SMT1105 ENGINEERING MATHEMATICS II

UNIT III

VECTOR CALCULUS

VECTOR CALCULUS

Part-A

Problem 1 Prove that $div(grad\,\phi) = \nabla^2\phi$

Solution:

$$div(grad\,\phi) = \nabla.\nabla\phi$$
$$= \nabla\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$$
$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$$
$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$
$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$
$$= \nabla^2\phi\,.$$

Problem 2 Find a, b, c, if $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

Solution:

$\vec{F}$ is irrotational if $\nabla\times\vec{F} = \vec{0}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$
$$= \vec{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$
$$+\vec{k}\left[\frac{\partial}{\partial x}(bx-3y+2z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$
$$= \vec{i}[c+1] + \vec{j}[a-4] + \vec{k}[b-2]$$

$\because \nabla\times\vec{F} = 0 \Rightarrow 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{i}[c+1] + \vec{j}[a-4] + \vec{k}[b-2]$

$\therefore c+1 = 0,\ a-4 = 0,\ b-2 = 0$

$\Rightarrow c = -1,\ a = 4,\ b = 2.$

Problem 3 If S is any closed surface enclosing a volume V and $\vec{r}$ is the position vector of a point, prove $\int\int_S \left(\vec{r}.\hat{n}\right) ds = 3V$

Solution:

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

By Gauss divergence theorem

$$\iint_S \overrightarrow{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \overrightarrow{F} dV \qquad \text{Here } \overrightarrow{F} = \nabla.\vec{r}$$

$$\iint_S \vec{r} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{r} dV$$

$$= \iiint_V \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right).\left(x\vec{i} + y\vec{j} + z\vec{k} \right) dV$$

$$= \iiint (1+1+1)\, dV$$

$$\iint_S \vec{r} \cdot \hat{n}\, ds = 3V.$$

Problem 4 If $\vec{r} = \vec{a}\cos nt + \vec{b}\sin nt$, where $\vec{a}, \vec{b}, n$ are constants show that $\vec{r} \times \frac{d\vec{r}}{dt} = n\left(\vec{a} \times \vec{b}\right)$

Solution:

Given $\vec{r} = \vec{a}\cos nt + \vec{b}\sin nt$

$$\frac{d\vec{r}}{dt} = -n\vec{a}\sin nt + n\vec{b}\cos nt$$

$$\vec{r} \times \frac{d\vec{r}}{dt} = \left(\vec{a}\cos nt + \vec{b}\sin nt\right) \times \left(-n\vec{a}\sin nt + n\vec{b}\cos nt\right)$$

$$= n\left(\vec{a} \times \vec{b}\right)\cos^2 nt - \left(\vec{b} \times \vec{a}\right)\sin^2 nt$$

$$= n\left(\vec{a} \times \vec{b}\right)\cos^2 nt + \left(\vec{a} \times \vec{b}\right)\sin^2 nt \qquad \left(\because \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}\right)$$

$$= n\left(\vec{a} \times \vec{b}\right)(1) = n\left(\vec{a} \times \vec{b}\right)$$

Problem 5 Prove that $div\left(curl \overrightarrow{A}\right) = 0$

Solution:

$$\nabla.\left(\nabla \times \mathrm{A}\right) = \nabla \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla \cdot \left[\vec{i}\left(\frac{\partial \mathrm{A}_3}{\partial y} - \frac{\partial \mathrm{A}_2}{\partial z} \right) - \vec{j}\left(\frac{\partial \mathrm{A}_3}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial z} \right) \right] + \vec{k}\left(\frac{\partial \mathrm{A}_2}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial y} \right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \mathrm{A}_3}{\partial y} - \frac{\partial \mathrm{A}_2}{\partial z} \right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathrm{A}_3}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial z} \right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathrm{A}_2}{\partial x} - \frac{\partial \mathrm{A}_1}{\partial y} \right)$$

$$=\left(\frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_2}{\partial x\partial z}\right)+\left(\frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_3}{\partial y\partial x}\right)+\left(\frac{\partial^2 A_2}{\partial z\partial x}-\frac{\partial^2 A_1}{\partial z\partial y}\right)$$

$\therefore div\left(curl\vec{A}\right)=0$

Problem 6 Find the unit normal to surface $xy^3z^2=4$ at $(-1,-1,2)$

Solution:

Let $\phi=xy^3z^2-4$

$\nabla\phi=y^3z^2\vec{i}+3xy^2z^2\vec{j}+2xy^3z\vec{k}$

$\nabla\phi_{(-1,-1,2)}=(-1)^3(2)^2\vec{i}+3(-1)(-1)^2(2)^2\vec{j}+2(-1)(-1)^3(2)\vec{k}$

$=-4\vec{i}-12\vec{j}+4\vec{k}$

Unit normal to the surface is $\hat{n}=\dfrac{\nabla\phi}{|\nabla\phi|}$

$$=\frac{-4\vec{i}-12\vec{j}+4\vec{k}}{\sqrt{16+144+16}}$$

$$=-\frac{-4\left(\vec{i}+3\vec{j}-\vec{k}\right)}{\sqrt{176}}$$

$$=\frac{-4\left(\vec{i}+3\vec{j}-\vec{k}\right)}{\sqrt{16\times 11}}=\frac{-\left(\vec{i}+3\vec{j}-\vec{k}\right)}{\sqrt{11}}.$$

Problem 7 Applying Green's theorem in plane show that area enclosed by a simple closed curve C is $\frac{1}{2}\int(xdy-ydx)$

Solution:

$$\int_C Pdx+Qdy=\int\int_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

$P=-y,\ Q=x$

$\dfrac{\partial P}{\partial y}=-1,\ \dfrac{\partial Q}{\partial x}=1$

$$\therefore\int(x\,dy-ydx)=\int\int_R(1+1)\,dxdy=2\int\int_R dx\,dy$$

= 2 Area enclosed by C

$\therefore$ Area enclosed by C $=\frac{1}{2}\int(xdy-ydx)$.

Problem 8 If $\vec{A}$ and $\vec{B}$ are irrotational show that $\vec{A}\times\vec{B}$ is solenoidal

Solution:

Given $\vec{A}$ is irrotational i.e., $\nabla\times\vec{A}=\vec{0}$

$\vec{B}$ is irrotational i.e., $\nabla\times\vec{B}=\vec{0}$

$$\nabla\left(\vec{A}\times\vec{B}\right)=\vec{B}.\left(\nabla\times\vec{A}\right)-\vec{A}.\left(\nabla\times\vec{B}\right)$$
$$=\vec{B}\cdot\vec{0}-\vec{A}\cdot\vec{0}=\vec{0}$$

$\therefore\ \vec{A}\times\vec{B}$ is solenoidal.

Problem 9 If $\vec{F}=grad\left(x^3+y^3+z^3-3xyz\right)$ find curl $\vec{F}$

Solution:

$$\vec{F}=\nabla\left(x^3+y^3+z^3-3xyz\right)$$
$$=\left(3x^2-3yz\right)\vec{i}+\left(3y^2-3xz\right)\vec{j}+\left(3z^2-3xy\right)\vec{k}$$
$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy\end{vmatrix}$$
$$=\vec{i}\left[\frac{\partial}{\partial y}\left(3z^2-3xy\right)-\frac{\partial}{\partial z}\left(3y^2-3xz\right)\right]-\vec{j}\left[\frac{\partial}{\partial x}\left(3z^2-3xy\right)-\frac{\partial}{\partial z}\left(3x^2-3yz\right)\right]$$
$$+\vec{k}\left[\frac{\partial}{\partial x}\left(3y^2-3xz\right)-\frac{\partial}{\partial y}\left(3x^2-3yz\right)\right]$$
$$=\vec{i}\left[-3x+3x\right]-\vec{j}\left[-3y+3y\right]+\vec{k}\left[-3z+3z\right]$$
$$=\vec{i}\,0+\vec{j}0+\vec{k}\,0=0.$$

Problem 10 If $\vec{F}=x^2\vec{i}+y^2\vec{j}$, evaluate $\int\vec{F}\cdot d\vec{r}$ along the straight line $y=x$ from $(0,0)$ to $(1,1)$.

Solution:

$$\vec{F}.d\vec{r}=\left(x^2\vec{i}+y^2\vec{j}\right)\cdot\left(dx\vec{i}+dy\right)$$
$$=x^2dx+y^2dy$$

Given $y=x$

$dy=dx$

$$\therefore\int_C\vec{F}\cdot d\vec{r}=\int_0^1\left(x^2dx+y^2dy\right)$$
$$=\int_0^1 x^2dx+x^2dx=2\int_0^1 x^2dx=2\left[\frac{x^3}{3}\right]_0^1=\frac{2}{3}$$

Problem 11 What is the unit normal to the surface $\phi\left(x,y,z\right)=C$ at the point $\left(x,y,z\right)$?

Solution:

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}.$$

Problem 12 State the condition for a vector $\vec{F}$ to be solenoidal

Solution:

$$\nabla.\vec{F} = div\vec{F} = 0$$

Problem 13 If $\vec{a}$ is a constant vector what is $\nabla\times\vec{a}$?

Solution:

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$$\nabla\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \vec{0}$$

Problem 14 Find grad ϕ at $(2,2,2)$ when $\phi = x^2 + y^2 + z^2 + 2$

Solution:

$$grad\phi = \nabla\phi$$

$$= \vec{i}\frac{\partial}{\partial x}\left(x^2 + y^2x^2 + 2\right) + \vec{j}\frac{\partial}{\partial y}\left(x^2 + y^2 + z^2 + 2\right) + \vec{k}\frac{\partial}{\partial z}\left(x^2 + y^2 + z^2 + 2\right)$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\phi_{(2,2,2)} = 4\vec{i} + 4\vec{j} + 4\vec{k}$$

Problem 15 State Gauss Divergence Theorem

Solution:

The surface integral of the normal component of a vector function F over a closed surface S enclosing volume V is equal to the volume integral of the divergence of $\vec{F}$ taken over V. i.e., $\iint_S \vec{F}.\vec{n}ds = \iiint_V \nabla.\vec{F}dV$

Part –B

Problem 1 Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

$$\phi = x^2yz + 4xz^2$$

$$\nabla\phi = \left(2xyz + 4z^2\right)\vec{i} + x^2 z\vec{j} + \left(x^2 y + 8xz\right)\vec{k}$$
$$\nabla\phi_{(1,-2,-1)} = \left[2(1)(-2)(-1) + 4(-1)^2\right]\vec{i} + (1)^2(-1)\vec{j} + \left[(1)^2(-2) + 8(1)(-1)\right]\vec{k}$$
$$= (4+4)\vec{i} - \vec{j} + (-2-8)\vec{k}$$
$$= 8\vec{i} - \vec{j} - 10\vec{k}$$

Directional derivative $\vec{a}$ is $= \dfrac{\nabla\phi.\vec{a}}{|\nabla\phi|}$

$$= \frac{\left(8\vec{i} - \vec{j} - 10\vec{k}\right).\left(2\vec{i} - \vec{j} - 2\vec{k}\right)}{\sqrt{4+1+4}}$$
$$= \frac{16+1+20}{3} = \frac{37}{3}.$$

Problem 2 Find the maximum directional derivative of $\phi = xyz^2$ at $(1,0,3)$.

Solution:

Given $\phi = xyz^2$

$$\nabla\phi = yz^2\vec{i} + xz^2\vec{j} + 2xyz\vec{k}$$
$$\nabla\phi_{(1,0,3)} = 0(3)^2\vec{i} + (1)(3)^2\vec{j} + 2(1)(0)(3)\vec{k} = 9\vec{j}$$

Maximum directional directive of ϕ is $\nabla\phi = 9\vec{j}$

Magnitude of maximum directional directive is $|\nabla\phi| = \sqrt{9^2} = 9$.

Problem 3 Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point $(2,-1,2)$.

Solution:

Let $\phi_1 = x^2 + y^2 + z^2 - 9$

$$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$
$$\nabla\phi_{1(2,-1,2)} = 2(2)\vec{i} + 2(-1)\vec{j} + 2(2)\vec{k} = 4\vec{i} - 2\vec{j} + 4\vec{k}$$
$$\phi_2 = x^2 + y^2 - z - 3$$
$$\nabla\phi_2 = 2x\vec{i} + 2y\vec{j} - \vec{k}$$
$$\nabla\phi_{2(2,-1,2)} = 4\vec{i} - 2\vec{j} - 2\vec{k}$$

If θ is the angle between the surfaces then

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$
$$= \frac{\left(4\vec{i} - 2\vec{j} + 4\vec{k}\right).\left(4\vec{i} - 2\vec{j} - 2\vec{k}\right)}{\sqrt{16+4+16}\sqrt{16+4+4}}$$

$$= \frac{16+4-8}{\sqrt{36}\sqrt{24}}$$

$$= \frac{12}{6\times 2\sqrt{6}} = \frac{1}{\sqrt{6}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right).$$

Problem 4 Find the work done, when a force $\vec{F} = \left(x^2 - y^2 + x\right)\vec{i} - \left(2xy + y\right)\vec{j}$ moves a particle from the origin to the point $(1,1)$ along $y^2 = x$.

Solution:

Given $\vec{F} = \left(x^2 - y^2 + x\right)\vec{i} - \left(2xy + y\right)\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F}.d\vec{r} = \left(x^2 - y^2 + x\right)dx - \left(2xy + y\right)dy$

Given $y^2 = x$

$2ydy = dx$

$$\therefore \vec{F}.d\vec{r} = \left(x^2 - x + x\right)dx - \left(2y^3 + y\right)dy$$

$$= x^2dx - \left(2y^3 + y\right)dy$$

$$\int_C \vec{F}\vec{dr} = \int_0^1 x^2dx - \int_0^1 \left(2y^3 + y\right)dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{2y^4}{4} + \frac{y^2}{2}\right]_0^1$$

$$= \left(\frac{1}{3} - 0\right) - \left[\left(\frac{2}{4} + \frac{1}{2}\right) - \left(0+0\right)\right]$$

$$= \frac{1}{3} - \left[\frac{1}{2} + \frac{1}{2}\right]$$

$$= \frac{1}{3} - 1 = \frac{-2}{3}$$

$$\therefore \text{Work done} = \int_C \vec{F}.\vec{dr} = \frac{2}{3}$$

Problem 5 Prove that $\vec{F} = \left(y^2 \cos x + z^3\right)\vec{i} + \left(2y\sin x - 4\right)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential.

Solution:

Given $\vec{F} = \left(y^2 \cos x + z^3\right)\vec{i} + \left(2y\sin x - 4\right)\vec{j} + 3xz^2\vec{k}$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y^2\cos x+z^3 & 2y\sin x-4 & 3xz^2\end{vmatrix}$$

$$=\vec{i}[0-0]-\vec{j}\left[3z^2-3z^2\right]+\vec{k}[2y\cos x-2y\cos]$$

$$=0\vec{i}-0\vec{j}+0\vec{k}=0$$

$\nabla\times\vec{F}=0$

Hence $\vec{F}$ is irrotational

$\vec{F}=\nabla\phi$

$$\left(y^2\cos x+z^3\right)\vec{i}\left(2y\sin x-4\right)\vec{j}+3xz^2\vec{k}=\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{yz}$$

Equating the coefficient $\vec{i},\vec{j},\vec{k}$

$$\frac{\partial\phi}{\partial x}=y^2\cos x+z^3\Rightarrow\int\partial\phi=\int y^2\cos x+z^3dx$$

$$\phi_1=y^2\sin x+z^3x+C_1$$

$$\frac{\partial\phi}{\partial x}=2y\sin x-4\Rightarrow\int\partial\phi=\int\left(2y\sin x-4\right)dy$$

$$\phi_2=2\left(\sin x\right)\frac{y^2}{2}-4y+C_2$$

$$\frac{\partial\phi}{\partial x}=3xz^2\Rightarrow\int\partial\phi=\int 3xz^2dy$$

$$\phi_3=3x\frac{z^3}{3}+C_3$$

$$\therefore\phi=y^2\sin x+xz^3-4y+C$$

Problem 6 If $\vec{F}=3xy\vec{i}-y^2\vec{j}$ evaluate $\int\vec{F}.\vec{dr}$ when C is curve in the xy plane $y=2x^2$, from $(0,0)$ to $(1,2)$

Solution:

$\vec{F}=3xy\vec{i}-y^2\vec{j}$

$\vec{dr}=dx\vec{i}+dy\vec{j}+dz\vec{k}$

$\vec{F}.\vec{dr}=3xydx-y^2dy$

Given $y=2x^2$

$dy=4xdx$

$$\therefore\vec{F}.d\vec{r}=3x(2x^2)dx-\left(2x^2\right)^2 4x\,dx$$

$$=6x^3dx-4x^4(4x)dx$$

$$=6x^3dx-16x^5dx$$

$$\int_C \vec{F} d\vec{r} = \int_0^1 \left(6x^3 - 16x^5\right) dx$$

$$= \left[6\frac{x^4}{4} - \frac{16x^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = -\frac{-7}{6}.$$

Problem 7 Find $\int_C \vec{F}.d\vec{r}$ when $\vec{F} = \left(x^2 + y^2\right)\vec{i} - 2xy\vec{j}$ where the cure C is the rectangle in the xy plane bounded by $x = 0,\ x = a,\ y = b,\ y = 0$.

Solution:

Given $\vec{F}\left(x^2 + y^2\right)\vec{i} - 2xy\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F} d\vec{r} = (x^2 + y^2)dx - 2xy\, dy$

C is the rectangle $OABC$ and C consists of four different paths.

OA (y = 0)

AB (x = a)

BC (y = b)

CO (x = 0)

$$\therefore \int_C \vec{F}.d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along

$OA,$	$y = 0,$	$dy = 0$
$AB,$	$x = a,$	$dx = 0$
$BC,$	$y = b,$	$dy = 0$
$CO,$	$x = 0,$	$dx = 0$

$$\therefore C\int_C \vec{F}.d\vec{r} = \int_{OA} x^2 dx \int_{AB} -2ay dy + \int_{BC} \left(x^2 + b^2\right) dx + \int_{CO} 0$$

$$= \int_0^a x^2 dx - 2a\int_0^b y dy + \int_a^0 \left(x^2 + b^2\right) dx$$

$$= \left[\frac{x^3}{3}\right]_0^a - 2a\left[\frac{y^2}{2}\right]_o^b + \left[\frac{x^3}{3} + b^2 x\right]_a^o$$

$$= \left(\frac{a^3}{3} - 0\right) - 2a\left(\frac{b^2}{2} - 0\right) + \left(\left(0 + 0\right) - \left(\frac{a^3}{3} + ab^2\right)\right) = -2ab^2.$$

Problem 8 If $\vec{F} = \left(4xy - 3x^2z^2\right)\vec{i} + 2x^2\vec{j} - 2x^3zk$ check whether the integral $\int_C \vec{F}.d\vec{r}$ is independent of the path C.

Solution:

Given

$\vec{F} = \left(4xy - 3x^2z^2\right)\vec{i} + 2x^2\vec{j} - 2x^3zk$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\int_C \vec{F}.d\vec{r} = \int_C \left(4xy - 3x^2z^2\right)dx + \int_C 2x^2dy - \int_C 2x^3zdz$

This integral is independent of path of integration if

$\vec{F} = \nabla\phi \Rightarrow \nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \vec{i}\left(0,0\right) - j\left(-6x^2z + 6x^2z\right) + \vec{k}\left(4x - 4x\right)$$

$$= 0\vec{i} - 0\vec{i} - 0\vec{j} + 0\vec{k} = 0.$$

Hence the line integral is independent of path.

Problem 9 Verify Green's Theorem in a plane for $\int_C \left(x^2(1+y)dx + (y^3 + x^3)dy\right)$ where C is the square bounded $x = \pm a, y = \pm a$

Solution:

Let $P = x^2(1+y)$

$\frac{\partial P}{\partial y} = x^2$

$Q = y^3 + x^3$

$\frac{\partial Q}{\partial x} = 3x^2$

By green's theorem in a plane

$$\int_C \left(Pdx + Qdy\right) = \iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

Now $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

$$= \int_{-a}^{a}\int_{-a}^{a} \left(3x^2 - x^2\right)dx\,dy$$

$$= \int_{-a}^{a} dy \int_{-a}^{a} 2x^2dx$$

$$= \left(y\right)_{-a}^{a}\left(\frac{2x^3}{3}\right)_{-a}^{a}$$

$$= \left(a + a\right)\frac{2}{3}\left(a^3 + a^3\right)$$

$$=\frac{8a^4}{3}-(1)$$

Now $\int_C (Pdx+Qdy)=\int_{AB}+\int_{BC}+\int_{CD}+\int_{DA}$

Along AB, $y=-a$, $dy=0$

X varies from $-a$ to a

$$\int_{AB}(Pdx+Qdy)=\int_{-a}^{a}\left(x^2(1+y)dx+\left(x^3+y^3\right)dy\right)$$

$$=\int_{-a}^{a}x^2(1-a)dx+0$$

$$=(1-a)\left[\frac{x^3}{3}\right]_{-a}^{a}$$

$$=\left(\frac{1-a}{3}\right)\left(a^3+a^3\right)=\frac{2a^3}{3}-\frac{2a^4}{3}$$

Along BC

$x=a$, $dx=0$

Y varies from $=-a$ to a

$$\int_{BC}(Pdx+Qdy)=\int_{-a}^{a}\left(x^2(1+y)dx+\left(x^3+y^3\right)dy\right)$$

$$=\int_{-a}^{a}(a^3+y^3)dy$$

$$=\left[a^3y+\frac{y^4}{4}\right]_{-a}^{a}$$

$$=\left(a^4+\frac{a^4}{4}\right)-\left(-a^4+\frac{a^4}{4}\right)=2a^4$$

Along CD

$y=a$, $dy=0$

X varies from a to $-a$

$$\int_{CD}(Pdx+Qdy)=\int_{a}^{-a}\left(x^2(1+y)dx+\left(x^3+y^3\right)dy\right)$$

$$=\int_{a}^{-a}x^2(1+a)dx$$

$$=(1+a)\left(\frac{x^3}{3}\right)_{a}^{-a}dx$$

$$=(1+a)\left[\frac{-a^3-a^3}{3}\right]$$

$$= -\frac{2a^3}{3} - \frac{2a^4}{3}$$

Along DA,

$x = -a,\ dx = 0$

Y Varies from a to $-a$

$$\int_{DA} (Pdx + Qdy) = \int_a^{-a} \left(x^2 (1+y) dx + \left(x^3 + y^3 \right) dy \right)$$

$$= \int_{+a}^{-a} \left(a^2 (1+y) dx + \left(y^3 - a^3 \right) dy \right)$$

$$= \left[\frac{y^4}{4} - a^3 y \right]_a^a$$

$$= \left(\frac{a^4}{4} + a^4 \right) - \left(\frac{a^4}{4} - a^4 \right) = 2a^4$$

$$\int_C (Pdx + Qdy) = \frac{2a^3}{3} - \frac{2a^4}{3} + 2a^4 - \frac{2a^3}{3} - \frac{2a^4}{3} + 2a^4$$

$$= 4a^4 - \frac{4}{3} a^4$$

$$= \frac{8a^4}{3} \ldots\ldots(2)$$

From (1) and (2)

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \frac{8a^4}{3}.$$

Hence Green’s theorem verified.

Problem 10

Verify Green’s theorem in a plane for $\int_C \left(3x^2 - 8y^2\right) dx + \left(4y - 6xy\right) dy$ where C is the boundary of the region defined by $x = y^2,\ y = x^2.$

Solution:

Green’s theorem states that

$$\int_C udx + vdy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial p}{\partial y} \right) dxdy$$

Given $\int_C \left(3x^2 - 8y^2\right) dx + \left(4y - 6xy\right) dy$

$P = 3x^2 - 8y^2$

$$\frac{\partial P}{\partial y} = -16y$$

$$Q = 4y - 6xy$$

$$\frac{\partial Q}{\partial x} = -6y$$

Evaluation of $\int_C Pdx + Qdy$

(i) Along OA

$$y = x^2 \Rightarrow dy = 2xdx$$

$$\int_{OA} Pdx + Qdy = \int_{OA} \left(3x^2 - 8x^4\right) dx + \left(4x^2 - 6x^3\right) 2xdx$$

$$= \int_0^1 \left(3x^2 - 8x^4 + 8x^3 - 12x^4\right) dx$$

$$= \int_0^1 \left(-20x^4 + 8x^3 + 3x^2\right) dx$$

$$= \left[-20\frac{x^5}{5} + 8\frac{x^4}{4} + \frac{3x^3}{3}\right]_0^1$$

$$= \frac{-20}{5} + \frac{8}{5} + \frac{3}{3}$$

$$= -4 + 2 + 1 = -1$$

Along AO

$$y^2 = x \Rightarrow 2ydy = dx$$

$$\int_{Ao} Pdx + Qdy = \int_{Ao} \left(3y^4 - 8y^2\right) 2y\, dy + \left(4y - 6y^3\right) dy$$

$$= \int_{AO} (6y^5 - 16y^3 + 4y - 6y^3) dy$$

$$= \int_1^0 \left(6y^5 - 22y^3 + 4y\right) dy$$

$$= \left[6\frac{y^6}{6} - 22\frac{y^4}{4} + \frac{4y^2}{2}\right]_1^0$$

$$= \left[y^6 - \frac{11}{2}y^4 + 2y^2\right]_1^0 = \frac{5}{2}$$

$$\therefore \int_C Pdx + Qdy = \int_{OA} + \int_{AO} = -1 + \frac{5}{2} = \frac{3}{2} \to (1)$$

Evaluation of $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy = \iint_R \left(-6y + 16y\right) dxdy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} 10y\,dx\,dy = \int_0^1 \left[10xy\right]_{x=y}^{x=\sqrt{y}} dy$$

$$= \int_0^1 10y\left(\sqrt{y} - y^2\right)dy$$

$$= 10\int_0^1 \left(y^{\frac{3}{2}} - y^3 \right) dy$$

$$= 10\left[\frac{y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^4}{4} \right]_0^1$$

$$= 10\left[\frac{2}{5} - \frac{1}{4} \right]$$

$$= 10\left[\frac{8-5}{20} \right]$$

$$= \frac{30}{20} = \frac{3}{2} \rightarrow (2)$$

For (1) and (2)

Hence Green's theorem is verified.

Problem 11 Verify Gauss divergence theorem for $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9,\ Z = 0$ and $Z = 2$.

Solution:

Gauss divergence theorem is

$$\iint_S \vec{F}.\vec{n}\ \ ds = \iiint_V div\vec{F}dV$$

$$div\,\vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2) = 2z$$

$$\iiint_V div\,\vec{F}\,dV = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^2 2z\,dzdy\,dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 2\left(\frac{z^2}{2}\right)_0^2 dydx$$

$$= 4\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dydx$$

= 4 (Area of the circular region)

$$= 4\left(\pi\,(3)^2\right)$$

$$= 36\pi ....(1)$$

$$\iint_S \vec{F}.\vec{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

S_1 is the bottom of the circular region, S_2 is the top of the circular region and S_3 is the cylindrical region

On $S_1, \vec{n} = -\vec{k}, \quad ds = dxdy, \quad z = 0$

$$\iint_{S_1} \vec{F}.\vec{n}\, ds = \iint -z^2 dxdy = 0$$

On S_2, $\vec{n} = \vec{k}$, $ds = dxdy$, $z = 2$

$$\iint_{s_2} \vec{F}.\vec{n}\, ds = \iint z^2 dx\, dy$$

$$= 4\iint dxdy$$

$$= 4\,(\text{Area of circular region})$$

$$= 4\left(\pi(3)^2\right) = 36\pi$$

On S_3, $\phi = x^2 + y^2 - 9$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4\left(x^2+y^2\right)}}$$

$$= \frac{x\vec{i} + y\vec{j}}{3}$$

$$\iint_{S_3} \vec{F}.\vec{n}\, ds = \iint \left(y\vec{i} + x\vec{j} + z^2\vec{k}\right)\left(\frac{x\vec{i} + y\vec{j}}{3}\right) ds$$

$$= \iint \frac{yx + yx}{3} ds = \frac{2}{3}\iint_S xy\, ds$$

Let $x = 3\cos\theta, \quad y = 3\sin\theta$

$ds = 3\ d\theta dy$

θ varies from 0 to 2π

z varies from 0 to 2π

$$= \frac{2}{3}\int_0^2\int_0^{2\pi} (9\sin\theta\cos\theta) 3\, d\theta\, dz$$

$$= \frac{18}{2}\int_0^2\int_0^{2\pi} \sin 2\theta\, d\theta\, dz$$

$$= 9\int_0^2 \left(-\frac{\cos 2\theta}{2}\right)_0^{2\pi} dz$$

$$= -\frac{9}{2}\int_0^2 [1-1]\ dz = 0$$

$$\therefore \iint_S \vec{F}.\vec{n}\, ds = 0 + 36\pi + 0 = 36\pi \text{..............(2)}$$

from (1) and (2)

$$\int_C \vec{F}.\vec{n}\ ds = \iiint_V div\ \vec{F} dV$$

Problem 12 Verify Stoke's theorem for the vector field defined by $\vec{F} = \left(x^2 - y^2\right)\vec{i} - 2xy\vec{j}$ in the rectangular region in the xy plane bounded by the lines $x = 0,\ x = a,\ y = 0,\ y = b.$

Solution:

$$\vec{F} = \left(x^2 - y^2\right)\vec{i} - 2xy\vec{j}$$

By Stoke's theorem $\int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\vec{n}\,ds$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-2y-2y] = -4y\vec{k}$$

As the region is in the xy plane we can take $\vec{n} = \vec{k}$ and $ds = dxdy$

$$\iint_S curl\,\vec{F}.\vec{n}\,ds = \iint -4y\vec{k}.\vec{k}\,dx\,dy$$

$$= -4\int_0^b\int_0^a y\ dx\,dy$$

$$= -4\left(\frac{y^2}{2}\right)_0^b (x)_0^a$$

$$= -2ab^2 ....(1)$$

$$\int_C \vec{F}.dr = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

<u>Along OA</u>

$y = 0 \Rightarrow dy = 0,$

x varies from 0 to a

$$\therefore \int_{OA} = \int_0^a \left(x^2 + y^2\right)dx - 2xy\ dy$$

$$= \int_0^a x^2 dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

<u>Along AB</u>

$x = a \Rightarrow dx = o,\ y$ varies from 0 to b

$$\int_{AB} = \int_0^b \left(a^2+y^2\right).0 - 2ay\ dy$$

$$= -2a\left(\frac{y^2}{2}\right)_0^b = -ab^2$$

<u>Along BC</u>

$y = b,\ dy = 0$

x varies from a to 0

$$\int_{BC} = \int_a^0 \left(x^2+b^2\right)dx - 0$$

$$= \left(\frac{x^3}{3}+b^2x\right)_a^0$$

$$= -\frac{a^3}{3} - ab^2$$

<u>Along CO</u>

$x = 0,\ dx = 0,$

y varies from b to 0

$$\int_{CO} = \int_b^0 \left(0+y^2\right)0 + 0 = 0$$

$$\therefore \int_c \vec{F}.d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0$$

$$= -2ab^2 ....(2)$$

For (1) and (2)

$$\int_C \vec{F}.d\vec{r} = \iint_S curl\vec{F}.\vec{n}ds$$

Here Stoke's theorem is verified.

Problem 13 Find $\iint_S \vec{F}.\vec{n}ds$ if $\vec{F} = \left(x+y^2\right)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ where S is the surface in the plane $2x+y+2z=6$ in the first octant.

Solution:

Let $\phi = 2x+y+2z-6$ be the given surface

Then $\nabla\phi = 2\vec{i}+\vec{j}+2\vec{k}$

$$|\nabla\phi| = \frac{2\vec{i}+\vec{j}+2\vec{k}}{\sqrt{4+1+4}} = \frac{2\vec{i}+\vec{j}+2\vec{k}}{3}$$

$\therefore$ The unit outward normal $\vec{n}$ to the surface S is $\hat{n} = \frac{1}{3}\left[2\vec{i}+\vec{j}+2\vec{k}\right]$

Let R be the projection of S on the xy plane

$$\therefore \iint_S \vec{F}.\vec{n}\,ds = \iint_{R_1} \vec{F}.\vec{n}\frac{dx\,dy}{\left|\vec{n}.\vec{k}\right|}$$

$$\vec{n}.\vec{k} = \frac{1}{3}\left(2\vec{i}+\vec{j}+2\vec{k}\right).\vec{k} = \frac{2}{3}$$

$$\vec{F}.\vec{n} = \left[\left(x+y^2\right)\vec{i} - 2x\vec{j} + 2yz\vec{k}\right].\frac{1}{3}\left(2\vec{i}+\vec{j}+2\vec{k}\right)$$

$$= \frac{2}{3}\left(x+y^2\right) - \frac{2}{3}x + \frac{4}{3}yz$$

$$= \frac{2}{3}\left(y^2+2yz\right)$$

$$= \frac{2}{3}y\left(y+2z\right)$$

$$= \frac{2}{3}y\left[y+6-y-2x\right]$$

$$= \frac{2}{3}y\left[6-2x\right]$$

$$= \frac{4}{3}y\left(3-x\right)$$

$$\therefore \iint_S \vec{F}.\vec{n}\,ds = \iint_S \vec{F}.\vec{n}\frac{dx\,dy}{\left|\vec{n}.\vec{k}\right|}$$

$$= \iint \frac{4}{3}y\left(3-x\right)\frac{dxdy}{2/3}$$

$$= 2\iint_{R_1}\left(3-x\right)dx\,dy$$

$$= 2\int_0^3 \int_0^{6-2-x} \left(3-x\right)\,dx\,dy$$

$$= 2\int_0^3 \left(3-x\right)\left(\frac{y^2}{2}\right)_0^{6-2-x} dx$$

$$= 4\int_0^3 \left(3-x\right)^3 dx$$

$$= 4\left[\frac{(3-x)^4}{-4}\right]_0^3$$

$= 81$ units.

Problem 14 Evaluate $\int_C \left[\left(x+y\right)dx + \left(2x-3xy\right)\right]$ where C is the boundary of the triangle with vertices $\left(2,0,0\right),\left(0,3,0\right)\,\&\,\left(0,0,6\right)$ using Stoke's theorem.

Solution:

Stoke's theorem is $\int_C \vec{F}.\vec{dr} = \iint_S curl\vec{F}.\hat{n}ds$ where S is the surface of the triangle and $\hat{n}$ is the unit vector normal to surface S.

Given $\vec{F}.\vec{dr} = (x+y)dx + (2x-z)dy + (y+z)dz$

$\therefore \vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}$

$\vec{dr} = \vec{i}\,dx + \vec{j}dy + \vec{k}\,dz$

$$curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \vec{i}(1+1) - \vec{j}(0-0) + \vec{k}(2-1)$$

$curl\,\vec{F} = 2\vec{i} + \vec{k}$

Equation of the plane ABC is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

$3x + 2y + z = 6$

Let $\phi = 3x + 2y + z - 6$

$\nabla\phi = 3\vec{i} + 2\vec{j} + \vec{k}$

Unit normal vector to the surface ABC $(or\,\phi)$ is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\vec{i} + 2\overleftarrow{j} + \vec{k}}{\sqrt{14}}$$

$$curl\,\vec{F}.\hat{n} = \left(2\vec{i} + \vec{k}\right).\left(\frac{3\vec{i} + 2\overleftarrow{j} + \vec{k}}{\sqrt{14}}\right) = \frac{6+1}{\sqrt{14}} = \frac{7}{\sqrt{14}}$$

Hence $\iint_S curl\vec{F}.\hat{n}ds = \iint_S \frac{7}{\sqrt{14}} ds$

$= \frac{7}{\sqrt{14}} \iint_R \frac{dxdy}{|\hat{n}\vec{k}|}$ where R is the projection of surface ABC on XOY plane

$$= \frac{7}{\sqrt{14}} \iint_R \frac{dxdy}{\frac{1}{\sqrt{14}}} \qquad \left(\because \vec{n}\,\vec{k} = \left(\frac{3i + 2j + k}{\sqrt{14}}\right).k = \frac{1}{\sqrt{14}}\right)$$

$$= 7\iint_R dxdy$$

$$= 7 \times \left(Area\ of\ \Delta^{le}\ OAB\right)$$

$$= 7 \times 3 = 21.$$

Problem 15 Verify Stoke's theorem for $\vec{F} = (y-z)\vec{i} + yz\vec{j} - xz\vec{k}$ where S is the surface bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$ above the XOY plane.

Solution:

Stoke's theorem is $\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.\hat{n}ds$

$$\vec{F} = (y-z)\vec{i} + yz\vec{j} - xz\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & yz & -xz \end{vmatrix}$$

$$= -y\vec{i} + (z-1)\vec{j} - \vec{k}$$

$$\iint_S \nabla \times \vec{F}.\hat{n}\ ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$

$\iint_{S_6}$ is not applicable, since the given condition is above the XOY plane.

$$\iint_{S_1} = \iint_{AEGD} \left[-y\vec{i} + (z-1)\vec{j} - \vec{k}\right].\vec{i}dydz$$

$$= \iint_{AEGD} -ydy\,dz$$

$$= \int_0^1\int_0^1 -y\,dy\,dz = \int_0^1 \left[-\frac{y^2}{2}\right]_0^1 dz$$

$$= -\frac{1}{2}(z)_0^1 = -\frac{1}{2}$$

$$\iint_{S_2} = \iint_{OBFC} \left[-y\vec{i} + (z-1)\vec{j} - \vec{k}\right].(\overline{-i})\ dydz$$

$$= \int_0^1\int_0^1 y\,dy\,dz = \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dz = \frac{1}{2}$$

$$\iint_{S_3} = \iint_{EBFG} \left[-y\vec{i} + (z-1)\vec{j} - \vec{k}\right]\vec{j}dxdz$$

$$= \int_0^1\int_0^1 (z-1)\ dx\,dz = \int_0^1 (xz-x)_0^1\, dz$$

$$= \left(\frac{z^2}{2} - z\right)_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$\iint_{S_4} = \iint_{OADC} \left[-y\vec{i} + (z-1)\vec{j} - \vec{k}\right]\left(-\vec{j}\ \right)dxdz$$

$$= \int_0^1 \int_0^1 (-z+1)\, dx\, dz$$

$$= \int_0^1 (-xz+x)_0^1 = \int_0^1 (-z+1)\, dz$$

$$= \left(\frac{-z^2}{2}+z\right)_0^1 = \frac{-1}{2}+1 = \frac{1}{2}$$

$$\iint_{S_5} = \iint_{DGFC} \left(-y\vec{i}+(z-1)\vec{j}-\vec{k}\right).\vec{k}dxdy$$

$$= \int_0^1 \int_0^1 (-1) dxdy = \int_0^1 (-x)_0^1\, dy$$

$$= \int_0^1 (-1)\, dy = (-y)_0^1 = -1$$

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5}$$

$$= -\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-1 = -1$$

$$L.H.S = \int_C \vec{F}.\vec{dr} = \int_{OA} + \int_{AE} + \int_{EB} + \int_{BO}$$

$$\int_{OA} = \int_{OA} (y-z)dx + yzdy - xzdz$$

$$= \int_{OA} 0 = 0 \qquad [\because y=0,\ z=0,\ dy=0,\ dz=0]$$

$$\int_{AE} = \int_{AE} (y-z)dx + yzdy - xzdz$$

$$= \int_{AE} 0 = 0 \qquad [\because x=1,\ z=0,\ dx=0,\ dz=0]$$

$$\int_{EB} = \int_{EB} (y-z)dx + yzdy - xzdz$$

$$= \int_1^0 1\, dx \qquad (y=1,\ z=0,)$$

$$= [x]_1^0 = 0-1 = -1$$

$$\int_{BO} = \int_{BO} (y-z)dx + yzdy - xzdz$$

$$= \int_{BO} 0 = 0 \qquad [x=0,\ z=0]$$

$$= 0$$

$$\therefore \int_C = \int_{OA} + \int_{AE} + \int_{EB} + \int_{BO}$$

$$= 0+0-1+0 = -1$$

∴ L.HS = R.HS.
Hence Stoke's theorem is verified.