UNIT –II

BETA AND GAMMA FUNCTIONS

Definite Integrals is defined as

$$\int_a^b f(x)\mathrm{dx} = [F(x)]_a^b = F(b) - F(a)$$

Properties of Definite Integrals

Property:1

$$\int_a^b f(x)\mathrm{dx} = \int_a^b f(y)\mathrm{dy} = \int_a^b f(t)\mathrm{dt}.$$

Property: 2

$$\int_a^b f(x)\mathrm{dx} = -\int_b^a f(x)\mathrm{dx}$$

Property: 3

$$\int_a^b f(x)\mathrm{dx} = \int_a^c f(x)\mathrm{dx} + \int_c^b f(x)\mathrm{dx}$$,if $a \le c \le b$

Property : 4

$$\int_0^a f(x)\mathrm{dx} = \int_0^a f(a-x)\mathrm{dx}$$. a : any real constant.

Proof : $\int_0^a f(a-x)\mathrm{dx}$

Let

$$a - x = t$$
$$dx = -dt$$
$$when x = 0; t = a$$
$$when x = a; t = 0$$

$$= \int_a^0 f(t)\frac{dt}{-1} = -\int_a^0 f(t)\mathrm{dt}$$

$$= \int_0^a f(t)\mathrm{dt}$$ (by prop 1)

$$= \int_0^a f(x)\mathrm{dx}$$

Property : 5

$$\int_0^{2a} f(x)\text{dx} = \int_0^a \left[f(x) + f(2a - x)\right]\text{dx}$$

Property : 6

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Property: 7

(i)If $f(-x) = f(x)$ **(Even Function)** **then** $\int_{-a}^{a} f(x)\,\text{dx} = 2\int_0^a f(x)\,\text{dx}$

(ii)If $f(-x) = -f(x)$ **(Odd Function)** **then** $\int_{-a}^{a} f(x)\,\text{dx} = 0$

Solved Problems

1. $Evaluate \int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,\text{dx}$

Solution Let $I = \int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,\text{dx}$ ………..(1)

Applying property 4 in (1),we have

$$I = \int_0^{\frac{\pi}{2}} \frac{a\sin(\frac{\pi}{2} - x) + b\cos(\frac{\pi}{2} - x)}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)}\,\text{dx}$$

$$I = \int_0^{\frac{\pi}{2}} \frac{a\cos x + b\sin x}{\sin x + \cos x}\,\text{dx}$$ ……………….(2)

Adding (1)and (2),we have

$$2I = \int_0^{\frac{\pi}{2}} \frac{(a+b)\sin x + (a+b)\cos x}{\sin x + \cos x}\,\text{dx}$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)(a+b)}{\sin x + \cos x}\,\mathrm{dx}$$

$$2I = \int_0^{\frac{\pi}{2}} (\mathrm{a}+\mathrm{b})\,\mathrm{dx}$$

$$2I = \frac{\pi}{2}(a+b)$$

$$I = \frac{\pi}{4}(a+b)$$

2. $Show that \int_0^{\frac{\pi}{2}} \log(\sin x)\mathrm{dx} = -\frac{\pi}{2}\log 2$

$$.Let.I = \int_0^{\frac{\pi}{2}} \log(\sin x)\mathrm{dx} \ldots\ldots\ldots\ldots(1)$$

$$I = \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - x\right)\mathrm{dx}$$

$$I = \int_0^{\frac{\pi}{2}} \log\cos(x)\mathrm{dx} \ldots\ldots\ldots\ldots(2)$$

(1)+(2) implies $2I = \int_0^{\frac{\pi}{2}} \log(\sin x)\mathrm{dx} + \int_0^{\frac{\pi}{2}} \log(\cos x)\mathrm{dx}$

$$2I = \int_0^{\frac{\pi}{2}} \log\sin x\cos x\mathrm{dx}$$

$$2I = \int_0^{\frac{\pi}{2}} \log\frac{\sin 2x}{2}\mathrm{dx}$$

$$2I = \int_0^{\frac{\pi}{2}} \log\sin 2x\mathrm{dx} - \int_0^{\frac{\pi}{2}} \log 2\mathrm{dx} \ldots\ldots\ldots\ldots\ldots(.I)$$

$Consider \int_0^{\frac{\pi}{2}} \log\sin 2x\mathrm{dx}$

Let θ=2x then dx =dθ/2

$x = 0$ *then* $\theta = 0$ *and* $x = \frac{\pi}{2}$ *then* $\theta = \pi$

$$= \int_0^{\pi} \log(\sin\theta)\frac{d\theta}{2}$$

$$(\text{since} \int_0^a f(x)\,dx = 2\int_0^{\frac{a}{2}} f(x)\,dx)$$

$$= 2\int_0^{\frac{\pi}{2}} \log(\sin\theta)\frac{d\theta}{2}$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin x)dx = I \quad \text{(by prop 1)}$$

Subsuiting in I,we have

$$2I = I - \int_0^{\frac{\pi}{2}} \log 2dx$$

$$I = -\int_0^{\frac{\pi}{2}} \log 2dx$$

$$I = -\log 2\int_0^{\frac{\pi}{2}} dx$$

$$I = -\frac{\pi}{2}\log 2 \; .$$

3.*Show that* $\int_0^{\frac{\pi}{4}} \log.(1+\tan x)dx = \frac{\pi}{8}\log 2$

$$Let\ I = \int_0^{\frac{\pi}{4}} \log.(1+\tan x)dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left[.1+\tan(\frac{\pi}{4}-x)\right]dx$$

$$= \int_0^{\frac{\pi}{4}} \log.\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]dx$$

$$= \int_0^{\frac{\pi}{4}} \log.\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

$$= \int_0^{\frac{\pi}{4}} \log .\left[\frac{2}{1+\tan x}\right] dx$$

$$I = \int_0^{\frac{\pi}{4}} [(\log 2 - \log(1+\tan x)] dx$$

$$= \int_0^{\frac{\pi}{4}} \log 2 dx - \int_0^{\frac{\pi}{4}} \log .(1+\tan x) dx$$

$$I = \int_0^{\frac{\pi}{4}} \log 2 dx - I$$

$$2I = \log 2 \int_0^{\frac{\pi}{4}} dx$$

$$2I = \frac{\pi}{4} \log 2$$

$$I = \frac{\pi}{8} \log 2$$

4. $Evaluate \int_0^{\pi} \log(1+\cos x) dx$

$$LetI = \int_0^{\pi} \log(1+\cos x) dx \quad \ldots\ldots\ldots\ldots(1)$$

$$I = \int_0^{\pi} \log(1+\cos(\pi - x) dx$$

$$I = \int_0^{\pi} \log(1-\cos x) dx \quad \ldots\ldots\ldots\ldots(2)$$

(1)+(2) implies $2I = \int_0^{\pi} \log(1+\cos x) dx + \int_0^{\pi} \log(1+\cos x) dx$

$$2I = \int_0^{\pi} \log(1-\cos x)(1+\cos x) dx$$

$$2I = \int_0^{\pi} \log(1-\cos^2 x) dx = \int_0^{\pi} \log \sin^2 x\, dx = 2\int_0^{\pi} \log \sin x\, dx$$

$$I = \int_0^{\pi} \log \sin x\, dx$$

By the property $f(2a-x) = f(x) then \int_0^{2a} f(2a-x)\, dx = 2\int_0^{a} f(x)\, dx$

$$= 2\int_0^{\frac{\pi}{2}} \log \sin x\, dx$$

$$=2\left[-\frac{\pi}{2}\log 2\right]$$

$$I=\pi\log\frac{1}{2}$$

2,1. Gamma Function

Definition :

Gamma function is defined as follows

$$\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}dx \quad ; \; n>0$$

1. $\Gamma(n+1)=n\Gamma(n)$

2. $\Gamma(1)=1$

3. $\Gamma(n+1)=n\,!$,n>0

4. $\Gamma(n).\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}.0<n<1$

5. $\Gamma(\frac{1}{2})=\sqrt{\pi}$

Proof :

$$WKT\;\; \Gamma(n)=\int_0^\infty t^{n-1}e^{-t}dx$$

$$\Gamma(\frac{1}{2})=\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt$$

$put\;\; t=x^2\quad dt=2x\;dx$

$No\ change\ in\ \lim its$

$$\Gamma(\frac{1}{2})=\int_0^\infty (x^2)^{-\frac{1}{2}}e^{-x^2}\,2x\;dx$$

$$\Gamma(\frac{1}{2})=2\int_0^\infty (\frac{1}{x})e^{-x^2}\,x\;dx$$

$$\Gamma(\frac{1}{2})=2\int_0^\infty e^{-x^2}\;dx$$

$$Similarly\ ,we\ have\;\; \Gamma(\frac{1}{2})=2\int_0^\infty e^{-y^2}\;dx$$

$$\left[\Gamma(\frac{1}{2})\right]^2 = \left[2\int_0^\infty e^{-x^2}\ dx\right]\left[2\int_0^\infty e^{-y^2}\ dy\right]$$

$$\left[\Gamma(\frac{1}{2})\right]^2 = \left[4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dx\ dy\right]$$

Transfor min *g* int *o polar coordinates*

$$\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{Bmatrix} \Rightarrow x^2 + y^2 = r^2$$

$dxdy = rdrd\theta$

$r : 0 \rightarrow \infty$

$\theta : 0 \rightarrow \dfrac{\pi}{2}$

$$\left[\Gamma(\frac{1}{2})\right]^2 = \left[4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\ dr\ d\theta\right]$$

$$= \left[4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\ d(\frac{r^2}{2})\ d\theta\right]$$

$$= \left[2\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\ d(r^2)\ d\theta\right]$$

$$= \left[2\int_0^{\frac{\pi}{2}} \left[\frac{e^{-r^2}}{-1}\right]_0^\infty d\theta\right]$$

$$= \left[2\int_0^{\frac{\pi}{2}} \left[\frac{e^{-\infty} - e^0}{-1}\right] d\theta\right]$$

$$= \left[2\int_0^{\frac{\pi}{2}} d\theta\right] = 2[\theta]_0^{\frac{\pi}{2}} = 2\left[\frac{\pi}{2}\right] = \pi$$

$$\left[\Gamma(\frac{1}{2})\right]^2 = \pi \Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$$

$6. \Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2} dx$

Pr*oof* :

$Sub\ x = t^2\ \ in\ the\ \ formula\ \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$

$then\ dx = 2\ t\ dt$

$$\Gamma(n) = \int_0^\infty t^{2(n-1)} e^{-t^2} (2t\ d\,t)$$

$$\Gamma(n) = 2\int_0^\infty t^{2n-1} e^{-t^2} dt$$

$$\Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2} dx \qquad (by\ \ property\ 1)$$

Beta Function

Definition

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, m > 0\ \&\ n > 0$$

Results:

$1. \beta(m,n) = \beta(n,m)$

Proof :

By definition of beta function, $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

Using property 4 of definite integral

$$\beta(m,n) = \int_0^1 (1-x)^{m-1} \left[(1-(1-x)\right]^{n-1} dx$$

$$\beta(m,n) = \int_0^1 (1-x)^{m-1} x^{n-1} dx$$

$$\beta(m,n) = \int_0^1 x^{n-1} (1-x)^{m-1} dx$$

$$\beta(m,n) = \beta(n,m)$$

2. $\beta(m,n) = 2\int_0^\infty \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

Proof:

$$WKT\ \ \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$Take\ \ \ x = \sin^2\theta \quad dx = 2\sin\theta\cos\theta\ d\theta$$

$$if\ \ x = 0\ then\ \theta = 0$$

$$if\ \ x = 1\ then\ \theta = \frac{\pi}{2}$$

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(\cos^2\theta\}^{n-1}\ 2\sin\theta\ \cos\theta\ d\theta$$

$$\beta(m,n) = 2\int_0^{\infty} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

3. $\beta(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

Relation between Beta and Gamma function

$$\beta(m,n) = \frac{\Gamma\ m\ \Gamma\ n}{\Gamma\ m+n}$$

Proof :WKT $\Gamma(n) = 2\int_0^{\infty} x^{2n-1}e^{-x^2}\,dx$

$$\Gamma(m)\Gamma(n) = \left[2\int_0^{\infty} x^{2m-1}e^{-x^2}\,dx\right]\left[2\int_0^{\infty} y^{2n-1}e^{-y^2}\,dx\right]$$

$$\Gamma(m)\Gamma(n) = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}x^{2m-1}y^{2n-1}dxdy$$

Transforming into polar coordinates

$$\left.\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right\} \Rightarrow x^2 + y^2 = r^2$$

$$dxdy = rdrd\theta$$

$$r : 0 \to \infty$$

$$\theta : 0 \to \frac{\pi}{2}$$

$$\Gamma(m)\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}r^{2m+2n-2}\cos^{2m-1}\theta\sin^{2n-1}\theta(rdrd\theta)$$

$$\Gamma(m)\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2} r^{2m+2n-1}\cos^{2m-1}\theta\sin^{2n-1}\theta dr d\theta$$

$$\Gamma(m)\Gamma(n) = \left[2\int_0^{\frac{\pi}{2}}\sin^{2n-1}\theta\cos^{2m-1}\theta d\theta)\right]\left[2\int_0^{\infty} r^{2(m+n)-1}e^{-r^2}\,dr\right]$$

$$\Gamma\, m\, \Gamma\, n = \beta(m,n)\Gamma\, m+n$$

$$\beta(m,n) = \frac{\Gamma\, m\, \Gamma\, n}{\Gamma\, m+n}$$

We can also prove Γ1/2 using the beta gamma relation

Put m=n=1/2

$$\beta(m,n) = \frac{\Gamma\, m\, \Gamma\, n}{\Gamma\, m+n}$$

$$\beta(\frac{1}{2},\frac{1}{2}) = \frac{\Gamma\,\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\, 1}$$

$$\left[\Gamma(\frac{1}{2})\right]^2 = \beta(\frac{1}{2},\frac{1}{2}) = \int_0^{\frac{\pi}{2}}\sin^{2(\frac{1}{2})-1}\theta\cos^{2(\frac{1}{2})-1}\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta = 2\,[\theta]_0^{\frac{\pi}{2}} = 2\left[\frac{\pi}{2}\right]$$

$$\left[\Gamma\left[\frac{1}{2}\right]\right]^2 = \pi$$

$$\Rightarrow \Gamma\left[\frac{1}{2}\right] = \sqrt{\pi}$$

Problems :

1.$Evaluate \int_0^1 x^6(1-x)^5\,dx$

$$WKT\,\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, m>0\ \&\ n>0$$

Taking m-1=6 and n-1=5 we get m=7 and n=6

$$\beta(7,6)=\frac{\Gamma 7\Gamma 6}{\Gamma 13}=\frac{6!x5!}{13!}=\frac{(6x5x4x3x2x1)(5x4x3x2x1)}{13x12x11x10x9x8x7x6x5x4x3x2x1}=\frac{1}{72072}$$

2. $Evaluate\beta(\frac{5}{2},\frac{7}{2})=\int_0^1 x^{m-1}(1-x)^{n-1}dx, m>0 \;\&\; n>0$

$$=\frac{\Gamma\frac{5}{2}\Gamma\frac{7}{2}}{\Gamma 6}=\frac{\frac{3}{2}.\frac{1}{2}.\Gamma\frac{1}{2}.\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}{5!}=\frac{3\pi}{256}$$

3. $Evaluate\int_0^{\frac{\pi}{2}}\sin^6\theta d\theta$

$$\int_0^{\frac{\pi}{2}}\sin^6\theta d\theta=\beta(\frac{7}{2},\frac{1}{2})=\frac{\Gamma\frac{7}{2}\Gamma\frac{1}{2}}{\Gamma 4}=\frac{1}{2}\frac{.\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{3!}=\frac{15\pi}{96}$$

4. $Evaluate\int_0^{\frac{\pi}{2}}\sin^6\theta\cos^5\theta d\theta$

$Take\,2m-1=6\,and\,2n-1=5$

$then\,m=\frac{7}{2}\,and\,n=3$

$$=\frac{1}{2}\beta(\frac{7}{2},3)$$

$$=\frac{1}{2}\frac{\Gamma\frac{7}{2}\Gamma 3}{\Gamma\frac{13}{2}}=\frac{1}{2}\frac{\Gamma\frac{7}{2}\Gamma 3}{\frac{11}{2}\frac{9}{2}\frac{7}{2}\Gamma\frac{7}{2}}=\frac{1}{2}\frac{2!}{\left[\frac{693}{8}\right]}=\frac{8}{693}$$

5. $Evaluate\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}d\theta$

$$Given\,I=\int_0^{\frac{\pi}{2}}\sin^{\frac{1}{2}}\theta\cos^{\frac{-1}{2}}\theta d\theta$$

$$=\frac{1}{2}\beta(\frac{3}{4},\frac{1}{4})=\frac{1}{2}\frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{\Gamma 1}$$

$$=\frac{1}{2}\Gamma\frac{3}{4}\Gamma\frac{1}{4}=\frac{1}{2}\frac{\pi}{\sin\frac{\pi}{4}}=\frac{\pi}{\sqrt{2}}\left[\Theta\;\Gamma(n).\Gamma(1-n)=\frac{\pi}{\sin n\pi}.\right]$$

$$6..Evaluate\int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy = \Gamma n, n > 0$$

$$Put\ \log\frac{1}{y} = t$$

$$\frac{1}{y} = e^t$$

$$y = e^{-t}$$

$$dy = -e^{-t}dt$$

$$Limits: y = 0 \Rightarrow t = \infty and y = 1 \Rightarrow t = 0$$

$$\therefore \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy = \int_\infty^0 (t)^{n-1}(-e^{-t})dt(byprop2)$$

$$\int_0^\infty (t)^{n-1}e^{-t}dt = \Gamma n$$

$$7.Evaluate\int_0^\infty e^{-(hx)^2}dx$$

$$put(hx)^2 = t$$

$$hx = t^{\frac{1}{2}}$$

$$dx = \frac{1}{2h}t^{-\frac{1}{2}}dt$$

$$\int_0^\infty e^{-(hx)^2}dx$$

$$when x = 0 \Rightarrow t = 0$$

$$when x = \infty \Rightarrow t = \infty$$

$$then\int_0^\infty e^{-(hx)^2}dx = \int_0^\infty e^{-t}\frac{t^{\frac{-1}{2}}}{2h}dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\frac{t^{\frac{1}{2}-1}}{h}dt$$

$$= \frac{\Gamma\frac{1}{2}}{2h} = \frac{\sqrt{\pi}}{2h}$$

8. Prove that $\int_0^\infty \frac{t^2}{1+t^4}\,dt = \frac{\pi}{2\sqrt{2}}$

$Put\ \ t = \sqrt{\tan\theta}$

$$dt = \frac{1}{2}\tan^{\frac{-1}{2}}\theta(\sec^2\theta)d\theta$$

$$dt = \frac{1+\tan^2\theta}{2\sqrt{\tan\theta}}\,d\theta$$

$$\therefore \int_0^\infty \frac{t^2}{1+t^4}\,dt = \int_0^{\frac{\pi}{2}} \frac{\tan\theta(1+\tan^2\theta)}{(1+\tan^2\theta)2\sqrt{\tan\theta}}d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\sqrt{\tan\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\cos^{\frac{-1}{2}}\theta\,d\theta$$

$$= \frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{4}\right)$$

$$= \frac{1}{4}\frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{\Gamma 1}$$

$$= \frac{1}{4}\Gamma\frac{3}{4}\Gamma\frac{1}{4}$$

$$= \frac{1}{4}\frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\pi}{2\sqrt{2}}$$

9. Evaluate $\int_0^\infty \frac{x^a}{a^x}\,dx \quad where \quad a > 1$

$Let \ \ a^x = e^t$

$t = x \log a \quad , by \ definition \ of \ \log arithm$

$$\therefore x = \frac{t}{\log a} \Rightarrow dx = \frac{dt}{\log a}$$

$when \ x = 0 \Rightarrow t = 0$

$when \ \ x = \infty \Rightarrow t = \infty$

$$\int_0^\infty \frac{x^a}{a^x} dx \ = \int_0^\infty \frac{\left(\frac{t}{\log \ a}\right)^a}{e^t} \frac{dt}{\log \ a}$$

$$= \frac{1}{(\log \ a)^{a+1}} \int_0^\infty e^{-t} t^a dt$$

$$= \frac{1}{(\log \ a)^{a+1}} \Gamma(a+1) .$$

$$10.\ \text{Prove that } \frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n}$$

$$WKT\ \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

$$Then\ \beta(m+1,n) = \frac{\Gamma m+1\ \Gamma n}{\Gamma(m+n+1)} = \frac{m\Gamma m \Gamma n}{\Gamma(m+n+1)}$$

$$\therefore \frac{\beta(m+1,n)}{m} = \frac{\Gamma m \Gamma n}{\Gamma(m+n+1)} \text{..........1}$$

$$similarly\ \beta(m,n+1) = \frac{\Gamma m \Gamma n+1}{\Gamma(m+n+1)} = \frac{\Gamma m\ n\Gamma n}{\Gamma(m+n+1)}$$

$$\frac{\beta(m,n+1)}{n} = \frac{\Gamma m \Gamma n}{\Gamma(m+n+1)} \text{.....................2}$$

$$also\ multiply\ by\ \frac{1}{m+n}\ on\ both\ sides\ in\ \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

$$\frac{\beta(m,n)}{m+n} = \frac{\Gamma m \Gamma n}{(m+n)\Gamma(m+n)}$$

$$\frac{\beta(m,n)}{m+n} = \frac{\Gamma m \Gamma n}{\Gamma(m+n+1)} \text{.................3}$$

from equation 1,2,3 we have

$$\frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n}$$

$$11.\ \text{Prove that } \frac{\beta(n,\frac{1}{2})}{2^{2n-1}} = \beta(n.n)\ \text{and hence deduce the duplication formula}$$

$$WKT\ \ \beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\ \ d\theta$$

$$\beta(n,\frac{1}{2}) = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\ d\theta$$

Also we have $\beta(n,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2n-1}\theta\ \ d\theta$

$$\beta(n,n) = 2\int_0^{\frac{\pi}{2}} (\sin\ \ \theta \cos\theta)^{2n-1}\ d\theta$$

$$\beta(n,n) = 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\theta}{2}\right)^{2n-1} d\theta$$

$$\beta(n,n) = \frac{2}{2^{2n-1}} \int_0^{\frac{\pi}{2}} \sin^{2n-1} 2\theta \; d\theta$$

$$let \;\; \phi = 2\theta \quad \therefore d\theta = \frac{1}{2} d\phi$$

$$\beta(n,n) = \frac{2}{2^{2n-1}} \int_0^{\frac{\pi}{2}} \sin^{2n-1} \phi \; \frac{d\phi}{2}$$

$$\beta(n,n) = \frac{1}{2^{2n-1}} \int_0^{\frac{\pi}{2}} \sin^{2n-1} \phi \; d\phi$$

$$\beta(n,n) = \frac{1}{2^{2n-1}} \beta(n,\frac{1}{2})$$

$$\beta(n\;,\frac{1}{2}) = 2^{2n-1} \beta(n,n)$$

$$\frac{\Gamma n \Gamma \frac{1}{2}}{\Gamma\left(n+\frac{1}{2}\right)} = 2^{2n-1} \beta(n,n)$$

$$\frac{\Gamma n \Gamma \frac{1}{2}}{\Gamma\left(n+\frac{1}{2}\right)} = 2^{2n-1} \frac{\Gamma n \Gamma n}{\Gamma 2n}$$

$$\Gamma 2n = 2^{2n-1} \frac{\Gamma n \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$$