Unit –II

Antenna Array

The study of a single small antenna indicates that the radiation fields are uniformly distributed and antenna provides wide beam width, but low directivity and gain. For example, the maximum radiation of dipole antenna takes place in the direction normal to its axis and decreases slowly as one moves toward the axis of the antenna. The antennas of such radiation characteristic may be preferred in broadcast services where wide coverage is required but not in point to point communication. Thus to meet the demands of point to point communication, it is necessary to design the narrow beam and high directive antennas, so that the radiation can be released in the preferred direction. The simplest way to achieve this requirement is to increase the size of the antenna, because a larger-size antenna leads to more directive characteristics. But from the practical aspect the method is inconvenient as antenna becomes bulky and it is difficult to change the size later. Another way to improve the performance of the antenna without increasing the size of the antenna is to arrange the antenna in a specific configuration, so spaced and phased that their individual contributions are maximum in desired direction and negligible in other directions. This way particularly, we get greater directive gain. This new arrangement of multi-element is referred to as an array of the antenna. The antenna involved in an array is known as element. The individual element of array may be of any form (wire. dipole. slot, aperture. etc.). Having identical element in an array is often simpler, convenient and practical, but it is not compulsory. The antenna array makes use of wave interference phenomenon that occurs between the radiations from the different elements of the array. Thus, the antenna array is one of the methods of combining the radiation from a group of radiators in such a way that the interference is constructive in the preferred direction and destructive in the remaining directions. The main function of an array is to produce highly directional radiation. The field is a vector quantity with both magnitude and phase. The total field (not power) of the array system at any point away from its centre is the vector sum of the field produced by the individual antennas. The relative phases of individual field components depend on the relative distance of the individual clement and in turn depend on the direction.

ARRAY CONFIGURATIONS

Broadly, array antennas can be classified into four categories:

(a) Broadside array

(b) End-fire array

(c) Collinear array

(d) Parasitic array

Broadside Array- This is a type of array in which the number of identical elements is placedon a supporting line drawn perpendicular to their respective axes.

Elements are equally spaced and fed with a current of equal magnitude and all in same phase. The advantage of this feed technique is that array fires in broad side direction (i.e. perpendicular to the line of array axis, where there are maximum radiation and small radiation in other direction). Hence the radiation pattern of broadside array is bidirectional and the array radiates equally well in either direction of maximum radiation. In Fig. 1 the elements are arranged in horizontal plane with spacing between elements and radiation is perpendicular to the plane of array (i.e. normal to plane of paper.) They may also be arranged in vertical and in this case radiation will be horizontal. Thus, it can be said that broadside array is a geometrical arrangement of elements in which the direction of maximum radiation is perpendicular to the array axis and to the plane containing the array clement. Radiation pattern of a broad side array is shown in Fig. 2. The bidirectional pattern of broadside array can be converted into unidirectional by placing an identical array behind this array at distance of $\lambda/4$ fed by current leading in phase by 90^0.

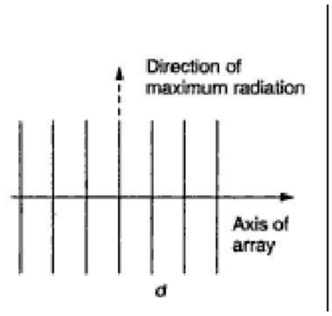


Fig. 1 Geometry of broadside array

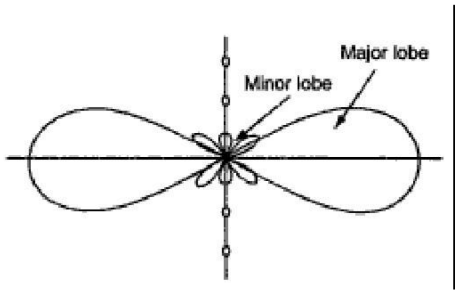


Fig. 2 Radiation pattern of broadside array

End Fire Array-The end fire array is very much similar to the broadside array from thepoint of view of arrangement. But the main difference is in the direction of maximum radiation. In broadside array, the direction of the maximum radiation is perpendicular to the axis of array; while in the end fire array, the direction of the maximum radiation is along the axis of array.

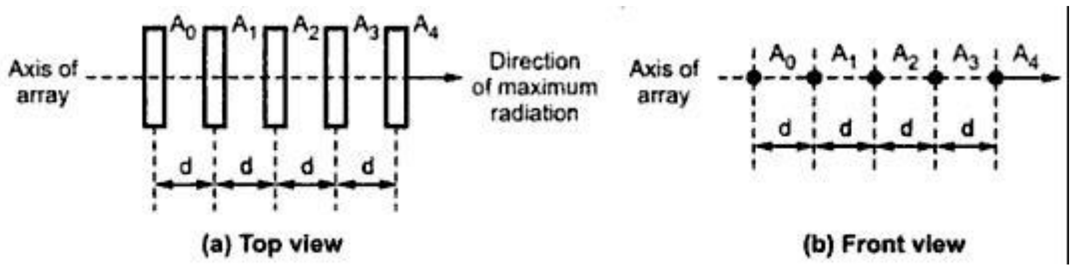


Fig. 3 End fire array

Thus in the end fire array number of identical antennas are spaced equally along a line. All the antennas are fed individually with currents of equal magnitudes but their phases vary progressively along the line to get entire arrangement unidirectional finally. i.e. maximum radiation along the axis of array.

Thus end fire array can be defined as an array with direction of maximum radiation coincides with the direction of the axis of array to get unidirectional radiation.

Collinear Array-In collinear array the elements are arranged co-axially, i.e., antennas areeither mounted end to end in a single line or stacked over one another. The collinear array is also a broadside array and elements are fed equally in phase currents. But the radiation pattern of a collinear array has circular symmetry with its main lobe everywhere normal to the principal axis. This is reason why this array is called broadcast or Omni-directional arrays. Simple collinear array consists of two elements: however, this array can also have more than two elements (Fig. 4). The performance characteristic of array does not depend directly on the number of elements in the array. For example, the power gain for collinear array of 2, 3, and 4 elements are respectively 2 dB, 3.2 dB and 4.4 dB respectively. The power gain of 4.4 dB obtained by this array is comparatively lower than the gain obtained by other arrays or devices. The collinear array provides maximum gain when spacing between elements is of the order of 0.3λ to 0.5λ; but this much spacing results in constructional and feeding difficulties. The elements are operated with their ends are much close to each other and joined simply by insulator.

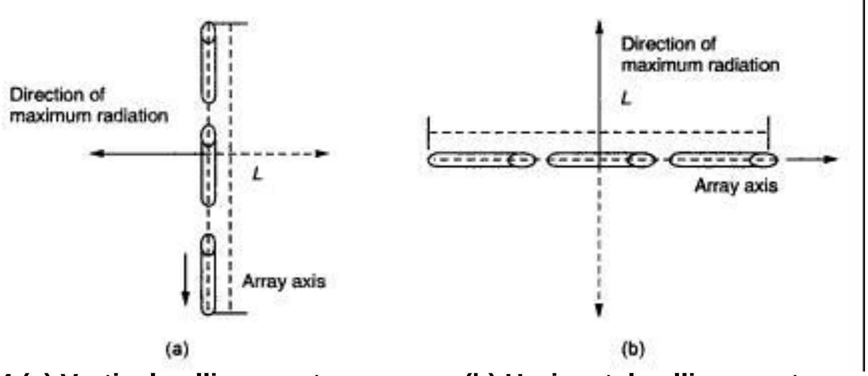


Fig. 4 (a) Vertical collinear antenna array (b) Horizontal collinear antenna array

Increase in the length of collinear arrays increases the directivity: however, if the number of elements in an array is more (3 or 4), in order to keep current in phase in all the elements, it is essential to connect phasing stubs between adjacent elements. A collinear array is usually mounted vertically in order to increase overall gain and directivity in the horizontal direction. Stacking of dipole antennas in the fashion of doubling their number with proper phasing produces a 3 dB increase in directive gain.

Parasitic Arrays-In some way it is similar to broad side array, but only one element is feddirectly from source, other element arc electromagnetically coupled because of its proximity to the feed element. Feed element is called driven element while other elements are called parasitic elements. A parasitic element lengthened by 5% to driven element act as reflector and another element shorted by 5% acts as director. Reflector makes the radiation maximum in perpendicular direction toward driven element and direction helps in making maximum radiation perpendicular to next parasitic element. The simplest parasitic array has three elements: reflector, driven element and director, and is used, for example in Yagi-Uda array antenna. The phase and amplitude of the current induced in a parasitic element depends upon its tuning and the spacing between elements and driven element to which it is coupled. Variation in spacing between driven element and parasitic elements changes the relative phases and this proves to be very convenient. It helps in making the radiation pattern unidirectional. A distance of $\lambda/4$ and phase difference of $\pi/2$ radian provides a unidirectional pattern. A properly designed parasitic array with spacing 0.1λ to 0.15λ provides a frequency bandwidth of the order of 2%, gain of the order of 8 dB and FBR of about 20 dB. It is of great practical importance, especially at higher frequencies between 150 and 100 MHz, for Yagi array used for TV reception.

The simplest array configuration is array of two point sources of same polarization and separated by a finite distance. The concept of this array can also be extended to more number of elements and finally an array of isotropic point sources can be formed.

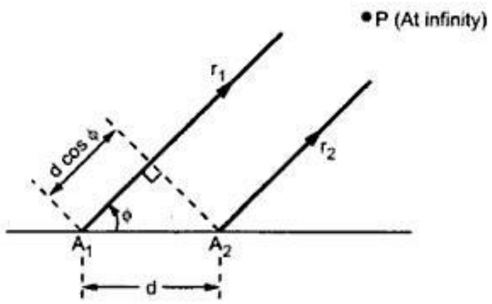
Based on amplitude and phase conditions of isotropic point sources, there are three types of arrays:

(a) Array with equal amplitude and phases

(b) Array with equal amplitude and opposite phases

(c) Array with unequal amplitude and opposite phases

Two Point Sources with Currents Equal in Magnitude and Phase



...(1)

Fig. 5 Two element array

Consider two point sources A_1 and A_2, separated by distance d as shown in the Fig. 5. Consider that both the point sources are supplied with currents equal in magnitude and phase. Consider point P far away from the array. Let the distance between point P and point sources A_1 and A_2 be r_1 and r_2 respectively. As these radial distances are extremely large as compared with the distance of separation between two point sources i.e. d, we can assume,

$r_1 = r_2 = r$

The radiation from the point source A_2 will reach earlier at point P than that from point source A_1 because of the path difference. The extra distance is travelled by the radiated wave from point source A_1 than that by the wave radiated from point source A_2.

Hence path difference is given

by, Path difference = d cos υ

The path difference can be expressed in terms of

wavelength as, Path difference = (d cos υ) / λ**...(2)**

Hence the phase angle υ is given by,

Phase angle υ = 2π (Path difference)

$$\therefore \quad \psi = 2\pi\left(\frac{d\cos\phi}{\lambda}\right)$$

$$\therefore \quad \boxed{\psi = \frac{2\pi}{\lambda} d\cos\phi \ \text{ rad}} \qquad \ldots(3)$$

But phase shift β = 2π/λ, thus equation (3) becomes,

$$\therefore \quad \boxed{\psi = \beta d\cos\phi \ \text{ rad}} \qquad \ldots(4)$$

Let E_1 be the far field at a distant point P due to point source A_l. Similarly let E_2 be the far field at point P due to point source A_2. Then the total field at point P be the addition of the two field components due to the point sources A_1 and A_2. If the phase angle between the two fields is υ = βdcosυ then the far field component at point P due to point source A_1 is given by,

$$E_1 = E_0 \cdot e^{-j\frac{\psi}{2}} \qquad \ldots(5)$$

Similarly the far field component at point P due to the point source A_2 is given by,

$$E_2 = E_0 \cdot e^{j\frac{\psi}{2}} \qquad \ldots(6)$$

Note that the amplitude of both the field components is E_0 as currents are same and the point sources are identical.

The total field at point P is given by,

$$E_T = E_1 + E_2 = E_0 \cdot e^{-j\frac{\psi}{2}} + E_0 \cdot e^{j\frac{\psi}{2}}$$

$$\therefore \qquad E_T = E_0\left(e^{-j\frac{\psi}{2}} + e^{j\frac{\psi}{2}}\right)$$

Rearranging the terms on R.H.S., we get,

$$\therefore \qquad E_T = 2E_0\left(\frac{e^{j\frac{\psi}{2}} + e^{-j\frac{\psi}{2}}}{2}\right) \qquad \ldots(7)$$

By trigonometric identity,

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta.$$

Hence equation (7) can be written as,

$$E_T = 2E_0 \cos\left(\frac{\psi}{2}\right) \qquad \ldots(8)$$

Substituting value of Ψ from equation (4), we get,.

$$\therefore \qquad \boxed{E_T = 2E_0 \cos\left(\frac{\beta d \cos\phi}{2}\right)} \qquad \ldots(9)$$

Above equation represents total field in intensity at point P. due to two point sources having currents of same amplitude and phase. The total amplitude of the field at point P is $2E_0$ while the phase shift is βdcosυ/2

The array factor is the ratio of the magnitude of the resultant field to the magnitude of the maximum field.

$$\therefore \qquad \text{A.F.} = \frac{|E_T|}{|E_{max}|}$$

But maximum field is Ernax =$2E_0$

$$\therefore \qquad \boxed{\text{A.F.} = \frac{|E_T|}{|2\,E_0|} = \cos\left(\pi \frac{d}{\lambda} \cos\phi\right)}$$

The array factor represents the relative value of the field as a function of υ defines the radiation pattern in a plane containing the line of the array.

Maxima direction

From equation (9), the total field is maximum when $\cos\left(\frac{\beta d \cos\phi}{2}\right)$ is maximum. As we know, the variation of cosine of a angle is ± 1. Hence the condition for maxima is given by,

$$\boxed{\cos\left(\frac{\beta d \cos\phi}{2}\right) = \pm 1}$$

Let spacing between the two point sources be $\lambda/2$. Then we can write,

$$\cos\left[\frac{\beta(\lambda/2)\cos\phi}{2}\right] = \pm 1 \qquad \ldots(10)$$

$$\text{i.e. } \cos\left[\frac{\frac{2\pi}{\lambda}\cdot\frac{\lambda}{2}\cos\phi}{2}\right] = \pm 1 \qquad \ldots \because \beta = \frac{2\pi}{\lambda}$$

$$\text{i.e. } \cos\left(\frac{\pi}{2}\cos\phi\right) = \pm 1$$

$$\text{i.e. } \frac{\pi}{2}\cos\phi_{max} = \cos^{-1}(\pm 1) = \pm n\pi, \text{ where } n = 0, 1, 2, \ldots\ldots$$

If n = 0, then

$$\frac{\pi}{2}\cos\phi_{max} = 0$$

$$\text{i.e. } \cos\phi_{max} = 0$$

$$\text{i.e. } \boxed{\phi_{max} = 90^\circ \text{ or } 270^\circ} \qquad \ldots(11)$$

Minima direction

Again from equation (9), total field strength is minimum when $\cos\left(\frac{\beta d\cos\phi}{2}\right)$ is minimum i.e. 0 as cosine of angle has minimum value 0. Hence the condition for minima is given by,

$$\therefore \boxed{\cos\left(\frac{\beta d\cos\phi}{2}\right) = 0} \qquad \ldots(12)$$

Again assuming d = $\lambda/2$ and $\beta=2\pi/\lambda$, we can write

$$\cos\left(\frac{\pi}{2}\cos\phi_{min}\right) = 0$$

$$\therefore \frac{\pi}{2}\cos\phi_{min} = \cos^{-1}0 = \pm(2n+1)\frac{\pi}{2}, \text{ where } n = 0, 1, 2, \ldots\ldots\ldots$$

$$\text{If } n = 0, \text{ then,}$$

$$\frac{\pi}{2}\cos\phi_{min} = \pm\frac{\pi}{2}$$

$$\text{i.e. } \cos\phi_{min} = \pm 1$$

$$\text{i.e. } \boxed{\phi_{min} = 0^\circ \text{ or } 180^\circ} \qquad \ldots(13)$$

Half power point direction:

When the power is half, the voltage or current is $1/\sqrt{2}$ times the maximum value.

Hence the condition for half power point is given by,

$$\boxed{\cos\left(\frac{\beta d\cos\phi}{2}\right) = \pm\frac{1}{\sqrt{2}}} \qquad \ldots(14)$$

Let d=λ/2 and β=2π/λ, then we can write,

$$\cos\left(\frac{\pi}{2}\cos\phi\right) = \pm\frac{1}{\sqrt{2}}$$

$$\text{i.e. } \frac{\pi}{2}\cos\phi = \cos^{-1}\left(\pm\frac{1}{\sqrt{2}}\right) = \pm(2n+1)\frac{\pi}{4}, \text{ where } n = 0, 1, 2, \ldots$$

If $n = 0$, then

$$\frac{\pi}{2}\cos\phi_{HPPD} = \pm\frac{\pi}{4}$$

$$\text{i.e. } \cos\phi_{HPPD} = \pm\frac{1}{2}$$

$$\text{i.e. } \phi_{HPPD} = \cos^{-1}\left(\pm\frac{1}{2}\right)$$

$$\therefore \quad \boxed{\phi_{HPPD} = 60^\circ \text{ or } 120^\circ} \qquad \ldots(15)$$

The field pattern drawn with E_T against υ for d=λ/2, then the pattern is bidirectional as shown in Fig 6. The field pattern obtained is bidirectional and it is a figure of eight.

If this pattern is rotated by 360^0 about axis, it will represent three dimensional doughnut shaped space pattern. This is the simplest type of broadside array of two point sources and it is called Broadside couplet as two radiations of point sources are in phase.

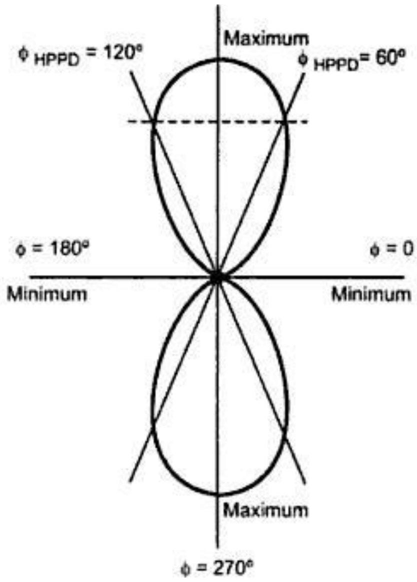


Fig. 6 Field pattern for two point source with spacing d=λ/2 and fed with currents equal in magnitude andphase.

Two Point Sources with Currents Equal in Magnitudes but Opposite in Phase

Consider two point sources separated by distance d and supplied with currents equal in magnitude but opposite in phase. Consider Fig. 5 all the conditions are exactly same except the phase of the currents is opposite i.e. 180°. With this condition, the total field at far point P is given by,

$$E_T = (-E_1) + (E_2) \qquad \text{...(1)}$$

Assuming equal magnitudes of currents, the fields at point P due to the point sources A_1 and A_2 can be written as,

$$E_1 = E_0 e^{-j\frac{\psi}{2}} \qquad \text{...(2)}$$

and

$$E_2 = E_0 e^{j\frac{\psi}{2}} \qquad \text{...(3)}$$

Substituting values of E_1 and E_2 in equation (1), we get

$$E_T = -E_0 \cdot e^{-j\frac{\psi}{2}} + E_0 \cdot e^{j\frac{\psi}{2}}$$

$$\therefore \quad E_T = E_0\left(-e^{-j\frac{\psi}{2}} + e^{j\frac{\psi}{2}}\right)$$

Rearranging the terms in above equation, we get,

$$\therefore \quad E_T = (j2)\ E_0\left(\frac{e^{j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}}{j2}\right) \qquad \text{...(4)}$$

By trigonometry identity,

$$\frac{e^{j\theta} - e^{-j\theta}}{2} = \sin\frac{\theta}{2}.$$

Equation (4) can be written as,

$$E_T = j2E_0 \sin\left(\frac{\psi}{2}\right) \qquad \text{...(5)}$$

Now as the condition for two point sources with currents in phase and out of phase is exactly same, the phase angle can be written as previous case.

Phase angle = βdcosυ ...(6) Substituting value of phase angle in equation (5), we get,

$$\boxed{E_T = j(2E_0)\sin\left(\frac{\beta d \cos\phi}{2}\right)} \qquad \text{...(7)}$$

Maxima direction

From equation (7), the total field is maximum when $\sin\left(\frac{\beta d \cos\phi}{2}\right)$ is maximum i.e. ±1 as the maximum value of sine of angle is ±1. Hence condition for maxima is given by,

$$\boxed{\sin\left(\frac{\beta d \cos\phi}{2}\right) = \pm 1} \qquad \ldots(8)$$

Let the spacing between two isotropic point sources be equal to $d=\lambda/2$

Substituting $d=\lambda/2$ and $\beta=2\pi/\lambda$, in equation (8), we get,

$$\sin\left(\frac{\pi}{2}\cos\phi\right) = \pm 1$$

$$\text{i.e.} \quad \frac{\pi}{2}\cos\phi = \pm(2n+1)\frac{\pi}{2}, \text{ where } n = 0, 1, 2, \ldots\ldots$$

If n = 0. then

$$\frac{\pi}{2}\cos\phi_{max} = \pm\frac{\pi}{2}$$

$$\text{i.e. } \cos\phi_{max} = \pm 1$$

$$\text{i.e.} \quad \boxed{\phi_{max} = 0^\circ \text{ and } 180^\circ} \qquad \ldots(9)$$

Minima direction

Again from equation (7), total field strength is minimum when $\sin\left(\frac{\beta d \cos\phi}{2}\right)$ is minimum i.e. 0.

Hence the condition for minima is given by,

$$\boxed{\sin\left(\frac{\beta d \cos\phi}{2}\right) = 0} \qquad \ldots(10)$$

Assuming $d=\lambda/2$ and $\beta=2\pi/\lambda$ in equation (10), we get,

$$\sin\left(\frac{\pi}{2}\cos\phi\right) = 0$$

$$\text{i.e.} \quad \frac{\pi}{2}\cos\phi = \pm n\pi, \text{ where } n = 0, 1, 2, \ldots\ldots$$

If n = 0, then

$$\frac{\pi}{2}\cos\phi_{min} = 0$$

$$\text{i.e. } \cos\phi_{min} = 0$$

$$\text{i.e.} \quad \boxed{\phi_{min} = +90^\circ \text{or} -90^\circ} \qquad \ldots(11)$$

Half Power Point Direction (HPPD)

When the power is half of maximum value, the voltage or current equals to 1/√2 times the respective maximum value. Hence the condition for the half power point can be obtained from equation (7) as,

$$\boxed{\sin\left(\frac{\beta d\cos\phi}{2}\right) = \pm\frac{1}{\sqrt{2}}} \qquad \ldots(12)$$

Let $d=\lambda/2$ and $\beta=2\pi/\lambda$, we can write,

$$\sin\left(\frac{\pi}{2}\cos\phi\right) = \pm\frac{1}{\sqrt{2}}$$

$$\text{i.e. } \frac{\pi}{2}\cos\phi = \pm(2n+1)\frac{\pi}{4}, \quad \text{where } n = 0, 1, 2.$$

If n = 0, we can write,

$$\frac{\pi}{2}\cos\phi_{HPPD} = \pm\frac{\pi}{4}$$

$$\text{i.e. } \cos\phi_{HPPD} = \pm\frac{1}{2}$$

$$\therefore \quad \boxed{\phi_{HPPD} = 60^\circ \text{ or } 120^\circ} \qquad \ldots(13)$$

Thus from the conditions of maxima, minima and half power points, the field pattern can be drawn as shown in the Fig. 7.

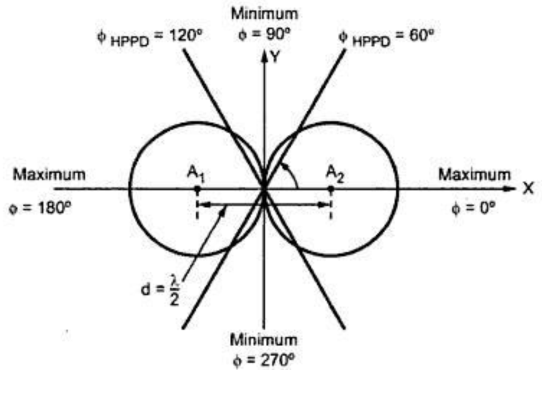


Fig. 7 Field pattern for two point sources with spacing d = d=λ/2 and fed with currents equal in magnitude but out of phase by 180^0.

As compared with the field pattern for two point sources with inphase currents, the maxima have shifted by 90° along X-axis in case of out-phase currents in two point source array. Thus the maxima are along the axis of the array or along the line joining two point sources. In first case, we have obtained vertical figure of eight. Now in above case, we have obtained horizontal figure of eight. As the maximum field is along the line joining the two point sources, this is the simple type of the end fire array.

Two point sources with currents unequal in magnitude and with any phase

Let us consider Fig. 5. Assume that the two point sources are separated by distance d and supplied with currents which are different in magnitudes and with any phase difference say α. Consider that source 1 is assumed to be reference for phase and amplitude of the fields E_1 and E_2, which are due to source 1 and source 2 respectively at the distant point P. Let us assume that E_1 is greater than E_2 in magnitude as shown in the vector diagram in Fig. 8.

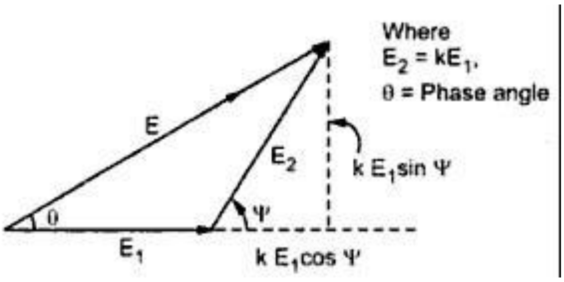


Fig. 8 Vector diagram of fields E_l and E_2

Now the total phase difference between the radiations by the two point sources at any far point P is given by,

$$\boxed{\Psi = \frac{2\pi}{\lambda}\cos\phi + \alpha} \qquad \ldots (1)$$

where α is the phase angle with which current I_2 leads current I_l. Now if $\alpha = 0$, then the condition is similar to the two point sources with currents equal in magnitude and phase. Similarly if α = 180", then the condition is similar to the two point source with currents equal in magnitude but opposite in phase. Assume value of phase difference as $0 < \alpha < 180^0$. Then the resultant field at point P is given by,

$$E_T = E_1 e^{j0} + E_2 e^{j\psi} \qquad \ldots\text{(source 1 is assumed to be}$$

$$\therefore \quad E_T = E_1 + E_2 e^{j\psi} \qquad \text{reference hence phase angle is 0)}$$

$$\therefore \quad E_T = E_1\left(1 + \frac{E_2}{E_1} e^{j\psi}\right)$$

$$\text{Let} \quad \frac{E_2}{E_1} = k \qquad \ldots(2)$$

Note that $E_1 > E_2$, the value of k is less than unity. Moreover the value of k is given by, $0 \leq k \leq 1$

$$\therefore \quad \boxed{E_T = E_1\,[1 + k\,(\cos\psi + j\sin\psi)]} \qquad \ldots (3)$$

The magnitude of the resultant field at point P is given by,

$$|E_T| = |E_1 [1 + k\cos\psi + j\,k\sin\psi]|$$

$$\therefore \qquad \boxed{|E_T| = E_1\sqrt{(1+k\cos\psi)^2 + (k\sin\psi)^2}} \qquad \ldots (4)$$

The phase angle between two fields at the far point P is given by,

$$\therefore \qquad \boxed{\theta = \tan^{-1}\frac{k\sin\psi}{1+k\cos\psi}} \qquad \ldots (5)$$

n Element Uniform Linear Arrays

At higher frequencies, for point to point communications it is necessary to have a pattern with single beam radiation. Such highly directive single beam pattern can be obtained by increasing the point sources in the arrow from 2 to n say. An array of n elements is said to be linear array if all the individual elements are spaced equally along a line. An array is said to be uniform array if the elements in the array are fed with currents with equal magnitudes and with uniform progressive phase shift along the line. Consider a general n element linear and uniform array with all the individual elements spaced equally at distance d from each other and all elements are fed with currents equal in magnitude and uniform progressive phase shift along line as shown in the Fig. 9.

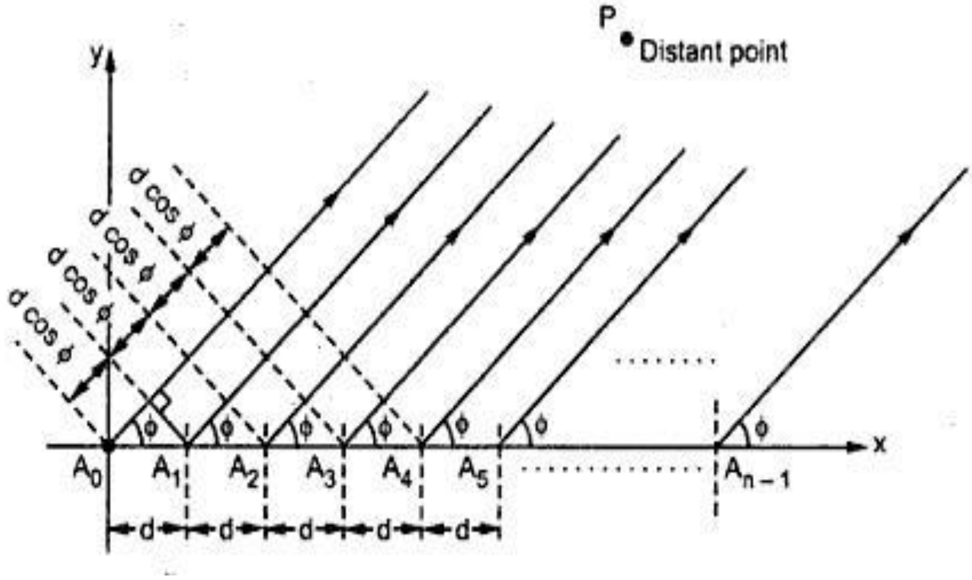


Fig. 9 Uniform, linear array of n elements

The total resultant field at the distant point P is obtained by adding the fields due to n individual sources vectorically. Hence we can write,

$$E_T = E_0 \cdot e^{j0} + E_0 e^{j\psi} + E_0 e^{2j\psi} + \ldots + E_0 e^{j(n-1)\psi}$$

$$\therefore \quad \boxed{E_T = E_0 [1 + e^{j\psi} + e^{j2\psi} + \ldots + e^{j(n-1)\psi}]} \qquad \ldots (1)$$

Note that $\psi = (\beta d \cos\psi + \alpha)$ indicates the total phase difference of the fields from adjacent sources calculated at point P. Similarly α is the progressive phase shift between two adjacent point sources. The value of α may lie between 0^0 and 180^0. If $\alpha = 0^0$ we get n element uniform linear broadside array. If $\alpha = 180^0$ we get n element uniform linear endfire array.

Multiplying equation (1) by $e^{j\psi}$, we get,

$$E_T e^{j\psi} = E_0 [e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \ldots + e^{jn\psi}] \qquad \ldots(2)$$

Subtracting equation (2) from (1), we get,

$$E_T - E_T e^{j\psi} = E_0 \left\{[1 + e^{j\psi} + e^{j2\psi} + \ldots + e^{j(n-1)\psi}] - [e^{j\psi} + e^{j2\psi} + \ldots + e^{jn\psi}]\right\}$$

$$E_T (1 - e^{j\psi}) = E_0 (1 - e^{jn\psi})$$

$$\therefore \quad \boxed{E_T = E_0 \left[\frac{1 - e^{jn\psi}}{1 - e^{j\psi}}\right]} \qquad \ldots (3)$$

Simply mathematically, we get

$$E_T = E_0 \left[\frac{e^{j\frac{n\psi}{2}}\left(e^{-j\frac{n\psi}{2}} - e^{j\frac{n\psi}{2}}\right)}{e^{j\frac{\psi}{2}}\left(e^{-j\frac{\psi}{2}} - e^{j\frac{\psi}{2}}\right)}\right]$$

According to trigonometric identity,

$$e^{-j\theta} - e^{j\theta} = -2j \sin\theta,$$

The resultant field is given by,

$$E_T = E_0 \left[\frac{\left(-j2\sin\frac{n\psi}{2}\right)e^{j\frac{n\psi}{2}}}{\left(-j2\sin\frac{\psi}{2}\right)e^{j\frac{\psi}{2}}}\right]$$

$$\therefore \quad \boxed{E_T = E_0 \left[\frac{\sin n\frac{\psi}{2}}{\sin\frac{\psi}{2}}\right] e^{j\left(\frac{n-1}{2}\right)\psi}} \qquad \ldots (4)$$

This equation (4) indicates the resultant field due to n element array at distant point P. The magnitude of the resultant field is given by, The phase angle θ of the resultant field at point P is given by,

$$\therefore \quad \boxed{\theta = \frac{(n-1)}{2}\psi = \frac{(n-1)}{2}\ \beta\, d \cos\phi + \alpha} \qquad \ldots (6)$$

Array of n elements with Equal Spacing and Currents Equal in Magnitude and Phase • Broadside Array

Consider 'n' number of identical radiators carries currents which are equal in magnitude and in phase. The identical radiators are equispaced. Hence the maximum radiation occurs in the directions normal to the line of array. Hence such an array is known as Uniform broadside array. Consider a broadside array with n identical radiators as shown in the Fig. 10.

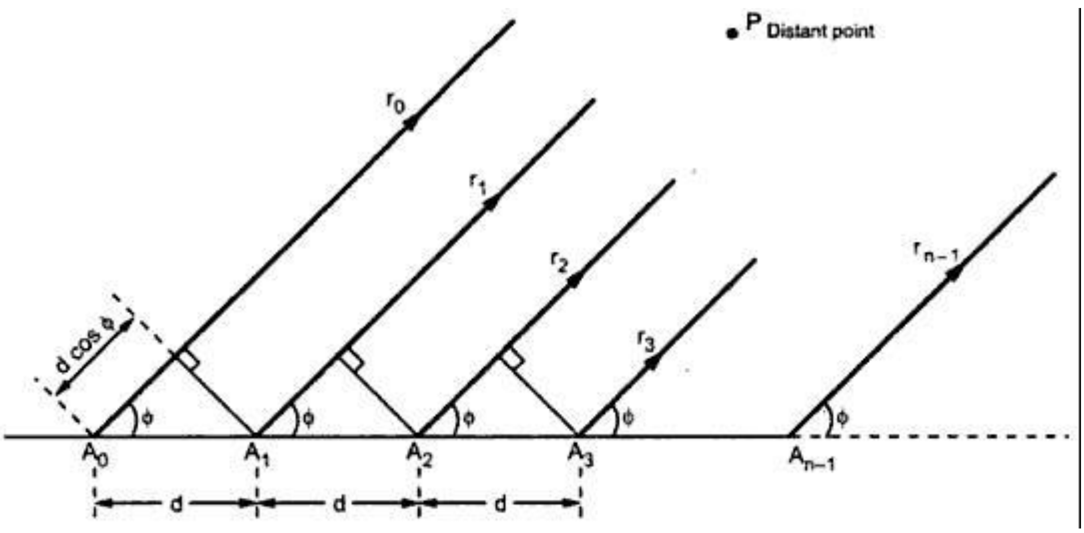


Fig 10 Array of n elements with Equal Spacing

The electric field produced at point P due to an element A_0 is given by,

$$E_0 = \frac{I\, dL \sin\theta}{4\pi\omega\varepsilon_0}\left[j\frac{\beta^2}{r_0}\right] e^{-j\beta r_0} \qquad \ldots (1)$$

As the distance of separation d between any two array elements is very small as compared to the radial distances of point P from A_0, A_1, ...A_{n-1}, we can assume r_0, r_1, ...r_{n-1} are approximately same.

Now the electric field produced at point P due to an element A_1 will differ in phase as r_0 and r_1 are not actually same. Hence the electric field due to A_1 is given by,

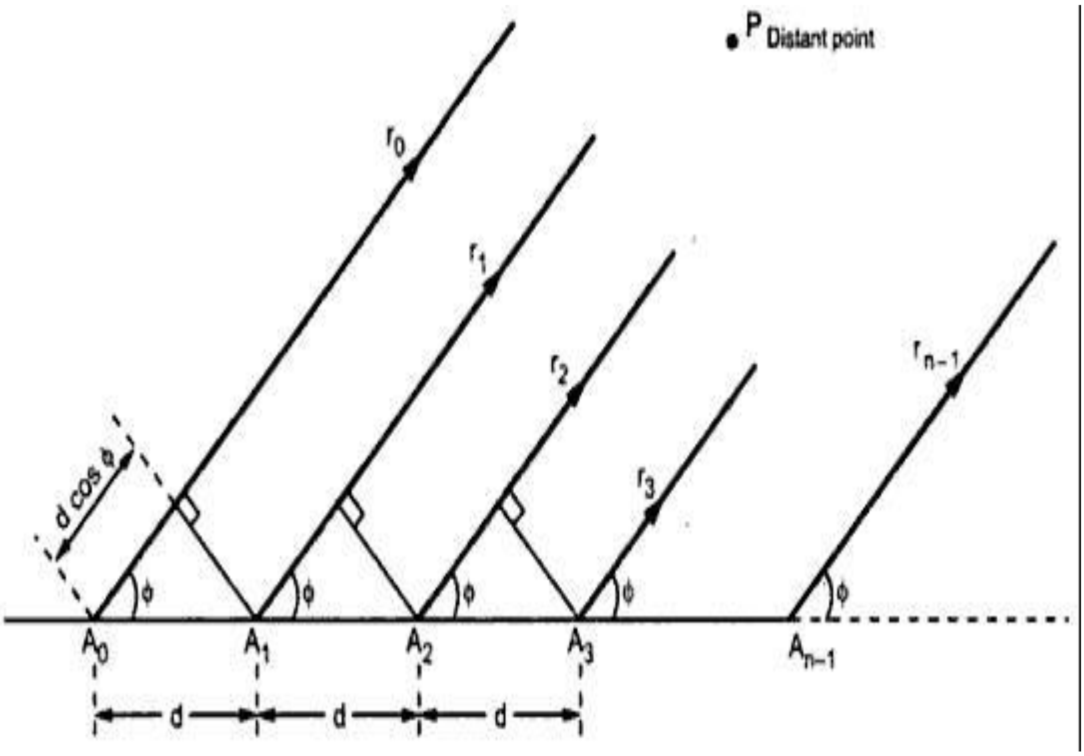
$$\therefore \quad \boxed{E_T = E_0 \left[\frac{\sin n\frac{\psi}{2}}{\sin\frac{\psi}{2}} \right]} \quad \dots (5)$$

Exactly on the similar lines we can write the electric field produced at point P due to an element A_2 as,

$$E_2 = \frac{I\, dL \sin\theta}{4\pi\omega\varepsilon_0}\left[j\frac{\beta^2}{r_2}\right] e^{-j\beta r_2}$$

$$\therefore \quad E_2 = \frac{I\, dL \sin\theta}{4\pi\omega\varepsilon_0}\left[j\frac{\beta^2}{r_1}\right] e^{-j\beta(r_1 - d\cos\phi)} \quad \dots r_2 = r_1 - d\cos\phi$$

$$\therefore \quad E_2 = \left\{\frac{I\, dL \sin\theta}{4\pi\omega\varepsilon_0}\left[j\frac{\beta^2}{r_1}\right] e^{-j\beta r_1}\right\} e^{j\beta d\cos\phi}$$



But the term inside the bracket represent E_1

$$\therefore \quad E_2 = E_1 e^{j\beta d\cos\phi}$$

From equation (2), substituting the value of E_1, we get,

$$E_2 = \left[E_0 e^{j\beta d\cos\phi}\right] e^{j\beta d\cos\phi}$$

$$\therefore \quad E_2 = E_0 \cdot e^{j2\beta d\cos\phi} \qquad \text{... (3)}$$

Similarly, the electric field produced at point P due to element A_{n-1} is given by,

$$E_{n-1} = E_0 \cdot e^{j(n-1)\beta d\cos\phi} \qquad \text{... (4)}$$

The total electric field at point P is given by,

$$E_T = E_0 + E_1 + E_2 + \ldots + E_{n-1}$$

$$\therefore \quad E_T = E_0 + E_0 e^{j\beta d\cos\phi} + E_0 e^{j2\beta d\cos\phi} + \ldots + E_0 e^{j(n-1)\beta d\cos\phi}$$

Let $\beta d\cos\upsilon = \upsilon$, then rewriting above equation,

$$E_T = E_0 + E_0 e^{j\psi} + E_0 e^{j2\psi} + \ldots + E_0 e^{j(n-1)\psi}$$

$$\therefore \quad E_T = E_0 \left[1 + e^{j\psi} + e^{j2\psi} + \ldots + e^{j(n-1)\psi}\right] \qquad \text{... (5)}$$

Consider a series given by $s = 1 + r + r^2 + \ldots + r^{n-1}$ where $r = e^{j\upsilon}$ ... (i)

Multiplying both the sides of the equation (i) by r, $s \cdot r = r + r^2 + \ldots + r^n$ ... (ii)

Subtracting equation (ii) from (i), we get. $s(1-r) = 1-r^n$

$$\therefore \quad s = \frac{1-r^n}{1-r} \qquad \text{... (iii)}$$

Using equation (iii), equation (5) can be modified as,

$$E_T = E_0\left[\frac{1-e^{jn\psi}}{1-e^{j\psi}}\right]$$

$$\therefore \qquad \frac{E_T}{E_0} = \frac{e^{jn\frac{\psi}{2}}\left[e^{-jn\frac{\psi}{2}} - e^{jn\frac{\psi}{2}}\right]}{e^{j\frac{\psi}{2}}\left[e^{-j\frac{\psi}{2}} - e^{j\frac{\psi}{2}}\right]} \qquad \text{... (6)}$$

From the trigonometric identities,

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\text{and } e^{-j\theta} - e^{j\theta} = -j\,2\sin\theta$$

Equation (6) can be written as,

$$\frac{E_T}{E_0} = \frac{e^{jn\frac{\psi}{2}}\left[-j\,2\sin\left(\frac{n\psi}{2}\right)\right]}{e^{j\frac{\psi}{2}}\left[-j\,2\sin\left(\frac{\psi}{2}\right)\right]}$$

$$\therefore \qquad \frac{E_T}{E_0} = e^{j\frac{(n-1)\psi}{2}}\left[\frac{\sin\left(\frac{n\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}\right] \qquad \text{... (7)}$$

The exponential term in equation (7) represents the phase shift. Now considering magnitudes of the electric fields, we can write,

$$\left|\frac{E_T}{E_0}\right| = \frac{\sin\frac{n\psi}{2}}{\sin\frac{\psi}{2}} \qquad \text{... (8)}$$

Properties of Broadside Array

1. **Major lobe**

   In case of broadside array, the field is maximum in the direction normal to the axis of the array. Thus the condition for the maximum field at point P is given by,

   $$\psi = 0 \text{ i.e. } \beta d\cos\phi = 0 \qquad \text{...(9)}$$

   $$\text{i.e. } \cos\phi = 0$$

   $$\text{i.e. } \boxed{\phi = 90^\circ \text{ or } 270^\circ} \qquad \text{...(10)}$$

   Thus $\upsilon = 90^0$ and 270^0 are called directions of principle maxima.

2. **Magnitude of major lobe**

The maximum radiation occurs when υ=0. Hence we can write,

$$|\text{Major lobe}| = \left|\frac{E_T}{E_0}\right| = \lim_{\psi \to 0}\left\{\frac{\frac{d}{d\psi}\left(\sin n\frac{\psi}{2}\right)}{\frac{d}{d\psi}\left(\sin\frac{\psi}{2}\right)}\right\}$$

$$= \lim_{\psi \to 0}\left\{\frac{\left(\cos n\frac{\psi}{2}\right)\left(n\frac{\psi}{2}\right)}{\left(\cos\frac{\psi}{2}\right)\left(\frac{\psi}{2}\right)}\right\}$$

$$\therefore \qquad \boxed{|\text{Major lobe}| = n} \qquad \ldots(11)$$

where, n is the number of elements in the array.

Thus from equation (10) and (11) it is clear that, all the field components add up together to give total field which is 'n' times the individual field when υ = 90^0 and 270^0.

3. **Nulls**

The ratio of total electric field to an individual electric field is given by,

$$\left|\frac{E_T}{E_0}\right| = \frac{\sin n\frac{\psi}{2}}{\sin\frac{\psi}{2}}$$

Equating ratio of magnitudes of the fields to zero,

$$\therefore \qquad \left|\frac{E_T}{E_0}\right| = \frac{\sin n\frac{\psi}{2}}{\sin\frac{\psi}{2}} = 0$$

The condition of minima is given by,

$$\therefore \qquad \boxed{\sin n\frac{\psi}{2} = 0; \quad \text{but } \sin\frac{\psi}{2} \neq 0} \qquad \ldots(12)$$

Hence we can write,

$$\sin n\frac{\psi}{2} = 0$$

$$\text{i.e. } n\frac{\psi}{2} = \sin^{-1}(0) = \pm m\pi, \text{ where } m = 1, 2, 3, \ldots\ldots$$

$$\text{Now } \psi = \beta d\cos\phi = \frac{2\pi}{\lambda}(d)\cos\phi$$

$$\therefore \quad \frac{n}{2}\left(\frac{2\pi}{\lambda}d\right)\cos\phi_{min} = \pm m\pi$$

$$\text{i.e.} \quad \frac{nd}{\lambda}\cos\phi_{min} = \pm m$$

$$\therefore \quad \boxed{\phi_{min} = \cos^{-1}\left(\pm\frac{m\lambda}{nd}\right)} \quad \ldots(13)$$

where, n= number of elements in the array d=

spacing between elements in meter

λ= wavelength in meter

m= constant= 1, 2 , 3....

Thus equation (13) gives direction of nulls

4. **Side Lobes Maxima**

The directions of the subsidary maxima or side lobes maxima can be obtained if in equation (8),

$$\sin\left(n\frac{\psi}{2}\right) = \pm 1$$

$$\therefore \quad \boxed{n\frac{\psi}{2} = \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots\ldots} \quad \ldots(14)$$

Hence sin(nυ/2), is not considered. Because if nυ/2=π/2 then sin nυ/2 =1 which is the direction of principle maxima.

Hence we can skip sin nυ/2 = ±π/2 value Thus, we get

$$\psi = \pm\frac{3\pi}{n}, \pm\frac{5\pi}{n}, \pm\frac{7\pi}{n}, \ldots\ldots\ldots\ldots$$

$$\text{Now} \quad \psi = \beta d\cos\phi = \left(\frac{2\pi}{\lambda}\right)d\cos\phi$$

Now equation for υ can be written as,

$$\frac{2\pi}{\lambda}d\cos\phi = \pm\frac{3\pi}{n}, \pm\frac{5\pi}{n}, \pm\frac{7\pi}{n}, \ldots\ldots\ldots\ldots$$

$$\therefore \quad \cos\phi = \frac{\lambda}{2\pi d}\left[\pm\frac{(2m+1)}{n}\pi\right] \quad \text{where } m = 1, 2, 3, \ldots\ldots\ldots$$

$$\therefore \quad \boxed{\phi = \cos^{-1}\left[\pm\frac{\lambda(2m+1)}{2nd}\right]} \quad \ldots(15)$$

The equation (15) represents directions of subsidary maxima or side lobes maxima.

5. Beamwidth of Major Lobe

Beamwidth is defined as the angle between first nulls. Alternatively beamwidth is the angle equal to twice the angle between first null and the major lobe maximum direction. Hence beamwidth between first nulls is given by,

$$\therefore \quad \boxed{BWFN = 2 \times \gamma, \quad \text{where } \gamma = 90 - \phi} \qquad \ldots(16)$$

But $\quad \phi_{min} = \cos^{-1}\left(\pm \frac{m\lambda}{nd}\right), \quad$ where $m = 1, 2, 3, \ldots\ldots$

Also $\quad 90 - \phi_{min} = \gamma \;$ i.e. $90 - \gamma = \theta_{min}$

Hence $\quad 90 - \gamma = \cos^{-1}\left(\pm \frac{m\lambda}{nd}\right)$

Taking cosine of angle on both sides, we get

$$\cos(90 - \gamma) = \cos\left[\cos^{-1}\left(\pm \frac{m\lambda}{nd}\right)\right]$$

$$\therefore \quad \sin\gamma = \pm \frac{m\lambda}{nd} \qquad \ldots(17)$$

If γ is very small, then $\sin\gamma \approx \gamma$. Substituting n above equation we get,

$$\gamma = \pm \frac{m\lambda}{nd} \qquad \ldots(18)$$

For first null i.e. m=1,

$$\gamma = + \frac{\lambda}{nd}$$

$$\therefore \quad BWFN = 2\gamma = \frac{2\lambda}{nd}$$

But nd$\approx$ (n-1)d if n is very large. This L= (nd) indicates total length of the array.

$$\therefore \quad \boxed{BWFN = \frac{2\lambda}{L} \text{ rad} = \frac{2}{\left(\frac{L}{\lambda}\right)} \text{ rad}} \qquad \ldots(19)$$

BWFN in degree is written as,

$$\boxed{BWFN = \frac{114.6\lambda}{L} = \frac{114.6}{\left(\frac{L}{\lambda}\right)} \text{ degrees}} \qquad \ldots(20)$$

Now HPBW is given by,

$$\boxed{\text{HPBW} = \frac{\text{BWFN}}{2} = \frac{1}{\left(\frac{L}{\lambda}\right)} \text{ rad}} \quad \ldots(21)$$

HPBW in degree is written as,

$$\therefore \quad \boxed{\text{HPBW} = \frac{57.3}{\left(\frac{L}{\lambda}\right)} \text{ degrees}} \quad \ldots(22)$$

6. **Directivity**

The directivity in case of broadside array is defined as,

$$G_{Dmax} = \frac{\text{Maximum radiation intensity}}{\text{Average radiation intensity}} = \frac{U_{max}}{U_{avg}} = \frac{U_{max}}{U_0} \quad \ldots(23)$$

where, U_0 is average radiation intensity which is given by,

$$U_0 = \frac{P_{rad}}{4\pi} = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} |E(\theta, \phi)|^2 \sin\theta \, d\theta \, d\phi \quad \ldots(24)$$

From the expression of ratio of magnitudes we can write,

$$\left|\frac{E_T}{E_0}\right| = n$$

or $\quad |E_T| = n|E_0|$

For the normalized condition let us assume $E_0 = 1$, then

$$|E_T| = n$$

Thus field from array is maximum in any direction θ when $\upsilon = 0$. Hence normalized field pattern is given by,

$$E_{Normalized} = \left|\frac{E_T}{E_{Tmax}}\right| = \frac{|E_0|}{n|E_0|} = \frac{1}{n}$$

Hence the field is given by,

$$\therefore \quad E_{Normalized} = \frac{\sin n\frac{\psi}{2}}{n\left(\sin\frac{\psi}{2}\right)} \quad \ldots(25)$$

where $\upsilon = \beta d \cos\upsilon$

Equation (23) indicated array factor, hence we can write electric field due to n array as

$$E = \frac{1}{n}\left[\frac{\sin\frac{n\beta d\cos\phi}{2}}{\sin\frac{\beta d\cos\phi}{2}}\right]$$

Assuming d is very small as compared to length of an array,

$$\sin\frac{\beta d\cos\phi}{2} \approx \frac{\beta d\cos\phi}{2}.$$

Then,

$$E = \frac{1}{n}\left[\frac{\sin\frac{n\beta d\cos\phi}{2}}{\sin\frac{\beta d\cos\phi}{2}}\right] \qquad \ldots(26)$$

Substituting value of E in equation (24) we get

$$U_0 = \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[\frac{\sin\frac{n\beta d\cos\phi}{2}}{\sin\frac{\beta d\cos\phi}{2}}\right]^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{4\pi}\int_{\phi=0}^{2\pi} d\phi \cdot \int_{\theta=0}^{\pi}\left[\frac{\sin\frac{n}{2}\beta d\cos\phi}{\frac{n}{2}\beta d\cos\phi}\right]^2 \sin\theta\, d\theta$$

$$= \frac{1}{4\pi}[2\pi]\cdot\int_{\theta=0}^{\pi}\left[\frac{\sin z}{z}\right]^2 \sin\theta\, d\theta \qquad \ldots(27)$$

Let

$$z = \frac{n}{2}\beta d\cos\theta$$

$$\therefore \qquad dz = -\frac{n}{2}\beta d\sin\theta\, d\theta$$

$$\therefore \qquad \sin\theta\, d\theta = -\frac{dz}{\frac{n}{2}\beta d}$$

Also when $\theta = \pi$, $z = -\frac{n}{2}\beta d$, and

when $\theta = 0$, $z = +\frac{n}{2}\beta d$

Rewritting above equation we get,

$$U_0 = \frac{1}{2} \int_{+\frac{n}{2}\beta d}^{-\frac{n}{2}\beta d} \left[\frac{\sin z}{z}\right]^2 \cdot \frac{dz}{-\frac{n}{2}\beta d}$$

$$\therefore \qquad U_0 = -\frac{1}{n\beta d} \int_{\frac{n}{2}\beta d}^{-\frac{n}{2}\beta d} \left[\frac{\sin z}{z}\right]^2 dz$$

For large array, n is large hence $n\beta d$ is also very large (assuming tending to infinity). Hence rewriting above equation.

$$U_0 = -\frac{1}{n\beta d} \int_{\infty}^{-\infty} \left[\frac{\sin z}{z}\right]^2 dz$$

Interchanging limits of integration, we get

$$U_0 = +\frac{1}{n\beta d} \int_{-\infty}^{\infty} \left[\frac{\sin z}{z}\right]^2 dz$$

By integration formula,

$$\int_{-\infty}^{\infty} \left[\frac{\sin z}{z}\right]^2 dz = \pi.$$

Using above property in above equation we can write,

$$\boxed{U_0 = \frac{1}{n\beta d}[\pi] = \frac{\pi}{n\beta d}} \qquad \ldots(28)$$

From equation (23), the directivity is given by,

$$G_{Dmax} = \frac{U_{max}}{U_0}$$

But $U_{max} = 1$ at $\upsilon = 90°$ and substituting value of U_0 from equation (28), we get,

$$\boxed{G_{Dmax} = \frac{1}{\left(\frac{\pi}{n\beta d}\right)} = \frac{n\beta d}{\pi}} \qquad \ldots(29)$$

But $\beta = 2\pi/\lambda$

Hence

The total length of the array is given by, L = (n - 1) d ≈ nd, if n is very large. Hence the directivity can be expressed in terms of the total length of the array as,

$$\boxed{G_{Dmax} = 2\left(\frac{L}{\lambda}\right)}$$

Array of n Elements with Equal Spacing and Currents Equal in Magnitude but with Progressive Phase Shift - End Fire Array

Consider n number of identical radiators supplied with equal current which are not in phase as shown in the Fig. 11. Assume that there is progressive phase lag of βd radians in each radiator.

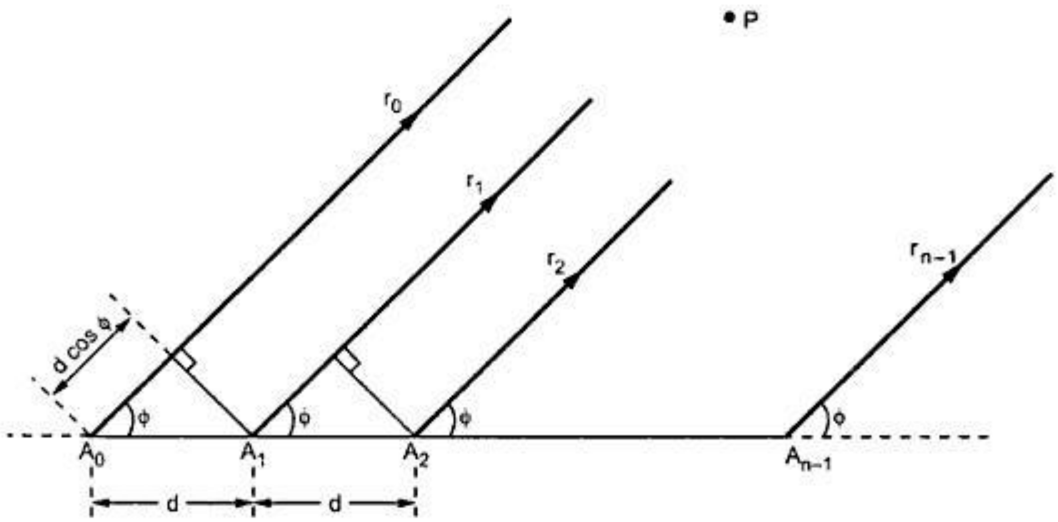


Fig.11 End fire array

Consider that the current supplied to first element A_0 be I_0. Then the current supplied to A_1 is given by,

$$I_1 = I_0 \cdot e^{-j\beta d}$$

Similarly the current supplied to A_2 is given by,

$$I_2 = I_1 \cdot e^{-j\beta d} = \left[I_0 \cdot e^{-j\beta d}\right] e^{-j\beta d} = I_0 \cdot e^{-j2\beta d}$$

Thus the current supplied to last element is

$$I_{n-1} = I_0 \, e^{-j(n-1)\beta d}$$

The electric field produced at point P, due to A_0 is given by,

$$E_0 = \frac{I\,dL\sin\theta}{4\pi\omega\varepsilon_0}\left[j\frac{\beta^2}{r_0}\right]e^{-j\beta r_0} \quad \dots (1)$$

The electric field produced at point P, due to A_1 is given by,

$$E_1 = \frac{I\,dL\sin\theta}{4\pi\omega\varepsilon_0}\left[j\frac{\beta^2}{r_1}\right]e^{-j\beta r_1}\cdot e^{-j\beta d}$$

But $r_1 = r_0 - d\cos\upsilon$

$$\therefore \quad E_1 = \frac{I\,dL\sin\theta}{4\pi\omega\varepsilon_0}\left[j\frac{\beta^2}{r_0}\right]e^{-j\beta(r_0 - d\cos\phi)}\cdot e^{-j\beta d}$$

$$\therefore \quad E_1 = \left[\frac{I\,dL\sin\theta}{4\pi\omega\varepsilon_0}\left[j\frac{\beta^2}{r_0}\right]e^{-j\beta r_0}\right]e^{j\beta d\cos\phi}\cdot e^{-j\beta d}$$

$$\therefore \quad E_1 = E_0\cdot e^{j\beta d(\cos\phi - 1)} \quad \dots (2)$$

Let $\upsilon = \beta d(\cos\upsilon - 1)$

$$\therefore \quad E_1 = E_0\, e^{j\psi} \quad \dots (3)$$

The electric field produced at point P, due to A_2 is given by,

$$E_2 = E_0\cdot e^{j2\psi} \quad \dots (4)$$

Similarly electric field produced at point P, due to A_{n-1} is given by,

$$E_{n-1} = E_0\, e^{j(n-1)\psi} \quad \dots (5)$$

The resultant field at point p is given by,

$$E_T = E_0 + E_1 + E_2 + \dots + E_{n-1}$$

$$\therefore \quad E_T = E_0 + E_0 e^{j\psi} + E_0 e^{j2\psi} + \dots + E_0 e^{j(n-1)\psi}$$

$$\therefore \quad E_T = E_0\left[1 + e^{j\psi} + e^{j2\psi} + \dots + e^{j(n-1)\psi}\right] \quad \dots (6)$$

$$E_T = E_0\cdot\frac{1 - e^{jn\psi}}{1 - e^{j\psi}}$$

$$\therefore \quad \frac{E_T}{E_0} = \frac{\sin\frac{n\psi}{2}}{\sin\frac{\psi}{2}}\cdot e^{j\frac{(n-1)}{2}\psi} \quad \dots (7)$$

Considering only magnitude we get,

$$\therefore \quad \boxed{\left|\frac{E_T}{E_0}\right| = \frac{\sin\frac{n\psi}{2}}{\sin\frac{\psi}{2}}} \quad \dots (8)$$

Properties of End Fire Array

1. Major lobe

For the end fire array where currents supplied to the antennas are equal in amplitude but the phase changes progressively through array, the phase angle is given by,

$\upsilon = \beta d(\cos\upsilon\ -1)$ **...(9)**

In case of the end fire array, the condition of principle maxima is given by,

$\upsilon =$ = 0 i.e.

$$\boxed{\beta d(\cos\phi - 1) = 0} \quad \ldots(10)$$

i.e. $\cos\upsilon$

$= 1$

i.e. $\upsilon = 0^0$ **...(11)**

Thus $\upsilon = 0^0$ indicates the direction of principle maxima.

2. Magnitude of the major lobe

The maximum radiation occurs when $\upsilon = 0$. Thus we can write,

$$|\text{Major lobe}| = \lim_{\psi \to 0}\left\{\frac{\frac{d}{d\psi}\left(\sin n\frac{\psi}{2}\right)}{\frac{d}{d\psi}\left(\sin\frac{\psi}{2}\right)}\right\} = \lim_{\psi \to 0}\left\{\frac{\left(\cos n\frac{\psi}{2}\right)\left(n\frac{\psi}{2}\right)}{\left(\cos\frac{\psi}{2}\right)\left(\frac{\psi}{2}\right)}\right\}$$

$$\therefore \quad \boxed{|\text{Major lobe}| = n} \quad \ldots(12)$$

where, n is the number of elements in the array.

3. Nulls

The ratio of total electric field to an individual electric field is given by,

$$\left|\frac{E_T}{E_0}\right| = \frac{\sin n\frac{\psi}{2}}{\sin\frac{\psi}{2}}$$

Equating ratio of magnitudes of the fields to zero,

$$\therefore \quad \left|\frac{E_T}{E_0}\right| = \frac{\sin n\frac{\psi}{2}}{\sin\frac{\psi}{2}} = 0$$

The condition of minima is given by,

$$\boxed{\sin n\frac{\Psi}{2} = 0, \quad \text{but } \sin\frac{\Psi}{2} \neq 0} \qquad \ldots(13)$$

Henc

e we can write,

$$\sin n\frac{\Psi}{2} = 0$$

$$\text{i.e. } n\frac{\Psi}{2} = \sin^{-1}(0) = \pm m\pi, \text{ where } m = 1, 2, 3, \ldots\ldots$$

Substituting value of υ from equation (9), we get,

$$\therefore \frac{n\beta d(\cos\phi - 1)}{2} = \pm m\pi$$

But β= 2π/λ

$$\therefore \frac{nd}{\lambda}(\cos\phi - 1) = \pm m \qquad \ldots(14)$$

Note that value of (cosυ-1) is always less than 1. Hence it is always negative. Hence only considering -ve values, R.H.S., we get

$$\frac{nd}{\lambda}(\cos\phi - 1) = -m$$

$$\text{i.e. } \cos\phi - 1 = -\frac{m\lambda}{nd}$$

$$\boxed{\phi_{min} = \cos^{-1}\left[1 - \frac{m\lambda}{nd}\right]} \qquad \ldots(15)$$

where, n= number of elements in the array d=

spacing between elements in meter

λ= wavelength in meter

m= constant= 1, 2 , 3....

Thus equation (15) gives direction of nulls

Consider equation(14),

$$\cos\phi_{min} - 1 = \pm\frac{m\lambda}{nd}$$

Expressing term on L.H.S. in terms of halfangles, we get,

$$2\sin^2\frac{\phi_{min}}{2} = \pm\frac{m\lambda}{nd} \qquad \ldots\left(\cos\theta - 1 = 2\sin^2\frac{\theta}{2}\right)$$

$$\therefore \quad \sin^2 \frac{\phi_{min}}{2} = \pm \frac{m\lambda}{2nd}$$

$$\therefore \quad \boxed{\phi_{min} = 2\sin^{-1}\left[\pm\sqrt{\frac{m\lambda}{2nd}}\right]} \qquad \ldots(16)$$

4. Side Lobes Maxima

The directions of the subsidary maxima or side lobes maxima can be obtained if in equation (8),

$$\sin\left(n\frac{\psi}{2}\right) = \pm 1$$

$$\therefore \quad \boxed{n\frac{\psi}{2} = \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots\ldots} \qquad \ldots(17)$$

Hence sin(nυ/2), is not considered. Because if nυ/2=±π/2 then sin nυ/2 =1 which is the direction of principle maxima.

Hence we can skip sin nυ/2 = ±π/2 value Thus, we get

$$\frac{n\psi}{2} = \pm(2m+1)\frac{\pi}{2}, \text{ where } m = 1, 2, 3, \ldots\ldots$$

Putting value of υ from equation (9) we get

$$\frac{n\beta d(\cos\phi - 1)}{2} = \pm(2m+1)\frac{\pi}{2}$$

$$\therefore \; n\beta d(\cos\phi - 1) = \pm(2m+1)\pi$$

Now equation for υ can be written as,But β = 2π/λ

$$n\left(\frac{2\pi}{\lambda}\right)d(\cos\phi - 1) = \pm(2m+1)\pi$$

$$\text{i.e. } \cos\phi - 1 = \pm(2m+1)\frac{\lambda}{2nd}$$

Note that value of (cosυ-1) is always less than 1. Hence it is always negative. Hence only considering -ve values, R.H.S., we get

$$\cos\phi - 1 = -(2m+1)\frac{\lambda}{2nd}$$

i.e. $$\cos\phi = 1-(2m+1)\frac{\lambda}{2nd}$$

i.e. $$\boxed{\phi = \cos^{-1}\left[1-\frac{(2m+1)\lambda}{2nd}\right]} \qquad \ldots(18)$$

5. **Beamwidth of Major Lobe**

Beamwidth is defined as the angle between first nulls. Alternatively beamwidth is the angle equal to twice the angle between first null and the major lobe maximum direction.

From equation (16) we get,

$$\phi_{min} = 2\sin^{-1}\left[\pm\sqrt{\frac{m\lambda}{2nd}}\right] \qquad \ldots(19)$$

$$\therefore \quad \sin\frac{\phi_{min}}{2} = \pm\sqrt{\frac{m\lambda}{2nd}}$$

$\upsilon_{min}$is very low

Hence sin $\upsilon_{min}/2 \approx \upsilon_{min}/2$

$$\frac{\phi_{min}}{2} = \pm\sqrt{\frac{m\lambda}{2nd}}$$

$$\phi_{min} = \pm\sqrt{\frac{4m\lambda}{2nd}} = \pm\sqrt{\frac{2m\lambda}{nd}} \qquad \ldots(20)$$

But nd$\approx$ (n-1)d if n is very large. This L= (nd) indicates total length of the array. So equation (20) becomes,

$$\phi_{min} = \pm\sqrt{\frac{2m\lambda}{L}} = \pm\sqrt{\frac{2m}{L/\lambda}} \qquad \ldots(21)$$

BWFN is given by,

$$\boxed{BWFN = 2\phi_{min} = \pm 2\sqrt{\frac{2m}{L/\lambda}}} \qquad \ldots(22)$$

BWFN in degree is expressed as

$$BWFN = \pm 2\sqrt{\frac{2m}{L/\lambda}} \times 57.3 = \pm 114.6\sqrt{\frac{2m}{L/\lambda}} \text{ degree}$$

For m=1,

$$\boxed{\text{BWFN} = \pm 2\sqrt{\frac{2}{L/\lambda}}\ \text{rad} = 114.6\sqrt{\frac{2}{L/\lambda}}\ \text{degree}} \qquad \ldots(23)$$

6. Directivity

The directivity in case of endfire array is defined as,

$$G_{Dmax} = \frac{\text{Maximum radiation intensity}}{\text{Average radiation intensity}} = \frac{U_{max}}{U_{avg}} = \frac{U_{max}}{U_0} \qquad \ldots(23)$$

where, U_0 is average radiation intensity which is given by, For endfire array, U_{max} = 1and $U_0 = \frac{\pi}{2n\beta d}$

$$\therefore \quad G_{Dmax} = \frac{1}{\frac{\pi}{2n\beta d}} = \frac{2n\beta d}{\pi}$$

$$\therefore \quad G_{Dmax} = 2n\left(\frac{2\pi}{\lambda}\right)\cdot\frac{d}{\pi}$$

$$\therefore \quad \boxed{G_{Dmax} = 4\left(\frac{nd}{\lambda}\right)} \qquad \ldots(24)$$

The total length of the array is given by, L = (n - 1) d ≈ nd, if n is very large. Hence the directivity can be expressed in terms of the total length of the array as,

$$\therefore \quad \boxed{G_{Dmax} = 4\left(\frac{L}{\lambda}\right)} \qquad \ldots(25)$$

Multiplication of patterns

In the previous sections we have discussed the arrays of two isotropic point sources radiating field of constant magnitude. In this section the concept of array is extended to non-isotropic sources. The sources identical to point source and having field patterns of definite shape and orientation. However, it is not necessary that amplitude of individual sources is equal. The simplest case of non-isotropic sources is when two short dipoles are superimposed over the two isotopic point sources separated by a finite distance. If the field pattern of each source is given by

$$E_0 = E_1 = E_2 = E' \sin\,\theta$$

Then the total far-field pattern at point P becomes

$$E_T = 2E_0 \cos\left(\frac{\psi}{2}\right) = 2E' \sin\theta \cos\left(\frac{\psi}{2}\right) \Rightarrow E_{Tn} = \sin\theta \cos\left(\frac{\psi}{2}\right) \qquad \text{...(1)}$$

$$E_{Tn} = E(\theta) \times \cos\left(\frac{\psi}{2}\right)$$

where

$$\psi = \left(\frac{2\pi d}{\lambda}\cos\theta + \alpha\right)$$

Equation (1) shows that the field pattern of two non-isotropic point sources (short dipoles) is equal to product of patterns of individual sources and of array of point sources. The pattern of array of two isotropic point sources, i.e., cos υ/2 is widely referred as an array factor. That is

E_T= E (Due to reference source) x Array factor

This leads to the principle of pattern multiplication for the array of identical elements. In general, the principle of pattern multiplication can he stated as follows:

The resultant field of an array of non-isotropic hut similar sources is the product of the fields of individual source and the field of an array of isotropic point sources, each located at the phase centre of individual source and hating the relative amplitude and phase. The total phase is addition of the phases of the individual source and that of isotropic point sources. The same is true for their respective patterns also.

The normalized total field (i.e., E_{Tn}), given in Eq. (1), can re-written as

$$E = E_1(\theta) \times E_2(\theta)$$

where $E_1(\theta)$ = sin θ = Primary pattern of array

$$E_2(\theta) = \cos\left(\frac{2\pi d}{\lambda}\cos\theta + \alpha\right)$$ = Secondary pattern of array.

Thus the principle of pattern multiplication is a speedy method of sketching the field pattern of complicated array. It also plays an important role in designing an array. There is no restriction on the number of elements in an array; the method is valid to any number of identical elements which need not have identical magnitudes, phase and spacing between then). However, the array factor varies with the number of elements and their arrangement, relative magnitudes, relative phases and element spacing. The array of elements having identical amplitudes, phases and spacing provides a simple array factor. The array factor does not depend on the directional characteristic of the array elements; hence it can be formulated by using pattern multiplication techniques. The proper selection of the individual radiating element and their excitation are also important for the performance of array. Once the array factor is derived using the point-source array, the total field of the actual array can be obtained using Eq. (2).

Binomial Array

In order to increase the directivity of an array its total length need to be increased. In this approach, number of minor lobes appears which are undesired for narrow beam applications. In has been found that number of minor lobes in the resultant pattern increases whenever spacing between elements is greater than $\lambda/2$. As per the demand of modern communication where narrow beam (no minor lobes) is preferred, it is the greatest need to design an array of only main lobes. The ratio of power density of main lobe to power density of the longest minor lobe is termed side lobe ratio. A particular technique used to reduce side lobe level is called tapering. Since currents/amplitude in the sources of a linear array is non-uniform, it is found that minor lobes can be eliminated if the centre element radiates more strongly than the other sources. Therefore tapering need to be done from centre to end radiators of same specifications. The principle of tapering are primarily intended to broadside array but it is also applicable to end-fire array. Binomial array is a common example of tapering scheme and it is an array of n-isotropic sources of non-equal amplitudes. Using principle of pattern multiplication, John Stone first proposed the binomial array in 1929, where amplitude of the radiating sources arc arranged according to the binomial expansion. That is. if minor lobes appearing in the array need to be eliminated, the radiating sources must have current amplitudes proportional to the coefficient of binomial series, i.e. proportional to the coefficient of binomial series, i.e.

$$(1+x)^n = 1 + (n-1)x + \frac{(n-1)(n-2)}{!2}x^2 + \frac{(n-1)(n-2)(n-3)}{!3}x^3 \pm \cdots \qquad \textbf{...(1)}$$

where n is the number of radiating sources in the array.

For an array of total length $n\lambda/2$, the relative current in the nth element from the one end is given by

$$= \frac{n!}{r!(n-r)!}$$

where r = 0, 1, 2, 3, and the above relation is equivalent to what is known as Pascal's triangle.

For example, the relative amplitudes for the array of 1 to 10 radiating sources are as follows:

No. of sources	*Pascal's triangle*
$n = 1$	1
$n = 2$	1 1
$n = 3$	1 2 1
$n = 4$	1 3 3 1
$n = 5$	1 4 6 4 1
$n = 6$	1 5 10 10 5 1
$n = 7$	1 6 15 20 15 6 1
$n = 8$	1 7 21 35 35 21 7 1
$n = 9$	1 8 28 56 70 56 28 8 1
$n = 10$	1 9 36 84 126 126 84 36 9 1

Since in binomial array the elements spacing is less than or equal to the half-wave length, the HPBW of the array is given by

$$\text{HPBW} = \frac{10.6}{\sqrt{n-1}} = \frac{1.06}{\sqrt{\frac{2L}{\lambda}}} = \frac{0.75}{\sqrt{L_\lambda}}$$

and directivity

$$D_0 = 1.77\sqrt{n} = 1.77\sqrt{1 + 2L_\lambda}$$

Using principle of multiplication, the resultant radiation pattern of an n-source binomial array is given by

In particular, if identical array of two point sources is superimposed one above other, then three effective sources with amplitude ratio 1:2:1 results. Similarly, in case three such elements are superimposed in same fashion, then an array of four sources is obtained whose current amplitudes are in the ratio of 1:3:3:1.

The far-field pattern can be found by substituting n = 3 and 4 in the above expression and they take shape as shown in Fig. 14(a) and (b).

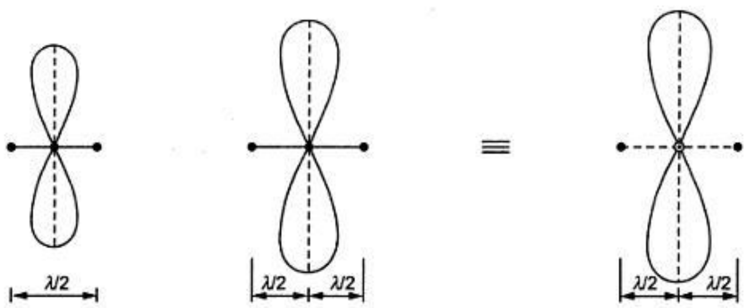


Fig. 14(a) Radiation pattern of 2-element array with amplitude ratio 1:2:1.

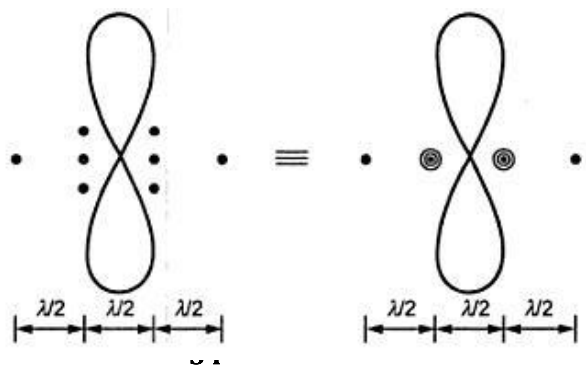


Fig 14(b) Radiation pattern of 3-element array with amplitude ratio 1:3:3:1.

It has also been noticed that binomial array offers single beam radiation at the cost of directivity, the directivity of binomial array is greater than that of uniform array for the same length of the array. In other words, in uniform array secondary lobes appear, but principle lobes are narrower than that of the binomial array.

Disadvantages of Binomial Array

(a) The side lobes are eliminated but the directivity of array reduced.

(b) As the length of array increases, larger current amplitude ratios are required.