)}{(2+1)(2-0)(2-3)}(1)+\frac{(1+1)(1-0)(1-2)}{(3+1)(3-0)(3-2)}(12)$$
$$\Rightarrow y(x=1)=2$$

To find polynomial $y(x)$, from (1) we get

$$y(x)=\frac{2}{3}(x^3-5x^2+6x)+\frac{1}{2}(x^3-4x^2+x+6)$$
$$-\frac{1}{6}(x^3-2x^2-3x)+\frac{1}{1}(x^3-x^2-2x)----(1)$$
$$y(x)=x^3(\frac{2}{3}+\frac{1}{2}-\frac{1}{6}+1)+x^2(\frac{-10}{3}+\frac{-4}{2}+\frac{2}{6}-1)+x\ (\frac{12}{3}+\frac{1}{2}+\frac{3}{6}-2)+(\frac{6}{2})$$
$$\Rightarrow y(x)=2x^3-6x^2+3x\ +3-----(2)$$

To compute $y(1)$ put $x = 1$ in (2), we get $y(x=1)=2-6+3+3=2$

Inverse interpolation

For a given set of values of x and y, the process of finding $x(dependent)$ given $y(independent)$ is called Inverse interpolation

$$x(y)=\frac{(y\ -y_1)(y\ -y_2)--(y\ -y_n)}{(y_0-y_1)(y_0-y_2)--(y_0-y_n)}x_0+\frac{(y\ -y_0)(y\ -y_2)--(y\ -y_n)}{(y_1-y_0)(y_1-y_2)--(y_1-y_n)}x_1$$
$$+\frac{(y\ -y_0)(y\ -y_1)--(y\ -y_n)}{(y_2-y_0)(y_2-y_1)--(y_2-y_n)}x_2+---+\frac{(y\ -y_0)(y\ -y_1)--(y\ -y_{n-1})}{(y_n-y_0)(y_n-y_1)--(y_n-y_{n-1})}x_n$$

Problem 4: Estimate the value of x given $y = 100$ from the following data, $y(3) = 6$ $y(5) = 24$, $y(7) = 58$, $y(9) = 108$, $y(11) = 174$

Solution: given

x	$x_0=3$	$x_1=5$	$x_2=7$	$x_3=9$	$x_n=11$
y	$y_0=6$	$y_1=24$	$y_2=58$	$y_3=108$	$y_n=174$

By applying Lagrange's inverse interpolation

$$x(y)=\frac{(y-y_1)(y-y_2)(y-y_3)(y-y_n)}{(y_0-x_1)(y_0-y_2)(y_0-y_3)(y_0-y_n)}x_0+\frac{(y-y_0)(y-y_2)(y-y_3)(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_n)}x_1$$
$$+\frac{(y-y_0)(y-y_1)(y-y_3)(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_n)}x_2+\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_n)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_n)}x_3$$
$$+\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)(y_n-y_{n-1})}x_n$$
$$\Rightarrow x(100)=\frac{(100-24)(100-58)(100-108)(100-174)}{(6-24)(6-58)(6-108)(6-174)}(3)+\frac{(100-6)(100-58)(100-108)(100-174)}{(24-6)(24-58)(24-108)(24-174)}(5)$$
$$+\frac{(100-6)(100-24)(100-108)(100-174)}{(58-6)(58-24)(58-108)(58-174)}(7)+\frac{(100-6)(100-24)(100-58)(100-174)}{(108-6)(108-24)(108-58)(108-174)}(9)$$
$$+\frac{(100-6)(100-24)(100-58)(100-108)}{(174-6)(174-24)(174-58)(174-108)}(11)$$
$$\Rightarrow x(100)=0.35344-1.51547+2.88703+7.06759-0.13686=8.65573$$

Newton's forward formula for Derivatives

$$y'(x)=\frac{dy}{dx}=\frac{1}{h}\left\{\Delta y_0+\frac{\Delta^2 y_0}{2}(2u-1)+\frac{\Delta^3 y_0}{6}(3u^2-6u+2)+\frac{\Delta^4 y_0}{24}(4u^3-18u^2+22u-6)+---\right\}$$
$$y''(x)=\frac{d^2y}{dx^2}=\frac{1}{h^2}\left\{\Delta^2 y_0+\frac{\Delta^3 y_0}{1}(u-1)+\frac{\Delta^4 y_0}{24}(12u^2-36u+22)+---\right\}\text{where } u=\frac{1}{h}(x-x_0)$$

Newton's backward formula for Derivatives

$$y'(x)=\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n+\frac{\nabla^2 y_n}{2}(2v+1)+\frac{\nabla^3 y_n}{6}(3v^2+6v+2)+\frac{\nabla^4 y_n}{24}(4v^3+18v^2+22v+6)+---\right\}$$
$$y''(x)=\frac{d^2y}{dx^2}=\frac{1}{h^2}\left\{\nabla^2 y_n+\frac{\nabla^3 y_n}{1}(v+1)+\frac{\nabla^4 y_n}{24}(12v^2+36v+22)+---\right\}\text{where } v=\frac{1}{h}(x-x_n)$$

Problem 5: Find the rate of growth of population in the year 1941&1961 from the following table

year	1931	1941	1951	1961	1971
Population	40.62	60.80	79.95	103.56	132.65

Solution: Here all the intervals are equal with h=x_1-x_0=10 we apply Newton interpolation

Difference Table: let x-year,y-Population

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	$40.62 = y_0$	$y_1 - y_0 = 20.18 = \Delta y_0$			
1941	$60.80 = y_1$	$y_2 - y_1 = 19.15 = \Delta y_1$	$-1.03 = \Delta^2 y_0$	$5.49 = \Delta^3 y_0$	
1951	$79.95 = y_2$	$y_3 - y_2 = 23.61 = \Delta y_2$	$4.46 = \Delta^2 y_1$	$1.02 = \nabla^2 y_n$	$-4.47 = \Delta^4 y_0 = \nabla^4 y_n$
1196	$103.56 = y_3$	$y_n - y_{n-1} = 20.18 = \nabla y_n$	$5.48 = \nabla^2 y_n$		
1971	$132.65 = y_n$				

Case (i): to find rate of growth of population $(\frac{dy}{dx})$ in the year $(x = 1941)$

Since $x = 1941$ is nearer to x_0 we apply Newton's forwarded formula for derivative

$$y'(x) = \frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{\Delta^2 y_0}{2}(2u-1) + \frac{\Delta^3 y_0}{6}(3u^2-6u+2) + \frac{\Delta^4 y_0}{24}(4u^3-18u^2+22u-6) + --- \right\}$$

$$where\, u = \frac{1}{h}(x - x_0) = \frac{1}{10}(1941-1931) = 1$$

$$\Rightarrow y'(x=1941) = \frac{dy}{dx} = \frac{1}{10}\left\{20.18 + \frac{-1.03}{2}(2-1) + \frac{5.49}{6}(3-6+2) + \frac{-4.47}{24}(4-18+22-6) + --- \right\}$$

The rate of growth of population $(\frac{dy}{dx})$ in the year $(x = 1941)$= $y'(1941) = 2.36425$

Case (ii): to find rate of growth of population $(\frac{dy}{dx})$in the year $(x = 1961)$

Since $x = 1961$ is nearer to x_n we apply Newton's backward formula for derivative

$$y'(x)=\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n+\frac{\nabla^2 y_n}{2}(2v+1)+\frac{\nabla^3 y_n}{6}(3v^2+6v+2)+\frac{\nabla^4 y_n}{24}(4v^3+18v^2+22v+6)+---\right\}$$

$$v=\frac{1}{h}(x-x_n)=\frac{1}{10}(1961-1971)=-1$$

$$\Rightarrow y'(x=1961)=\frac{dy}{dx}=\frac{1}{10}\left\{29.09+\frac{5.48}{2}(-2+1)+\frac{1.02}{6}(3-6+2)+\frac{-4.47}{24}(-4+18-22+6)+---\right\}$$

The rate of growth of population ($\frac{dy}{dx}$) in the year ($x=1961$)= $y'(1961)=2.65525$

Problem 6 A rod is rotating in a plane, estimate the angular velocity and angular acceleration of the rod at time 6 secs from the following table

Time-t(sec)	0	0.2	0.4	0.6	0.8	1.0
Angle-θ(radians)	0	0.12	0.49	1.12	2.02	3.20

Solution: Here all the intervals are equal with h=x_1-x_0=0.2 we apply Newton interpolation

Difference Table: let x- time (sec),y-Angle (radians)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$0=y_0$	$y_1-y_0=0.12=\Delta y_0$			
	$0.12=y_1$	$y_2-y_1=0.37=\Delta y_1$	$0.25=\Delta^2 y_0$	$0.01=\Delta^3 y_0$	
	$0.49=y_2$	$y_3-y_2=0.63=\Delta y_2$	$0.26=\Delta^2 y_1$	$0.01=\Delta^3 y_1$	$0=\Delta^4 y_0$
	$1.12=y_3$	$y_4-y_3=0.90=\Delta y_3$	$0.27=\Delta^2 y_2$	$0.01=\nabla^2 y_n$	$0=\nabla^4 y_n$
	$2.02=y_4$	$y_n-y_{n-1}=1.18=\nabla y_n$	$0.28=\nabla^2 y_n$		
	$3.20=y_n$				

Case (i): to find Angular velocity ($\frac{dy}{dx}$) in time ($x=0.6\ sec$)

Since $x=0.6\ sec$ is nearer to x_n we apply Newton's backward formula for derivative

$$y'(x)=\frac{dy}{dx}=\frac{1}{h}\left\{\nabla y_n+\frac{\nabla^2 y_n}{2}(2v+1)+\frac{\nabla^3 y_n}{6}(3v^2+6v+2)+\frac{\nabla^4 y_n}{24}(4v^3+18v^2+22v+6)+---\right\}$$

$$v=\frac{1}{h}(x-x_n)=\frac{1}{0.2}(0.6-1.0)=-2$$

$$y'(x=0.6)=\frac{dy}{dx}=\frac{1}{0.2}\left\{1.18+\frac{0.28}{2}(-4+1)+\frac{0.01}{6}(12-12+2)+\frac{0}{24}(4v^3+18v^2+22v+6)+---\right\}$$

$$\Rightarrow The\,angular\,velocity\;y'(x=0.6)=3.81665\,radian/\sec$$

Case (ii): to find Angular acceleration $(\frac{d^2y}{dx^2})$ in time $(x = 0.6\ sec)$

Since $x = 0.6\ sec$ is nearer to x_n we apply Newton's backward formula for derivative

$$y''(x)=\frac{d^2y}{dx^2}=\frac{1}{h^2}\left\{\nabla^2 y_n+\frac{\nabla^3 y_n}{1}(v+1)+\frac{\nabla^4 y_n}{24}(12v^2+36v\;+22)+---\right\}$$

$$where\; v=\frac{1}{h}(x-x_n)=\frac{1}{0.2}(0.6-1.0)=-2$$

$$\Rightarrow y''(x=0.6)=\frac{1}{0.2^2}\left\{0.28+\frac{0.01}{1}(-2+1)+0\right\}$$

$$y''(0.6)=6.75\;radian/\sec^2$$

Numerical Integration

Trapezoidal rule

$$\int_{x_0}^{x_0+nh} y(x)dx=\frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+y_3+y_4+—)\;where\,h=\frac{1}{n}(x_n-x_0), n-number\;of\;\text{int}ervals$$

Simpson's $1/3$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\{(y_0+y_n)+2(y_2+y_4+y_6+-)+4(y_1+y_3+y_5+—)\ \}$$

$$where\, h = \frac{1}{n}(x_n - x_0), n - number\, of\, \text{int}\, ervals$$

Simpson's $^3/_8$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{3h}{8}\{(y_0+y_n)+2(y_3+y_6+y_9+-)+3(y_1+y_2+y_4+y_5+—)\ \}$$

$$where\, h = \frac{1}{n}(x_n - x_0), n - number\, of\, \text{int}\, ervals$$

Remarks:

1) Geometrical interpretation of $\int_{x_0}^{x_n} y(x)dx$ is approximated by the sum of area of the trapezium
2) Simpson's $^1/_3$ rule is applicable when number of intervals are multiples of 2 and Simpson's $^3/_8$ rule is applicable when number of intervals are multiples of 3
3) The error in trapezoidal rule is $\frac{b-a}{12}h^2M$ where $M = max\{y_0'', y_1'', ...\}$which is of order h^2
4) The error in Simpson's $^1/_3$ rule rule is $\frac{b-a}{180}h^4M$ where $M = max\{y_0'''', y_2'''', ...\}$which is of order h^4

Problem7: Evaluate $\int_1^6 \frac{1}{1+x^2}dx$ using (i) Trapezoidal rule (ii) Simpson's $^1/_3$ rule (iii) Simpson's $^3/_8$ rule and Compare your answer with actual value.

Solution: Given $\int_0^6 \frac{1}{1+x^2}dx = \int_{x_0}^{x_0+nh} y(x)dx \Rightarrow y(x) = \frac{1}{1+x^2}, x_0 = 0, x_0 + nh = 6 ----(1)$

Choose the number of interval (n)=6 so that we can apply all rules

x	$x_0 = 0$	$x_1 = x_0 + h = 1$	$x_2 = x_1 + h = 2$	$x_3 = 3$	$x_4 = 4$	$x_5 = 5$	$x_n = 6$
$y(x) = \frac{1}{1+x^2}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$

case(i) Trapezoidal rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + y_3 + y_4 + \text{---})$$

$$\Rightarrow \int_0^6 \frac{1}{1+x^2}dx = \frac{1}{2}\left\{(1+\frac{1}{37}) + 2(\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26})\right\} = 1.410799$$

Case (ii) Simpson's $1/3$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\{(y_0 + y_n) + 2(y_2 + y_4 + y_6 + -) + 4(y_1 + y_3 + y_5 + \text{---})\}$$

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{1}{3}\left\{(1+\frac{1}{37}) + 2(\frac{1}{5} + \frac{1}{17}) + 4(\frac{1}{2} + \frac{1}{10} + \frac{1}{26})\right\} = 1.36617$$

Case(iii) Simpson's $3/8$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{3h}{8}\{(y_0 + y_n) + 2(y_3 + y_6 + y_9 + -) + 3(y_1 + y_2 + y_4 + y_5 + \text{---})\}$$

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{3}{8}\left\{(1+\frac{1}{37}) + 2(\frac{1}{10}) + 3(\frac{1}{2} + \frac{1}{5} + \frac{1}{17} + \frac{1}{26})\right\} = 1.35708$$

Comparison

Exact value $\int_0^6 \frac{1}{1+x^2}dx = \left[\tan^{-1}(x)\right]_{x=0}^{x=6} = \tan^{-1}(6) - \tan^{-1}(0) = 1.40565$

Hence trapezoidal rule gives better approximation than Simpson's rule.

Problem 8: By dividing the range into 10 equal part Determine the value of $\int_0^{\pi} \sin x\, dx$ using (i) Trapezoidal rule (ii) Simpson's $1/3$ rule (iii) Simpson's $3/8$ rule and Compare your answer with actual value.

Solution: Given $\int_0^{\pi} \sin x\, dx = \int_{x_0}^{x_0+nh} y(x)dx \Rightarrow y(x) = \sin x, x_0 = 0, x_0 + nh = \pi and\ n = 10 ----(1)$

$$given\ number\ of\ \text{int}\, ervals(n) = 10, (1) \Rightarrow h = \frac{1}{n}(x_n - x_0) = \frac{1}{10}(\pi - 0) = \frac{\pi}{10}$$

x	$x_0 = 0$	$x_1 = x_0 + h = \frac{\pi}{10}$	$x_2 = x_1 + h = \frac{2\pi}{10}$	$x_3 = \frac{3\pi}{10}$	$x_4 = \frac{4\pi}{10}$	$x_5 = \frac{5\pi}{10}$	$x_6 = \frac{6\pi}{10}$
$y(x) = \sin(x)$	$\sin(0)$ $= 0$	$\sin(\frac{\pi}{10})$ $= 0.30901$	$\sin(\frac{2\pi}{10})$ $= 0.58779$	$\sin(\frac{3\pi}{10})$ $= 0.80901$	$\sin(\frac{4\pi}{10})$ $= 0.95106$	$\sin(\frac{5\pi}{10})$ $= 1.0$	$\sin(\frac{6\pi}{10})$ $= 0.95106$
x	$x_7 = \frac{7\pi}{10}$	$x_8 = \frac{8\pi}{10}$	$x_9 = \frac{9\pi}{10}$	$x_n = \pi$			
$y(x) = \sin(x)$	$\sin(\frac{7\pi}{10})$ $= 0.80902$	$\sin(\frac{8\pi}{10})$ $= 0.58779$	$\sin(\frac{9\pi}{10})$ $= 0.30902$	$\sin(\frac{10\pi}{10})$ $= 0$			

Case (i) Trapezoidal rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + y_3 + y_4 + \text{—})$$

$$\Rightarrow \int_0^6 \frac{1}{1+x^2}dx = \frac{1}{2}\{(0+0) + 2(0.30901 + 0.58779 + 0.80901 + 0.95106 + 1.0 + 0.95106 + 0.80901 + 0.58779 + 0.3090$$

$$\Rightarrow \int_0^6 \frac{1}{1+x^2}dx = 1.98352$$

Case (ii) Simpson's $^1/_3$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\left\{(y_0+y_n)+2(y_2+y_4+y_6+-)+4(y_1+y_3+y_5+—)\right\}$$

$$\Rightarrow \int_0^6 \sin(x)dx = \frac{\pi}{30}\{(0+0)+2(0.58779+0.95106+0.95106+0.58779)+4(0.30901+0.80901+1.0+0.80901+$$

$$\Rightarrow \int_0^6 \sin(x)dx = 2.00010$$

Case (iii) Simpson's $^3/_8$ rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{3h}{8}\left\{(y_0+y_n)+2(y_3+y_6+y_9+-)+3(y_1+y_2+y_4+y_5+—)\right\}$$

$This\ rule\ cannot\ be\ applied\ \sin ce\ \ n\ is\ not\ \ a\ multipole\ of\ 3$

Comparison

Exact value $\int_0^{\pi} \sin(x)dx = \left[-\cos(x)\right]_{x=0}^{x=\pi} = -\left[\cos(\pi)-\cos(0)\right] = 2.0$

Hence, Simpson's $^1/_3$ rule gives better approximation than trapezoidal rule

UNIT V

5.1 Numerical Solution to Ordinary Differential Equation

Introduction

An ordinary differential equation of order *n* in of the form $F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$, where $y^{(n)} = \frac{d^n y}{dx^n}$.

We will discuss the Numerical solution to first order linear ordinary differential equations by Taylor series method, and Runge - Kutta method, given the initial condition $y(x_0) = y_0$.

5.1.1 Taylor Series method

Consider the first order differential equation of the form $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$.

The solution of the above initial value problem is obtained in two types

- Power series solution
- Point wise solution

(i) Power series solution

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots$$

(ii) Point wise solution

$$y(x) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$$

Problems:

1. Using Taylor series method find y at $x = 0.1$ if $\frac{dy}{dx} = y + 1, y(0) = 1$.

Solution:

Given $\frac{dy}{dx} = y + 1$ and $x_0 = 0, y_0 = 1, h = 0.1$

Taylor series formula for $y(0.1)$ is

$$y(x) = y(x_0) + \frac{h}{1!}y'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \cdots$$

$y'(x) = y + 1$	$y'(0) = y(0) + 1 = 1 + 1 = 2$
$y''(x) = y'$	$y''(0) = y'(0) = 2$
$y'''(x) = y''$	$y'''(0) = y''(0) = 2$

Substituting in the Taylor's series expansion:

$$y(0.1) = y(0) + hy'(0) + \frac{h^2}{2!}y''(0) + \cdots$$

$$= 1 + 0.1 \times 2 + \frac{0.01}{2} \times 2 + \frac{0.001}{6} \times 2 + \cdots$$

$$y(0.1) = 1.2103$$

2. Find the Taylor series solution with three terms for the initial value problem $\frac{dy}{dx} = x^2 + y, y(1) = 1$

Solution:

Given $\frac{dy}{dx} = x^2 + y, x_0 = 1, y_0 = 1$

$y'(x) = x^2 + y$	$y'(1) = 1 + 1 = 2$
$y''(x) = 2x + y'$	$y''(1) = 2 + 2 = 4$
$y'''(x) = 2 + y''$	$y'''(1) = 2 + 4 = 6$

$y'^{v}(x) = y'''$	$y'^{v}(1) = 6$

The Taylor's series expansion about a point $x = x_0$ is given by

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!}y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots$$

Hence at $x = 1$

$$y(x) = y(1) + \frac{(x - 1)}{1!}y'(1) + \frac{(x - 1)^2}{2!}y''(1) + \frac{(x - 1)^3}{3!}y'''(1) + \cdots$$

$$y(x) = 1 + 2\frac{(x - 1)}{1!} + 4\frac{(x - 1)^2}{2!} + 6\frac{(x - 1)^3}{3!} + \cdots$$

5.1.2 Runge-Kutta method

Runge-kutta methods of solving initial value problem do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formula possesses the advantage of requiring only the function values at some selected points. These methods agree with Taylor series solutions up to the term in h^r where r is called the order of that method.

Fourth-order Runge-Kutta method

Let $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ be given.

Working rule to find $y(x_1)$

The value of $y_n = y(x_n)$ where $x_n = x_{n-1} + h$ where *h* is the incremental value for *x* is obtained as below:

Compute the auxiliary values

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Compute the incremental value for *y*

$$\Delta y = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

The iterative formula to compute successive value of *y* is $y_{n+1} = y_n + \Delta y$

Problems

1. Find the value of y at $x = 0.2$. Given $\frac{dy}{dx} = x^2 + y, y(0) = 1$, using R-K method of order IV.

Sol:

Here $f(x, y) = x^2 + y, y(0) = 1$
Choosing $h = 0.1, x_0 = 0, y_0 = 1$
Then by R-K fourth order method,

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf\,(x_0, y_0) = 0$$
$$k_2 = hf\,(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.00525$$
$$k_3 = hf\,(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.00525$$
$$k_4 = hf\,(x_0 + h, y_0 + k_3) = 0.0110050$$

$y(0.1) = 1.0053$

To find $y(0.2)$ given $x_2 = x_1 + h = 0.2, y_1 = 1.0053$

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_1, y_1) = 0.0110$$
$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.01727$$
$$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.01728$$
$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.02409$$

y(0.2)=1.0227

5.2 Numerical Solution to Partial Differential Equations

5.2.1 Solution of Laplace Equation and Poisson equation

Partial differential equations with boundary conditions can be solved in a region by replacing the partial derivative by their finite difference approximations. The finite difference approximations to partial derivatives at a point (x_i, y_i) are given below:

$$u_x(x_i, y_i) = \frac{u(x_{i+1}, y_i) - u(x_i, y_i)}{h}$$

$$u_y(x_i, y_i) = \frac{u(x_i, y_{i+1}) - u(x_i, y_i)}{k}$$

$$u_{xx}(x_i, y_i) = \frac{u_x(x_{i+1}, y_i) - u_x(x_i, y_i)}{h} = \frac{u(x_{i+1}, y_i) - 2u(x_i, y_i) + u(x_{i-1}, y_i)}{h^2}$$

$$u_{yy}(x_i, y_i) = \frac{u_y(x_i, y_{i+1}) - u_y(x_i, y_i)}{k} = \frac{u(x_i, y_{i+1}) - 2u(x_i, y_i) + u(x_i, y_{i-1})}{k^2}$$

Graphical Representation

The xy-plane is divided into small rectangles of length h and breadth k by drawing the lines $x = ih$ and $y = ik$, parallel to the coordinate axes. The points of intersection of these lines are called grid points or mesh points or lattice points. The grid points (x_i, y_j) is denoted by (i,j) and is surrounded by the neighbouring grid points $(i-1,j)$ to the left, $(i+1,j)$ to the right, $(i,j+1)$ above and $(i,j-1)$ below.

Note

The most general linear P.D.E of second order can be written as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = f(x,y)$$

where *A*,*B*,*C*,*D*,*E*,*F* are in general functions of *x* and *y*.

A partial differential equation in the above form is said to be

- Elliptic if $B^2 - 4AC < 0$
-
- Parabolic if $B^2 - 4AC = 0$

Hyperbolic if $B^2 - 4AC > 0$

Standard Five Point Formula (SFPF)

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

Diagonal Five Point Formula (DFPF)

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}\right]$$

Solution of Laplace equation $u_{xx}+u_{yy}=0$

Leibmann's Iteration Process

We compute the initial values of $u_1, u_2, \ldots, u_9$ by using standard five point formula and diagonal five point formula .First we compute u_5 by standard five point formula (SFPF).

$$u_5 = \frac{1}{4}[b_7 + b_{15} + b_{11} + b_3]$$

We compute $u_1,u_3,u_7.u_9$ by using diagonal five point formula (DFPF)

$$u_1 = \frac{1}{4}[b_1 + u_5 + b_3 + b_{15}]$$

$$u_3 = \frac{1}{4}[u_5 + b_5 + b_3 + b_7]$$

$$u_7 = \frac{1}{4}[b_{13} + u_5 + b_{15} + b_{11}]$$

$$u_9 = \frac{1}{4}[b_7 + b_{11} + b_9 + u_5]$$

Finally we compute u_2 ,u_4 ,u_6 ,u_8 by using standard five point formula.

$$u_2 = \frac{1}{4}[u_5 + b_3 + u_1 + u_3]$$

$$u_4 = \frac{1}{4}[u_1 + u_5 + b_{15} + u_7]$$

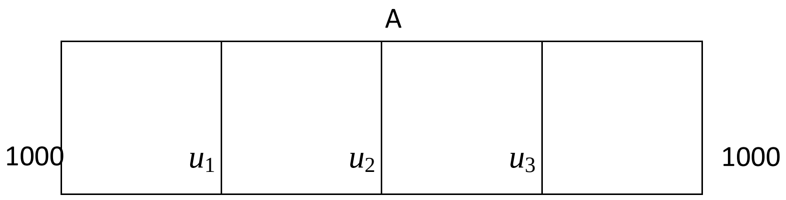
$$u_6 = \frac{1}{4}[u_3 + u_9 + u_5 + b_7]$$

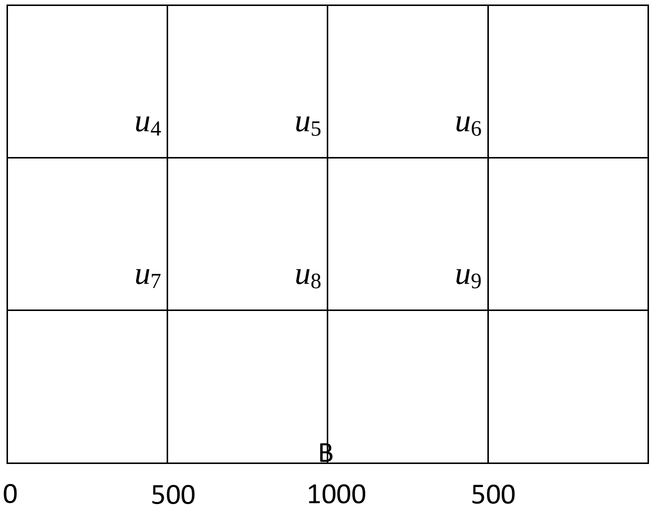
$$u_8 = \frac{1}{4}[u_7 + b_{11} + u_9 + u_5]$$

Solve the system of simultaneous equations obtained by finite difference method to get the value at the interior mesh points. This process is called *Leibmann's method.*

Problems

1.Solve the equation $\Delta^2 u = 0$ for the following mesh, with boundary values as shown using Leibmann's iteration process.





Sol:

Let $u_1, u_2 \ldots\ldots u_9$ be the values of u at the interior mesh points of the given region. By symmetry about the vertical lines AB and the horizontal line CD, we observe

$$u_1 = u_3 = u_9 = u_7; u_2 = u_8; u_4 = u_6$$

Hence it is enough to find $u_1, u_2, u_4,$

Using SFPF u_5 =1500

Using DFPF u_1=1125 u_2 =1187.5 u_4 =1437.5

Gauss-seidel scheme

$$u_1 = \frac{1}{4}[1500 + u_2 + u_4]$$

$$u_2 = \frac{1}{4}[2u_1 + u_5 + 1000]$$

$$u_4 = \frac{1}{4}[2000 + u_5 + u_4]$$

$$u_5 = \frac{1}{4}[2u_2 + 2u_4]$$

The iteration values are tabulated as follows

Iteration No k	u_1	u_2	u_4	u_5
0	**1500**	**1125**	**1187.5**	**1437.5**
1	**1031.25**	**1125**	**1375**	**1250**
2	**1000**	**1062.5**	**1312.5**	**1187.5**
3	**968.75**	**1031.25**	**1281.25**	**1156.25**
4	**953.1**	**1015.3**	**1265.6**	**1140.6**
5	**945.3**	**1007.8**	**1257.8**	**1132.8**
6	**941.4**	**1003.9**	**1253.9**	**1128.9**
7	**939.4**	**1001.9**	**1251.9**	**1126.9**
8	**938.4**	**1000.9**	**1250.9**	**1125.9**
9	**937.9**	**1000.4**	**1250.4**	**1125.4**
10	**937.7**	**1000.2**	**1250.2**	**1125.2**
11	**937.6**	**1000.1**	**1250.1**	**1125.1**
12	**937.6**	**1000.1**	**1250.1**	**1125.1**

$u_1 = u_3 = u_7 = u_9 = 937.6, u_2 = u_8 = 1000.1, u_4 = u_6 = 1250.1, u_5 = 1125.1$

Solution of Poisson equation

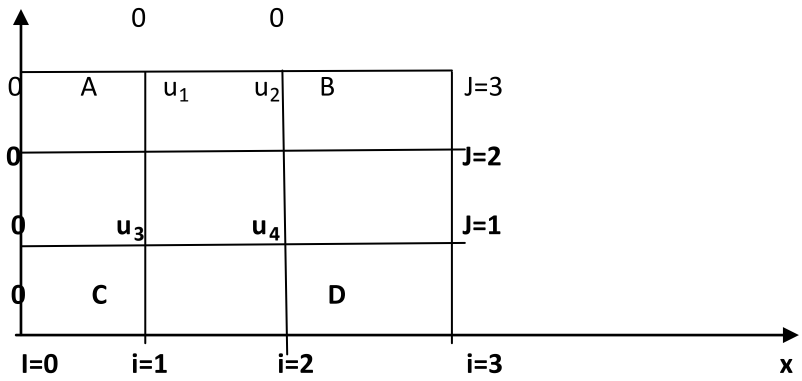
An equation of the type $\Delta^2 u = f(x, y)$ i.e., is called Poisson's equation where $f(x, y)$ is a function of x and y. Substituting the finite difference approximations to the partial differential coefficients, we get $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$

Problem: 1

Solve the poisson equation $\Delta^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides

x=0,y=0,x=3,y=3 and u=0 on the boundary .assume mesh length h=1 unit.

Sol:



Here the mesh length $\Delta x = h = 1$

Applying the formula below at the interior point of the mesh we get a system of simultaneous equations $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$

$u_2 + u_3 - 4u_1 = -150$
$u_1 + u_4 - 4u_3 = -120$
$u_2 + u_3 - 4u_4 - 150$
$u_1 = u_4 = 75, u_2 = 82.5, u_3 = 67.5$

5.3 Solution of One dimensional heat equation

In this chapter, we will discuss the finite difference solution of one dimensional heat flow equation by Explicit and implicit method

Explicit Method(Bender-Schmidt method

Consider the one dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ .This equation is an example of parabolic equation.

$$_{+} = \lambda u_{i+1,j} + 1 - 2\lambda u_{i,j} + \lambda u_{i-1,j} \quad u_{i,j\,1} \qquad (1)$$

$$\lambda = \frac{k}{{}^{2}}$$

Expression (1) is called the explicit formula and it valid for $0 < \lambda \le \frac{1}{2}$

If λ=1/2 then (1) is reduced into

$$u_{i,j+} = \frac{1}{2}[u_{i+1,j} + \lambda u_{i-1,j\,1} \quad 2 \quad]$$

(2)

This formula is called Bender-Schmidt formula.

Implicit method (Crank- Nicholson method)

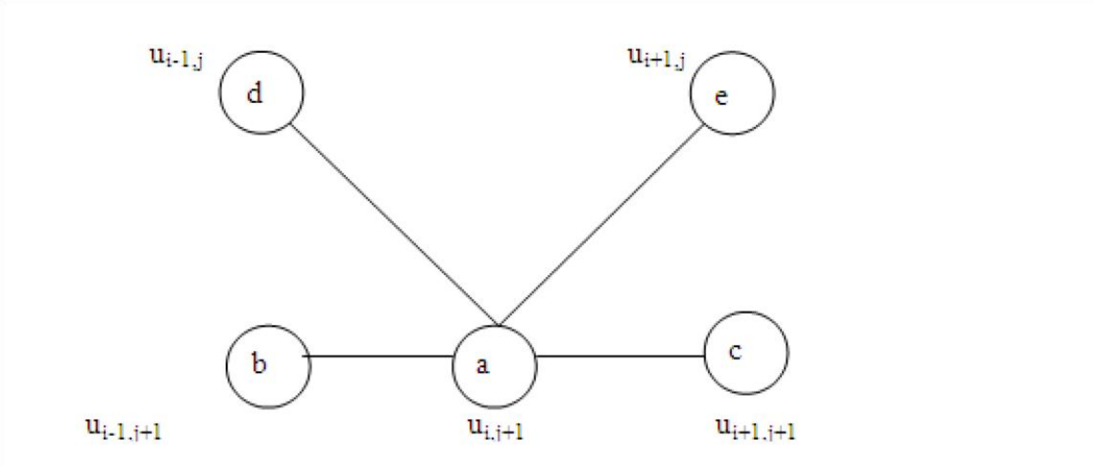
$$-\lambda u_{i-1,j+1} + 2(1+\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + 2(1-\lambda)u_{i,j} + \lambda u_{i+1,j}$$

This expression is called Crank-Nicholson’s implicit scheme. We note that Crank Nicholson’s scheme converges for all values of λ

When λ=1, i.e., k=ah^2 the simplest form of the formula is given by

$$\Rightarrow u_{i,j+1} = \frac{1}{4}[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j}]$$

The use of the above simplest scheme is given below.



The value of u at A=Average of the values of u at B, C, D, E

Note

In this scheme, the values of u at a time step are obtained by solving a system of linear equations in the unknowns u_i.

Solved Examples

1.Solve $u_{xx} = 2u_t$ **when u(0,t)=0,u(4,t)=0 and with initial condition u(x,0)=x(4-x) upto t=sec assuming** $\Delta x = h = 1$

Sol:

By Bender-Schmidt recurrence relation ,

$$u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + \lambda u_{i-1,j}] \quad (1)$$

$$= \frac{ah}{2}$$

k

For applying eqn(1) ,we choose 2

Here a=2,h=1.Then k=1

By initial conditions, u(x,0)=x(4-x) ,we have

$u_{i,0} = i(4 - i)\quad i = 1,2,3$

$u_{1,0} = 3, u_{2,0} = 4, u_{3,0} = 3$

By boundary conditions, u(0,t)=0,u_0=0,u(4,0)=0 $\Rightarrow u_{4,j} = 0 \forall j$

Values of u at t=1

$$u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$$

$$u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = 2$$

$$u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = 3$$

$$u_{3,1} = \frac{1}{2}[u_{2,0} + u_{4,0}] = 2$$

The values of u up to t=5 are tabulated below.

j\i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0		0.75	0.5	

2.Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions

$= x^2(25 - x^2)$ u(0,t)=u(5,t)=0 and

$u(x,0)$ taking h=1 and k=1/2,tabulate the values of u upto t=4 sec.

Solution

Here a=1,h=1

For λ=1/2,we must choose k=ah^2/2

K=1/2

By boundary conditions

$u(0,t) = 0 \Rightarrow u_{0,j} = 0 \forall j$ $u(5,t)$ 0 $u_{5,j}$

0 j $u(x, 0) = \Rightarrow x^2$ (25 x^2) ^{2} (25 i^2

),i 0,1,2,3,4,5 $-$

$\Rightarrow \quad = \quad - \quad =$

$= \quad = \quad =$

$u_{i,0}$ i u1,0 24,u2,0

84,u3,0 144,u4,0 = =

144,u5,0 0

By Bender-schmidt realtion,

$$u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + u_{i-1,j}]$$

The values of u upto 4 sec are tabulated as follows

j\i	0	1	2	3	4	5
0	0	24	84	144	144	0
0.5	0	42	84	144	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0
3.5	0	17.5312	26.0625	28.4062	16.125	0
4	0	13.0312	22.9687	21.0938	14.2031	0

5.4 Solution of One dimensional wave equation

Introduction

$_{tt} = a^2 u_{xx}$

The one $+\lambda^2 a^2 u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ dimensional wave equation is of hyperbolic type. In this chapter, we discuss the finite difference solution of the one dimensional wave equation u_{tt} $a^2 u_{xx}$.

Explicit method to solve u

$u_{i,j\,1} = 2(1-\lambda^2 a^2)u_{i,j}$ **(1)**

Where λ=k/h

Formula (1) is the explicit scheme for solving the wave equation.

Problems

1.Solve numerically ,$4u_{xx}$ u_{tt} with the boundary conditions u(0,t)=0,u(4,t)=0 and the initial conditions $u_t(x,0)$ 0 & $u(x,0)$ $x(4$ $x)$, taking h=1.Compute u upto t=3sec.

Solution

Here a^2=4

A=2 and h=1

We choose k=h/a$\Rightarrow$k=1/2

The finite difference scheme is

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$
$$u(0,t) = 0 \Rightarrow u_{0,j} \ \& \ u(4,t) = 0 \Rightarrow u_{4,j} = 0 \forall j$$
$$u(x,0) = x(4-x) \Rightarrow u_{i,0} = i(4-i), i = 0,1,2,3,4$$
$$u_{0,0} = 0, u_{1,0} = 3, u_{2,0} = 3, u_{4,0} = 0$$
$$u_{1,1} = 4 + 0/2 = 2$$
$$u_{2,1} = 3, u_{3,1} = 2$$

The values of u for steps t=1,1.5,2,2.5,3 are calculated using (1) and tabulated below.

j\i	**0**	**1**	**2**	**3**	**4**
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	-2	3	-2	0
4	0	-3	-4	3	0
5	0	-2	-3	-2	0
6	0	0	0	0	0