Unit 2
DIMENSIONAL ANALYSIS AND FLUID FLOW IN CLOSED CONDUICTS

TYPICAL MODEL STUDIES

Flow through Closed Conduits

When there is a fluid flow through closed conduits (such as pipe flows), the dominant forces are inertial and viscous because there is no fluid interface. Compressibility effects can be neglected for low Mach numbers (less than 0.3). In these class of problems, geometric similarity between the model and prototype must be maintained. Geometric similarity characteristics are described by series of length terms $l_1, l_2, l_3 ......, l_i$ and l, where l is some particular length dimension of the system. It leads to series of pi terms of the form as,

$$\Pi_i = \frac{l_i}{l} \text{ where } i = 1, 2, ...$$

In addition, other parameter of importance is the surface roughness (ε). The corresponding pi term representing the surface roughness is ε/l.

For low Mach number flows, Reynolds number must also match. Complete similarity requirements are as follows.

$$\frac{l_{im}}{l_m} = \frac{l_i}{l};\ \frac{\varepsilon_m}{l_m} = \frac{\varepsilon_i}{l};\ \frac{\rho_m V_m l_m}{\mu_m} = \frac{\rho V l}{\mu} \tag{1}$$

In order to find the pressure differential per unit length, the other dependent pi term is expressed as,

$$\Pi_1 = \frac{\Delta p}{\rho . V^2}$$

The prototype pressure drop can be obtained from the relation as,

$$\Delta p = \left(\frac{\rho}{\rho_m}\right)\left(\frac{V}{V_m}\right)^2 \Delta p_m \tag{2}$$

Flow around Immersed Bodies

Typical examples that fall under this category are the flow around aircraft, automobiles etc. The general formulations for these classes of problems are expressed in terms of dependent pi terms i.e.

$$\text{Dependent } pi \text{ terms} = \phi\left(\frac{l_i}{l}, \frac{\varepsilon}{l}, \frac{\rho V l}{\mu}\right) \tag{3}$$

For complete similarity requirements Eq. (1) must be satisfied. The parameter of interest in this type of problems is the drag coefficient (C_D) and is defined by,

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 l^2}$$

If the similarity conditions are met, then

$$\left(\frac{F_D}{\frac{1}{2}\rho V^2 l^2}\right)_{\text{model}} = \left(\frac{F_D}{\frac{1}{2}\rho V^2 l^2}\right)_{\text{prototype}} \tag{4}$$

Flow with a Free Surface

The flows in canals, rivers, spillways, around ships are some of the examples involving a free surface. In these flow patterns, following forces and numbers are of important. They are,

- Gravitational force (Froude number)
- Surface tension (Weber number)
- Inertia force (Reynolds number)

Thus general formulation for problems involving the flow with a free surface is expressed as,

$$\text{Dependent } pi \text{ terms} = \varphi\left(\frac{l_i}{l}, \frac{\varepsilon}{l}, \frac{\rho V l}{\mu}, \frac{V}{\sqrt{g\,l}}, \frac{\rho V^2 l}{\sigma}\right) \tag{5}$$

Since gravity is the driving force, Froude number similarity must be maintained i.e.

$$\frac{V_m}{\sqrt{g_m\, l_m}} = \frac{V_p}{\sqrt{g_p\, l_p}}$$

The model and prototype is expected to operate in same gravitational field. So,

$$\frac{V_m}{V_p} = \sqrt{\frac{l_m}{l_p}} \tag{6}$$

In many of the practical problems, involving free surface flows, both surface tension and viscous effects are small and strict adherence to Weber and Reynolds number is not required.

Turbo-machinery models

There are certain situations of practical interest in which there may be more than one dependent parameter. So, scaling is done for the dimensional groups with multiple dependent parameters. A typical example is centrifugal pump. The performance parameters of interest for a centrifugal pump include the pressure rise (or head) developed, the power input, and machine efficiency measured under specific operating conditions. These performance curves are usually generated by varying independent parameters. The independent variables are volume flow rate (Q), angular speed (ω), impeller diameter (D), and fluid properties like density (ρ), viscosity (μ). Dependent variables include the performance quantities.

The dependence of head (h) developed and power (P) can be written in terms of independent parameters as,

$$\begin{aligned} h &= \phi\left(Q, \rho, \omega, D, \mu\right) \\ P &= \varphi\left(Q, \rho, \omega, D, \mu\right) \end{aligned} \tag{18}$$

Using Pi theorem, the dimensionless parameters can be written as,

$$\frac{h}{\omega^2 D^2} = g_1\left(\frac{Q}{\omega D^3}, \frac{\rho\omega D^2}{\mu}\right) \tag{19}$$

$$\frac{P}{\rho\omega^3 D^5} = g_2\left(\frac{Q}{\omega D^3}, \frac{\rho\omega D^2}{\mu}\right) \tag{20}$$

The dimensionless parameters are,

- $\text{Flow coefficient} = \dfrac{Q}{\omega D^3}$
- $\text{Power coefficient} = \dfrac{P}{\rho\,\omega^3 D^5}$
- $\text{Reynolds number} = \dfrac{\rho\,\omega D^2}{\mu}$

Complete similarity in pump performance tests requires identical flow coefficients and Reynolds number. Thus form Eqs. (19) and (20), when

$$\frac{Q_1}{\omega_1 D_1^3} = \frac{Q_2}{\omega_2 D_2^3} \tag{21}$$

It follows that,

$$\frac{h_1}{\omega_1^2 D_1^2} = \frac{h_2}{\omega_2^2 D_2^2} \tag{22}$$

and

$$\frac{P_1}{\rho_1 \omega_1^3 D_1^5} = \frac{P_2}{\rho_2 \omega_2^3 D_2^5} \tag{23}$$

These useful scaling relationships are known as "pump laws". If operating conditions of one pump are known, the operating conditions for any geometrically similar machine can be found by changing D , ω in Eqs. (21) to (23).

Another dimensionless parameter is the "specific speed" which is defined as the speed required for a pump to produce unit head at unit volume flow rate i.e.

$$N_s = \frac{\omega Q^{0.5}}{h^{0.75}} \tag{24}$$

A constant specific speed describes all operating conditions of geometrically similar pumps with similar flow conditions.

<u>Example 1</u>

Water flows through a large valve having 0.8m diameter at a rate 85m^3/s. It is to be tested in a geometrically similar model of 10cm diameter with water as working fluid. Determine the required flow rate in the model.

<u>Solution</u>

In order to ensure the dynamic similarity, the Reynolds number must match for the model and the prototype i.e.

$$\left(R_e\right)_{\text{model}} = \left(R_e\right)_{\text{prototype}}$$

$$\frac{V_m D_m}{\nu_m} = \frac{V_p D_p}{\nu_p}$$

Since, water is used as the working fluid for the model and the prototype, so $\nu_m = \nu_p$. Hence,

$$\frac{V_m}{V_p} = \frac{D_p}{D_m}$$

The flow rate for the model and prototype can be related as,

$$\frac{Q_m}{Q_p} = \left(\frac{D_p}{D_m}\right)\left(\frac{[\pi/4]D_m^2}{[\pi/4]D_p^2}\right) = \frac{D_m}{D_p}$$

Hence,

$$Q_m = Q_p \frac{D_m}{D_p} = 85 \times \left(\frac{0.1}{0.8}\right) = 10.6\,\mathrm{m}^3/\mathrm{s}$$

Example-2

In order to estimate the drag force on an airplane that cruises at 100m/s, wind-tunnel test is carried out on a 1: 15 scale model. The airplane cruises at an altitude where there is 10% drop in standard atmospheric pressure. The wind tunnel test is also carried out at 100m/s and the measured drag force in one of the test is 5N. Determine the required air pressure in the tunnel and drag force on the prototype (assuming the same air temperature for the model and prototype).

Solution

For a geometrically similar model and prototype, the Reynolds number must match i.e.

$$(R_e)_{\text{model}} = (R_e)_{\text{prototype}}$$

$$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_p V_p D_p}{\mu_p}$$

In this case, $\frac{V_m}{V_p} = 1$; $\frac{l_m}{l_p} = \frac{1}{15}$, so that

$$\frac{\rho_m}{\rho_p} = \left(\frac{\mu_m}{\mu_p}\right)\left(\frac{V_p}{V_m}\right)\left(\frac{l_p}{l_m}\right) = \left(\frac{\mu_m}{\mu_p}\right)(1)(10) = 15\left(\frac{\mu_m}{\mu_p}\right)$$

It shows that same fluid with $\rho_m = \rho_p$; $\mu_m = \mu_p$ cannot be used to maintain the Reynolds number similarity. However, one possible solution is to use same fluid (air) for which

$\mu_m = \mu_p$, but increase the pressure to increase the density to a limit such that variation in viscosity is not significant. In that case,

$$\frac{\rho_m}{\rho_p} = 15$$

Using the ideal gas equation of state, $p = \rho.R.T$, we get,

$$\frac{\rho_m}{\rho_p} = \frac{p_m}{p_p} \text{ (Assuming constant temperature)}$$

The prototype cruises at an altitude where there is 10% drop in standard atmosphere i.e. at a pressure of 0.9atm. So, required pressure in the wind tunnel will be,

$$p_m = 15 \times 0.9 = 13.5\text{atm.}$$

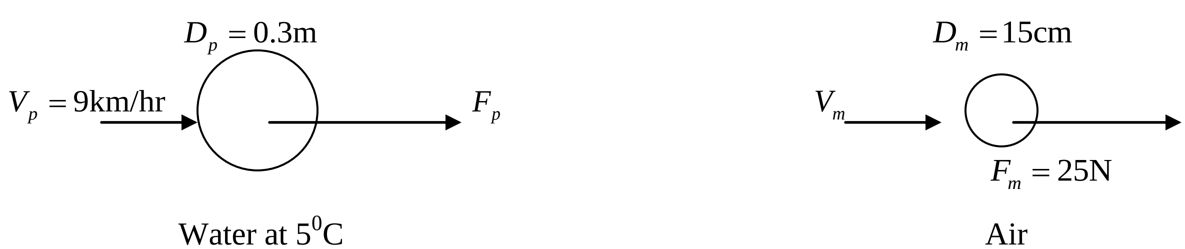
Using Eq. (4), the drag force on the prototype is estimated as,

$$\left(F_D\right)_{\text{prototype}} = \left(\frac{\rho_m}{\rho_p}\right)\left(\frac{V_p}{V_m}\right)^2\left(\frac{l_p}{l_m}\right)^2\left(F_D\right)_{\text{model}} = 15\left(F_D\right)_{\text{model}} = 75\text{N}$$

<u>Example 3</u>

A sonar transducer is a prototype sphere of 0.3m-diameter towing at 9km/hr in seawater at 5^0C. The drag on this transducer is to be predicted from wind tunnel data on a model of 15cm diameter. Determine the required test speed in the wind tunnel. If the drag of the model at test condition is 25N, estimate the drag on the prototype.

<u>Solution</u>



In order to estimate test speed and drag on the prototype, the following condition must be satisfied, i.e.

$$\frac{F}{\rho V^2 D^2} = g\left(\frac{\rho V D}{\mu}\right)$$

For ensuring the dynamic similarity, the test should run such that

$$\left(R_e\right)_{\text{model}} = \left(R_e\right)_{\text{prototype}}$$

For seawater at 5^0C, $\rho_p = 1000\,\text{kg}/\text{m}^3$, $\nu_p = 1.57\times10^{-6}\,\text{m}^2/\text{s}$; At prototype conditions, $V_p = 9\,\text{km/hr} = 2.5\,\text{m/s}$. So,

$$\left(R_e\right)_{\text{prototype}} = \frac{V_p\,D_p}{\nu_p} = 4.77\times10^5$$

The model must be tested at same Reynolds number, i.e.

$$\left(R_e\right)_{\text{model}} = \frac{V_m\,D_m}{\nu_m} = 4.77\times10^5$$

Now, for air at standard temperature and pressure conditions (STP), $\rho_m = 1.2\,\text{kg}/\text{m}^3$, $\nu_m = 1.5\times10^{-5}\,\text{m}^2/\text{s}$. So,

$$V_m = \left(R_e\right)_{\text{model}}\left(\frac{\nu_m}{D_m}\right) = 47.7\,\text{m/s}$$

At these conditions, the model and prototype are dynamic similar. Hence,

$$\frac{F_m}{\rho_m V_m^2 D_m^2} = \frac{F_p}{\rho_p V_p^2 D_p^2}$$

so that

$$F_p = F_m\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p^2}{V_m^2}\right)\left(\frac{D_p^2}{D_m^2}\right) = 229\text{N}$$

Example 4

A spillway model of 1:10 scale is constructed to study the flow characteristics for a prototype dam of width 10m and to carry water at a flow rate of 60m^3/s. Determine the required model width and flow rate. What operating time for the model corresponds to a 24hr period in prototype? The effects of surface tension and viscosity may be neglected.

Solution

If w_m, w_p are the width of the model and prototype respectively, then

$$\frac{w_m}{w_p} = \frac{1}{15}, \text{ i.e. } w_m = \frac{10}{15} = 0.67\text{m}$$

Since surface tension and viscosity are insignificant, so there is no need for considering Weber number and Reynolds number. However, the Froude number must match i.e.

$$\frac{V_m}{V_p} = \sqrt{\frac{l_m}{l_p}}$$

The flow rate is given by,

$$\frac{Q_m}{Q_p} = \frac{V_m\, A_m}{V_p\, A_p} = \left(\frac{l_m}{l_p}\right)^{\frac{5}{2}}$$

so that,

$$Q_m = \left(\frac{1}{10}\right)^{\frac{5}{2}} \times 60 = 0.19\,\mathrm{m}^3/\mathrm{s}$$

The time scale can be obtained as,

$$\frac{V_p}{V_m} = \left(\frac{l_p}{t_p}\right)\left(\frac{t_m}{l_m}\right)$$

i.e.

$$\frac{t_m}{l_m} = \left(\frac{V_p}{V_m}\right)\left(\frac{l_m}{l_p}\right) = \sqrt{\frac{l_m}{l_p}}$$

so that

$$t_m = \sqrt{\frac{1}{10}} \times 24\mathrm{hr} = 7.6\mathrm{hr}$$

<u>Example 5</u>

A centrifugal pump has a specific speed of 5000 when operated at 1170rpm and volume flow rate of 300gallons/min. In order to increase flow rate, it is fitted with same size motor operating at 1750rpm. Determine the flow rate, head developed, power required by the pump at this conditions. Show that specific speed remains constant at this speed.

<u>Solutions</u>

From “pump laws”, $\dfrac{Q_1}{\omega_1\, D_1^3} = \dfrac{Q_2}{\omega_2\, D_2^3}$,

$$\Rightarrow Q_2 = Q_1\left(\frac{\omega_2}{\omega_1}\right)\left(\frac{D_2^3}{D_1^3}\right) = 300\times\left(\frac{1750}{1170}\right)(1)^{3} = 449\,\text{gallons/min}$$

The pump head can be calculated from specific speed, i.e.

$$N_s = \frac{\omega Q^{0.5}}{h^{0.75}}$$

$$\Rightarrow h_1 = \left(\frac{\omega_1 Q_1^{0.5}}{N_s}\right)^{1.333} = \left(\frac{1170\times\sqrt{300}}{5000}\right)^{1.333} = 6.462\text{m}$$

Again from "pump laws" for head developed,

$$h_2 = h_1\left(\frac{\omega_2}{\omega_1}\right)^2\left(\frac{D_2}{D_1}\right)^2 = 6.462\times\left(\frac{1750}{1170}\right)^2\times(1)^3 = 14.45\text{m}$$

Pump power at 1170rpm is given by,

$$P_1 = \rho\, g\, Q_1\, H_1 = 1000\times 9.81\times 300\times 6.309\times 10^{-5}\times 6.462 = 1.2\text{kW}$$

$$(1\text{gallon/min} = 6.309\times10^{-5}\text{m}^3\text{/s})$$

From pump laws, pump power at 1750rpm is,

$$P_2 = P_1\left(\frac{\rho_2}{\rho_1}\right)\left(\frac{\omega_2}{\omega_1}\right)^3\left(\frac{D_2}{D_1}\right)^5 = 1.2\times(1)\times\left(\frac{1750}{1170}\right)^2\times(1)^5 = 4.23\text{kW}$$

Specific speed at 1750rpm is given by,

$$N_s = \frac{\omega Q^{0.5}}{h^{0.75}} = \frac{1750\times\sqrt{449}}{(14.45)^{0.75}} = 5003$$

EXERCISES

1. The capillary rise (h) of a liquid in a tube varies with the tube diameter (d), gravity (g), fluid density (ρ), surface tension (σ) and contact angle (θ). Find the dimensionless relation. If h = 9cm in a given experiment, then what will be the h in a similar case for which the diameter and surface tension is halved, density being twice and contact angle being the same.

2. A large hydraulic turbine is to generate 300kW at 1000rpm under a head of 40m. For initial testing, a 1:4 scale model of the turbine operates under a head of 10m. Find the power generated by the model.