

Unit 1

FLUID PROPERTIES & EQUATIONS OF MOTION

Physical Properties of Fluids

Introduction:

A treatment of fluid flow requires an understanding of the physical properties of a fluid which affect the motion, and an appreciation of the units in which these properties are measured. The three most important fluid properties are the *density*, *viscosity* and *surface tension*. Density is a familiar concept and is treated briefly, although there is one subtlety that requires attention. Viscosity is a concept that is generally understood intuitively, but some detailed consideration is necessary in order to establish a precise and useful definition. Each of these will be defined and viewed in terms of molecular concepts, and their dimensions will be examined in terms of mass, length and time (M, L and T).

Density:

The *density* ρ of a fluid is defined as its mass per unit volume and indicates its inertia or resistance to an accelerating force. Thus

$$\rho = \frac{\text{mass}}{\text{volume}} [=] \frac{M}{L^3} \quad (2.1)$$

in which the notation “[=]” is consistently used to indicate the *dimensions* of a quantity. It is usually understood in Eqn. 2.1 that the volume is chosen so that it is neither so small that it has no chance of containing a representative selection of molecules, nor is it so large that (in the case of gases) changes of pressure cause significant changes of density throughout the volume. A medium characterized by a density is called a *continuum*, and follows the classical laws of mechanics – including Newton’s law of motion.

Densities of liquids. Density depends on the mass of an individual molecule and the number of such molecules that occupy a unit of volume. For liquids, density depends primarily on the particular

liquid and, to a much smaller extent, on its temperature. Representative densities are given in table 2.1. The accuracy of the values given in tables 2.1 through 2.3 is adequate for the calculation needed for this course. However, if highly accurate values are needed, particularly at extreme conditions, then specialized information should be sought elsewhere.

Degrees A.P.I. (American Petroleum Institute) are related to specific gravity s by the formula:

$$^{\circ}\text{A.P.I.} = \frac{141.5}{s} - 131.5 \quad (2.2)$$

Note that for water, $^{\circ}\text{A.P.I.} = 10$, with correspondingly *higher* values for liquids that are *less* dense. Thus, for the crude oil listed in table 2.1, Eqn. 2.2 indeed gives $141.5 / 0.851 - 131.5 = 35^{\circ}\text{A.P.I.}$

Table 2.1 1.1 *Specific Gravities, Densities, and Expansion Coefficients of Liquids at 20° C*

<i>Liquid</i>	<i>Sp. Gr.</i> s	<i>Density, ρ</i>		α $^{\circ}\text{C}^{-1}$
		kg/m^3	lb_m/ft^3	
Acetone	0.792	792	49.4	0.00149
Benzene	0.879	879	54.9	0.00124
Crude oil, 35°API	0.851	851	53.1	0.00074
Ethanol	0.789	789	49.3	0.00112
Glycerol	1.26 (50 °C)	1,260	78.7	—
Kerosene	0.819	819	51.1	0.00093
Mercury	13.55	13,550	845.9	0.000182
Methanol	0.792	792	49.4	0.00120
<i>n</i> -Octane	0.703	703	43.9	—
<i>n</i> -Pentane	0.630	630	39.3	0.00161
Water	0.998	998	62.3	0.000207

Density of gases. For ideal gases, $pV = nRT$, where p is the *absolute* pressure, V is the volume of the gas, n is the number of moles, R is the gas constant, and T is the *absolute* temperature. If M_w is the molecular weight of the gas, it follows:

$$\rho = \frac{nM_w}{V} = \frac{M_w p}{RT} \quad (2.3)$$

Thus, the density of an ideal gas depends on the molecular weight, absolute pressure, and absolute temperature. Values of the gas constant R are given in table 2.2 for various systems of units. Note the degrees Kelvin, formerly represented by “°K” is now more simply denoted by “K”.

Table 2.2 *Values of the Gas Constant, R*

<i>Value</i>	<i>Units</i>
8.314	J/g-mole K
0.08314	liter bar/g-mole K
0.08206	liter atm/g-mole K
1.987	cal/g-mole K
10.73	psia ft ³ /lb-mole °R
0.7302	ft ³ atm/lb-mole °R
1,545	ft lb _f /lb-mole °R

For a *nonideal* gas, the compressibility factor Z (a function of p and T) is introduced into the denominator of Eqn. (2.3), giving:

$$\rho = \frac{nM_w}{V} = \frac{M_w p}{ZRT} \quad (2.4)$$

Thus, the extent to which Z deviates from unity gives a measure of the nonideality of the gas.

The *isothermal compressibility* of a gas is defined as:

$$\beta = -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_T \quad (2.5)$$

and equals – at constant temperature – the fractional *decrease* in volume caused by a unit *increase* in the pressure. For an ideal gas, $\beta = 1/p$, the reciprocal of the absolute pressure.

The *coefficient of thermal expansion* α of a material is its isobaric fractional increase in volume per unit rise in temperature:

$$\alpha = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_p \quad (2.6)$$

Since, for a given mass, density is inversely proportional to volume, it follows that for moderate temperature ranges (over which α is essentially constant) the density of most liquids is approximately a linear function of temperature:

$$\rho \doteq \rho_o [1 - \alpha(T - T_o)] \quad (2.7)$$

where ρ_o is the density at a reference temperature T_o . For an ideal gas, $\alpha = 1/T$, the reciprocal of the absolute temperature.

The *specific gravity* s of a fluid is the ratio of the density ρ to the density ρ_{sc} of a reference fluid at some standard condition:

$$s = \frac{\rho}{\rho_{sc}} \quad (2.8)$$

For liquids, ρ_{sc} is usually the density of water at 4°C, which equals 1.000 g / ml or 1,000 kg / m³. For *gases*, ρ_{sc} is sometimes taken as the density of air at 60°F and 14.7 psia, which is approximately 0.0759 lb_m / ft³, and sometimes at 0°C and one atmosphere absolute; since there is no *single* standard for gases, care must obviously be taken when interpreting published values. For *natural gas*, consisting primarily

of methane and other hydrocarbons, the *gas gravity* is defined as the ratio of the molecular weight of the gas to that of air (28.8 lb_m / lb-mole).

Values of the molecular weight M_w are listed in the table 2.3 for several commonly occurring gases, together with their densities at standard conditions of atmospheric pressure and 0°C.

Table 2.3

*3 Gas Molecular Weights and Densities
(The Latter at Atmospheric Pressure and 0 °C)*

<i>Gas</i>	M_w	<i>Standard Density</i>	
		kg/m ³	lb _m /ft ³
Air	28.8	1.29	0.0802
Carbon dioxide	44.0	1.96	0.1225
Ethylene	28.0	1.25	0.0780
Hydrogen	2.0	0.089	0.0056
Methane	16.0	0.714	0.0446
Nitrogen	28.0	1.25	0.0780
Oxygen	32.0	1.43	0.0891

2.3. Continuum Hypothesis:

The density provides a convenient means of illustrating the *continuum hypothesis*, which is one of the fundamental principles utilized in fluid mechanics. Density is defined as mass per unit volume, and it is commonly determined by measuring either the mass of a known volume or the volume displaced by a known mass. We recognize, however that density can vary with position in a process, and a more precise definition is to consider a series of small volumes Δv , with corresponding small masses ΔM , and to define the density ρ by the limiting process.

$$\rho = \lim_{\Delta v \rightarrow 0} \frac{\Delta M}{\Delta v} \quad (2.9)$$

Alternatively, if the density ρ is a function of position, $\rho(x, y, z)$, then the mass contained in a specified volume is the integral of the density over that volume:

$$M = \int \rho(x,y,z) dv \equiv \iiint \rho(x,y,z) dx dy dz \quad (2.10)$$

Now, Eqns. 2.9 and 2.10 have implicitly used the continuum hypothesis, for they assume that the density is a concept that has meaning for any volume, no matter how small. Yet we know that when the volume is of molecular dimensions we have very rapid spatial variations between dense and non-dense regions, depending on how close we are to a molecule. Hence, the notion of a density function that can be defined by a limiting process such as that of Eqn. 2.9 is questionable. *The continuum hypothesis assumes that mathematical limits for volumes tending to zero are reached over a scale that remains large compared to molecular dimensions.* Thus, variations on a molecular scale can be ignored, since they average out over all length scales of interest. Clearly, no result derived using the continuum hypothesis should be expected to apply a rarified gas, where the distance between molecules may be large.

2.4. Viscosity:

The *viscosity* is a measure of resistance to flow. If we tip over a glass of water on the table, the water will spill out before we can stop it. If we tip over a jar of honey, we probably can set it upright again before much honey flow out; this is because the honey has much greater resistance to flow, more viscosity, than water. A more precise definition of viscosity is possible in terms of following experiment:

Consider two long, solid plates separated by a thin film of fluid (see Fig 2.1). If we slide the upper plate steadily in the x direction with velocity V_o , a force is needed to overcome the friction in the fluid between the plates. This force is different for different velocities, plate sizes, fluids, and distances between the plates. We can eliminate the effect of different plate sizes, however, by measuring the force per unit area of the plate, which we define as the shear stress τ .

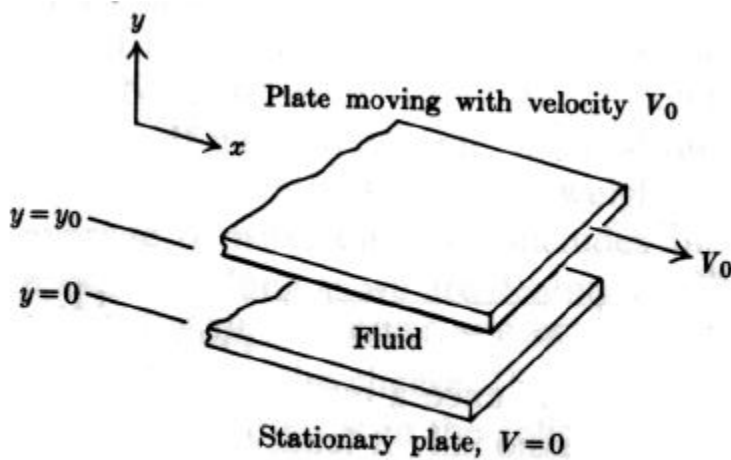


Fig. 2.1

the experiment.

It has been demonstrated experimentally that at low values of V_0 the velocity profile in the fluid between the plates is linear, i.e.,

$$V = \frac{V_0 y}{y_0} \quad (2.11)$$

so that

$$\frac{dV}{dy} = \frac{V_0}{y_0} \quad (2.12)$$

It has also been demonstrated experimentally that for most fluids the results of this experiment can be shown most conveniently on a plot τ versus dV / dy (see Fig. 2.2). As shown here, dV / dy is simply is velocity divided by a distance. In more complex geometries, it is the limiting value of such a ratio at a point. It is commonly called the *shear rate*, *rate of strain*, and *rate of shear deformation*, which all means exactly the same thing. Four different kinds of curve are shown as experiment results in the figure. All four are observed in nature. The behavior most common in nature is that represented by the straight line through the origin. This line is called *Newtonian* because it is described by Newton's law of viscosity

$$\tau = \mu \frac{dV}{dy} \quad (2.13)$$

This equation says that the shear stress τ is linearly proportional to the velocity gradient dV / dy . It is also the definition of viscosity, because we can rearrange it as

$$\mu = \frac{\tau}{dV/dy} \quad (2.14)$$

Here μ is called the *viscosity* or the coefficient of *viscosity*.

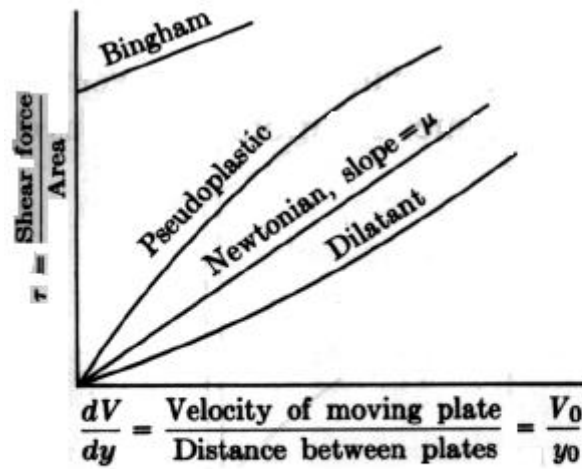


Fig. 2.2

.....e outcomes of the sliding-plate experiment at constant temperature and pressure.

For fluids such as air, the value of μ is very low; therefore, their observed behavior is represented in Fig 2.2 by a straight line through the origin, very close to the dV / dy axis. For fluid such as corn syrup, the value of μ is very large, and the straight line through the origin is close to the τ axis.

Fluids that exhibit this behavior in the sliding plate experiment (i.e., fluids that obey Newton's law of viscosity) are called *Newtonian fluids*. All others are called *nonnewtonian fluids*. Which fluids are Newtonian? All gases are Newtonian. All liquids for which we can write a simple chemical formula are Newtonian, such as water, benzene, ethyl alcohol, carbon tetrachloride, and hexane. Most solutions of simple molecules are Newtonian, such as aqueous solutions of inorganic salts and of sugar. Which fluids are nonnewtonian? Generally, nonnewtonian fluids are complex mixtures: slurries, pastes, gels, polymer solutions, etc. Most nonnewtonian fluids are composed of molecules or particles that are much larger

than water molecules, such as the sand grains in a mud or the collagen molecules in gelatin, which are thousands or millions of times larger than water molecules.

In discussing nonnewtonian fluids, we must agree on what we mean by viscosity. If we retain the definition given by Eqn. 2.12, then the viscosity can no longer be considered a constant independent of dV/dy for a given temperature, but must be considered a function of dV/dy . This is shown in figure 2.3.

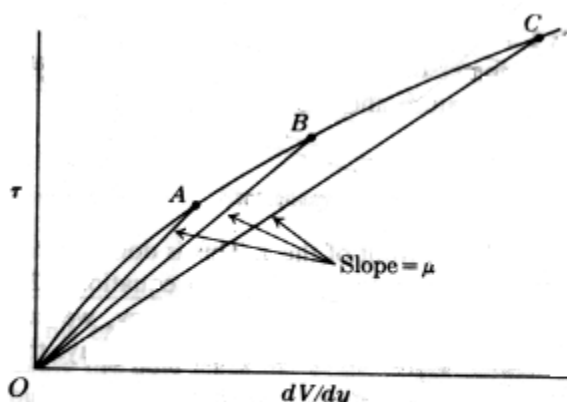


Fig. 2.3

τ of nonnewtonian fluids at constant temperature.

Here each of lines OA, OB, and OC have slope μ , so the viscosity is decreasing with increasing dV/dy . Using this definition, we can observe that there are three common types of nonnewtonian fluid (Fig 2.2):

1. *Bingham fluids*, sometimes called *bingham plastics*, resist a small shear stress indefinitely but flow easily under larger shear stresses. One may say that at low stresses the viscosity is infinite and that at higher stresses the viscosity decreases with increasing velocity gradient. Examples are bread dough, tooth paste, jellies, and some slurries.
2. *Pseudoplastic fluids* show a viscosity that decreases with increasing velocity gradient. Examples are most slurries, muds, polymer solutions, solutions of natural gums, and blood.
3. *Dilatant fluids* show a viscosity that increases with increasing velocity gradient. They are uncommon, but starch suspensions behave in this way.

The basic unit of viscosity is the poise, where $1P = 1g / (cm.s) = 0.1 Pa.s = 6.72 \times 10^{-2} lbm / (ft.s)$. It is widely used for materials such as high-polymer solutions and molten polymers. However, it is too large a unit for most common fluids. By sheer coincidence the viscosity of pure water at about 68°F is

0.01 P; for that reason the common unit of viscosity in the United States is the centipoises, where $1 \text{ cP} = 0.01 \text{ P} = 0.01 \text{ g} / (\text{cm} \cdot \text{s}) = 6.72 \times 10^{-4} \text{ lb}_m / (\text{ft} \cdot \text{s}) = 0.001 \text{ Pa} \cdot \text{s}$. Hence, the viscosity of a fluid, expressed in centipoises, is the same as the ratio of its viscosity to that of water at room temperature.

Kinematic Viscosity:

In many engineering problems, viscosity appears only in the relation of viscosity divided by density. Therefore, to save writing, we define

$$\text{Kinematic viscosity} = \nu = \frac{\mu}{\rho} \quad (2.15)$$

The most common unit of kinematic viscosity is the centistokes (cSt):

$$1 \text{ cSt} = \frac{1 \text{ cP}}{1 \text{ g/cm}^3} = 1.08 \times 10^{-5} \frac{\text{ft}^2}{\text{s}} = 10^{-6} \frac{\text{m}^2}{\text{s}}$$

(The kinematic viscosity of water at room temperature is 1cSt.) To avoid confusion over which viscosity is being used, some writers refer to the viscosity μ as the absolute viscosity.

Surface Tension:

Surface tension is the tendency of the surface of a liquid to behave like a stretched elastic membrane. There is a natural tendency for liquids to minimize their surface area. The obvious case is that of a liquid droplet on a horizontal surface that is not wetted by the liquid – mercury on glass, or water on a surface that also has a thin oil film on it. For small droplets, such as those on the left of the Fig. 2.4 the droplet adopts a shape that is almost perfectly spherical, because in this configuration there is the least surface area for a given volume.



Fig. 2.4 .9 The larger droplets are flatter because gravity becoming more important than surface tension.

For larger droplets, the shape becomes somewhat flatter because of the increasingly important gravitational effect, which is roughly proportional to a^3 , where a is the approximate droplet radius, whereas the surface area is proportional only to a^2 . Thus, the ratio of gravitational to surface tension effects depends roughly on the value of $a^3/a^2 = a$, and is therefore increasingly important for the larger droplets, as shown to the right in the Fig 2.4. Overall, the situation is very similar to that of a water-filled balloon, in which the water accounts for the gravitational effect and the balloon acts like the surface tension.

A fundamental property is the *surface energy*, which is defined with reference to the Fig2.5a. A molecule I, situated in the *interior* of the liquid, is attracted equally in all directions by its neighbors. However, a molecule S, situated in the *surface*, experiences a net attractive force into the bulk of the liquid. (The molecule S.) Therefore, work has to be done against such a force in bringing an interior molecule to the surface. Hence, an energy σ , called the *surface energy*, can be attributed to a unit area of the surface.

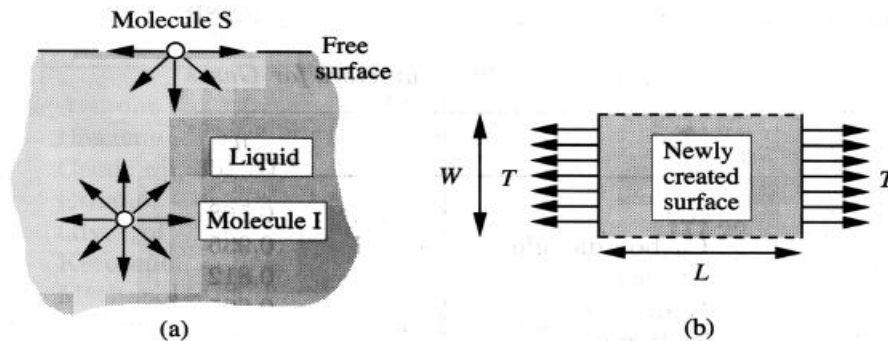


Fig. 2.5 (a) Molecules in the interior and surface of a liquid; (b) newly created surface caused by moving the tension T through a distance L .

An equivalent viewpoint is to consider the surface tension T existing per unit distance of a line drawn in the surface, as shown in the Fig 2.5b. Suppose that such a tension has moved a distance L , thereby creating an area WL of fresh surface. The work done is the product of the force, TW , and the distance L through which it moves, namely TWL , and this must equal the newly acquired surface energy σWL . Therefore, $T = \sigma$; both quantities have units of force per unit distance, such as N / m, which is equivalent to energy per unit area, such as J/m².

Units and Systems of Units:

Mass, weight, and force. The *mass* M of an object is a measure of the amount of matter it contains and will be constant, since it depends on the number of constituent molecules and their masses. On the other hand, the *weight* w of the object is the gravitational force on it, and is equal to Mg , where g is the local gravitational acceleration. Mostly, we shall be discussing phenomena occurring at the surface of the earth, where g is approximately $32.174 \text{ ft/s}^2 = 9.807 \text{ m/s}^2 = 980.7 \text{ cm/s}^2$. These values are simply taken as 32.2, 9.81 and 981, respectively.

Table 2.4

1.7 Representative Units of Force

<i>System</i>	<i>Units of Force</i>	<i>Customary Name</i>
SI	kg m/s^2	newton
CGS	g cm/s^2	dyne
FPS	$\text{lb}_m \text{ ft/s}^2$	poundal

Newton's second law of motion states that a force F applied to a mass M will give it an acceleration a :

$$F = Ma \quad (2.16)$$

from which is apparent that force has dimensions ML/T^2 . Table 2.4 gives the corresponding *units* of force in the SI (meter / kilogram / second), CGS (centimeter / gram / second), and FPS (foot / pound / second) systems.

The poundal is now an archaic unit, hardly ever used. Instead, the *pound force*, lb_f , is much more common in the English system; it is defined as the gravitational force on 1 lb_m , which, if left to fall freely, will do so with an acceleration of 32.2 ft/s^2 . Hence:

$$1 \text{ lb}_f = 32.2 \text{ lb}_m \frac{\text{ft}}{\text{s}^2} = 32.2 \text{ poundals} \quad (2.17)$$

When using lb_f in the ft, lb_m , s (FPS) system, the following conversion factor, commonly called " g_c ", will almost invariably be needed:

$$g_c = 32.2 \frac{lb_m ft}{lb_f s^2} \quad (2.18)$$

Some writers incorporate g_c into their equations, but this approach may be confusing since it virtually implies that one particular set of units is being use, and hence tends to rob the equations of their generality. Why not, for example, also incorporate the conversion factor of $144 \text{ in}^2/\text{ft}^2$ into equations where pressure is expressed in lb_f/in^2 ? We prefer to omit all conversion factors in equations, and introduce them only as needed in evaluating expressions numerically. If the reader is in any doubt, units should *always* be checked when performing calculations.

SI Units. The most systematically developed and universally accepted set of units occurs in the *SI* units or *Système International d'Unités*; the subset we mainly need is shown in table 2.5

The basic units are again the meter, kilogram, and second (m, kg, and s); from these, certain *derived* units can also be obtained. Force ($\text{kg m} / \text{s}^2$) has already been discussed; energy is the product of force and length; power amounts to energy per unit time; surface tension is energy per unit area or force per unit length, and so on. Some of the units have names, and these, together with their abbreviations, are also given in table 2.5

Table 2.5 8 SI Units

<i>Physical Quantity</i>	<i>Name of Unit</i>	<i>Symbol for Unit</i>	<i>Definition of Unit</i>
<i>Basic Units</i>			
Length	meter	m	—
Mass	kilogram	kg	—
Time	second	s	—
Temperature	degree Kelvin	K	—
<i>Supplementary Unit</i>			
Plane angle	radian	rad	—
<i>Derived Units</i>			
Acceleration			m/s^2
Angular velocity			rad/s
Density			kg/m^3
Energy	joule	J	$\text{kg m}^2/\text{s}^2$
Force	newton	N	kg m/s^2
Kinematic viscosity			m^2/s
Power	watt	W	$\text{kg m}^2/\text{s}^3$ (J/s)
Pressure	pascal	Pa	kg/m s^2 (N/m^2)
Velocity			m/s
Viscosity			kg/m s

Tradition dies hard, and certain other “metric” units are so well established that they may be used as *auxiliary* units; these are shown in table 2.6. The *gram* is the classic example. Note that the basic SI unit of mass (kg) is even represented in terms of the gram, and has not yet been given a name of its own!

Table 2.6

Table 1.9 Auxiliary Units Allowed in Conjunction with SI Units

<i>Physical Quantity</i>	<i>Name of Unit</i>	<i>Symbol for Unit</i>	<i>Definition of Unit</i>
Area	hectare	ha	10^4 m^2
Kinematic viscosity	stokes	St	$10^{-4} \text{ m}^2/\text{s}$
Length	micron	μm	10^{-6} m
Mass	tonne	t	$10^3 \text{ kg} = \text{Mg}$
	gram	g	$10^{-3} \text{ kg} = \text{g}$
Pressure	bar	bar	10^5 N/m^2
Viscosity	poise	P	10^{-1} kg/m s
Volume	liter	l	10^{-3} m^3

Table 2.7 shows some of the acceptable prefixes that can be used for accommodating both small and large quantities. For example, to avoid an excessive number of decimal places, 0.000001 s is normally better expressed as 1 μs (one microsecond). Note also, for example, that 1 μkg should be written as 1 mg-one prefix being better than two.

Table 2.7

10 Prefixes for Fractions and Multiples

<i>Factor</i>	<i>Name</i>	<i>Symbol</i>	<i>Factor</i>	<i>Name</i>	<i>Symbol</i>
10^{-12}	pico	p	10^3	kilo	k
10^{-9}	nano	n	10^6	mega	M
10^{-6}	micro	μ	10^9	giga	G
10^{-3}	milli	m	10^{12}	tera	T

Some of the more frequently used conversion factors are given in the table

Example 2.1 – Units Conversion

Part 1. Express 65 mph in (a) ft/s , and (b) m/s

Solution:

$$(a) \ 65 \frac{\text{miles}}{\text{hr}} \times \frac{1}{3,600} \frac{\text{hr}}{\text{s}} \times 5,280 \frac{\text{ft}}{\text{mile}} = 95.33 \frac{\text{ft}}{\text{s}}$$

$$(b) \ 95.33 \frac{\text{ft}}{\text{s}} \times 0.3048 \frac{\text{m}}{\text{ft}} = 29.06 \frac{\text{m}}{\text{s}}$$

Part 2. The density of 35°API crude oil is 53.1 lb_m/ft^3 at 68°F and its viscosity is 32.8 $\text{lb}_m/\text{ft hr}$. What are its density, viscosity, and kinematic viscosity in SI units?

Solution:

$$\rho = 53.1 \frac{\text{lb}_m}{\text{ft}^3} \times 0.4536 \frac{\text{kg}}{\text{lb}_m} \times \frac{1}{0.3048^3} \frac{\text{ft}^3}{\text{m}^3} = 851 \frac{\text{kg}}{\text{m}^3}$$

$$\mu = 32.8 \frac{\text{lb}_m}{\text{ft hr}} \times \frac{1}{2.419} \frac{\text{centipoise}}{\text{lb}_m/\text{ft hr}} \times 0.01 \frac{\text{poise}}{\text{centipoise}} = 0.361 \text{ poise}$$

Or, converting to SI units, noting that P is the symbol for poise, and evaluating ν :

$$\mu = 0.361 \text{ P} \times 0.1 \frac{\text{Kg/m s}}{\text{P}} = 0.0136 \frac{\text{Kg}}{\text{m s}}$$

$$\nu = \frac{\mu}{\rho} = \frac{0.0136 \text{ kg/m s}}{851 \text{ kg/m}^3} = 1.60 \times 10^{-5} \frac{\text{m}^2}{\text{s}} (= 0.160 \text{ St})$$

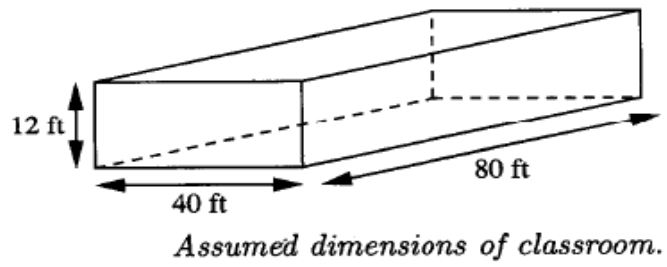
Example 2.2 – Mass of Air in a Room

Estimate the mass of air in your classroom, which is 80 ft wide, 40 ft deep, and 12 ft high. The gas constant is $R = 10.73 \text{ psia} \cdot \text{ft}^3 / \text{lb} - \text{mole} \cdot ^\circ\text{R}$.

Solution:

The volume of the classroom is:

$$V = 80 \times 40 \times 12 = 3.84 \times 10^4 \text{ ft}^3$$



If the air is approximately 20% oxygen and 80% nitrogen, its mean molecular weight is $M_w = 0.8 \times 28 + 0.2 \times 32 = 28.8 \text{ lb}_m / \text{lb} - \text{mole}$. From the gas law, assuming an absolute pressure of $p = 14.7 \text{ psia}$ and a temperature of $70^\circ\text{F} = 530^\circ\text{R}$, the density is:

$$\rho = \frac{M_w p}{RT} = \frac{28.8(\text{lb}_m / \text{lb} - \text{mole}) \times 14.7(\text{psia})}{10.73(\text{psia} \cdot \text{ft}^3 / \text{lb} - \text{mole} \cdot ^\circ\text{R}) \times 530(^{\circ}\text{R})} = 0.0744 \text{ lb}_m / \text{ft}^3$$

Hence the mass of air is:

$$M = \rho V = 0.0744(\text{lb}_m / \text{ft}^3) \times 3.84 \times 10^4(\text{ft}^3) = 2,860 \text{ lb}_m$$

I. FLUID MECHANICS

I.1 Basic Concepts & Definitions:

Fluid Mechanics - Study of fluids at rest, in motion, and the effects of fluids on boundaries.

Note: This definition outlines the key topics in the study of fluids:

(1) fluid statics (fluids at rest), (2) momentum and energy analyses (fluids in motion), and (3) viscous effects and all sections considering pressure forces (effects of fluids on boundaries).

Fluid - A substance which **moves** and **deforms continuously** as a result of an **applied shear stress**.

The definition also clearly shows that viscous effects are not considered in the study of fluid statics.

Two important properties in the study of fluid mechanics are:

Pressure and Velocity

These are defined as follows:

Pressure - The **normal stress** on **any** plane through a fluid element **at rest**.

Key Point: The direction of pressure forces will always be perpendicular to the

surface of interest.

Velocity - The rate of change of position at a point in a flow field. It is used not only to specify flow field characteristics but also to specify flow rate, momentum, and viscous effects for a fluid in motion.

I.4 Dimensions and Units

This text will use both the International System of Units (S.I.) and British Gravitational System (B.G.).

A key feature of both is that neither system uses g_c . Rather, in both systems the combination of units for mass * acceleration yields the unit of force, i.e. Newton's second law yields

$$\text{S.I.} - 1 \text{ Newton (N)} = 1 \text{ kg m/s}^2$$

$$\text{B.G.} - 1 \text{ lbf} = 1 \text{ slug ft/s}^2$$

This will be particularly useful in the following:

<u>Concept</u>	<u>Expression</u>	<u>Units</u>
momentum	$\dot{m} V$	$\text{kg/s} * \text{m/s} = \text{kg m/s}^2 = \text{N}$ $\text{slug/s} * \text{ft/s} = \text{slug ft/s}^2 = \text{lbf}$
manometry	$\rho g h$	$\text{kg/m}^3 * \text{m/s}^2 * \text{m} = (\text{kg m/s}^2) / \text{m}^2 = \text{N/m}^2$ $\text{slug/ft}^3 * \text{ft/s}^2 * \text{ft} = (\text{slug ft/s}^2) / \text{ft}^2 = \text{lbf/ft}^2$
dynamic viscosity	μ	$\text{N s / m}^2 = (\text{kg m/s}^2) \text{s / m}^2 = \text{kg/m s}$ $\text{lbf s / ft}^2 = (\text{slug ft/s}^2) \text{s / ft}^2 = \text{slug/ft s}$

Key Point: In the B.G. system of units, the unit used for mass is the slug and not the lbm. and 1 slug = 32.174 lbm. Therefore, be careful not to use conventional values for fluid density in English units without appropriate conversions, e.g., for water: $\rho_w = 62.4 \text{ lb/ft}^3$ (do not use this value). Instead, use $\rho_w = 1.94 \text{ slug/ft}^3$.

For a unit system using g_c , the manometer equation would be written as
 xample:

Given: Pump power requirements are given by

$$\dot{W}_p = \text{fluid density} \times \text{volume flow rate} \times g \times \text{pump head} = \rho Q g h_p$$

For $\rho = 1.928 \text{ slug/ft}^3$, $Q = 500 \text{ gal/min}$, and $h_p = 70 \text{ ft}$,

Determine: The power required in kW.

$$\dot{W}_p = 1.928 \text{ slug/ft}^3 \times 500 \text{ gal/min} \times 1 \text{ ft}^3/\text{s} / 448.8 \text{ gpm} \times 32.2 \text{ ft/s}^2 \times 70 \text{ ft}$$

$$\dot{W}_p = 4841 \text{ ft} \cdot \text{lbf/s} \times 1.3558 \times 10^{-3} \text{ kW/ft} \cdot \text{lbf/s} = 6.564 \text{ kW}$$

Note: We used the following, $1 \text{ lbf} = 1 \text{ slug ft/s}^2$, to obtain the desired units

Recommendation: In working with problems with complex or mixed system units, at the start of the problem convert all parameters with units to the base units being used in the problem, e.g. for S.I. problems, convert all parameters to kg, m, & s; for B.G. problems, convert all parameters to slug, ft, & s. Then convert the final answer to the desired final units.

Review examples on unit conversion in the text.

1.5 Properties of the Velocity Field

Two important properties in the study of fluid mechanics are:

Pressure and Velocity

The basic definition for velocity has been given previously. However, one of its most important uses in fluid mechanics is to specify both the volume and mass flow rate of a fluid.

Volume flow rate:

$$\dot{Q} = \int_{CS} \bar{V} \cdot \bar{n} dA = \int_{CS} V_n dA$$

where V_n is the normal component of velocity at a point on the area across which fluid flows.

Key Point: Note that only the normal component of velocity contributes to flow rate across a boundary.

Mass flow rate:

$$\dot{m} = \int_{CS} \rho \bar{V} \cdot \bar{n} dA = \int_{CS} \rho V_n dA$$

NOTE: While not obvious in the basic equation, V_n must also be measured relative to any motion at the flow area boundary, i.e., if the flow boundary is moving, V_n is measured **relative to** the moving boundary.

1.6 Thermodynamic Properties

All of the usual thermodynamic properties are important in fluid mechanics

P - Pressure	(kPa, psi)
T- Temperature	(°C, °F)
ρ – Density	(kg/m ³ , slug/ft ³)

Alternatives for density

γ - specific weight = weight per unit volume (N/m³, lbf/ft³)

$\gamma = \rho g$	H ₂ O:	$\gamma = 9790 \text{ N/m}^3 = 62.4 \text{ lbf/ft}^3$
	Air:	$\gamma = 11.8 \text{ N/m}^3 = 0.0752 \text{ lbf/ft}^3$

S.G. - specific gravity = ρ / ρ_{ref} where ρ_{ref} is usually at 4°C, but some references will use ρ_{ref} (ref) at 20°C

liquids ρ_{ref} = $\rho_{\text{water at 1 atm, 4°C}}$ for liquids = 1000 kg/m³

gases ρ_{ref} = $\rho_{\text{air at 1 atm, 4°C}}$ for gases = 1.205 kg/m³

Example: Determine the static pressure difference indicated by an 18 cm column of fluid (liquid) with a specific gravity of 0.85.

$$\Delta P = \gamma h = \text{S.G.} \cdot \rho_{\text{ref}} g h = 0.85 \cdot 9790 \text{ N/m}^3 \cdot 0.18 \text{ m} = 1498 \text{ N/m}^2 = 1.5 \text{ kPa}$$

Ideal Gas Properties

Gases at low pressures and high temperatures have an equation-of-state (the relationship between pressure, temperature, and density for the gas) that is closely approximated by the ideal gas equation-of-state.

The expressions used for selected properties for substances behaving as an ideal gas are given in the following table.

Ideal Gas Properties and Equations

Property	Value/Equation
1. Equation-of-state	$P = \rho R T$
2. Universal gas constant	$\bar{R} = 49,700 \text{ ft}^2/(\text{s}^2 \text{ } ^\circ\text{R}) = 8314 \text{ m}^2/(\text{s}^2 \text{ } ^\circ\text{K})$
3. Gas constant	$R = \bar{R} / M_{\text{gas}}$
4. Constant volume specific heat	
5. Internal energy	$du = C_v(T) dT \quad u = f(T) \text{ only}$
6. Constant Pressure specific heat	$C_p = \left(\frac{\partial h}{\partial T} \right)_p = \frac{dh}{dT} = C_p(T) = \frac{k R}{k - 1}$
7. Enthalpy	$h = u + P v, \quad dh = C_p(T) dT \quad h = f(T) \text{ only}$
8. Specific heat ratio	$k = C_p / C_v = k(T)$

Properties for Air

$$(R_{\text{air}} = 1716 \text{ ft}^2/(\text{s}^2 \text{ } ^\circ\text{R}) = 287 \text{ m}^2/(\text{s}^2 \text{ } ^\circ\text{K}))$$

$$\text{at } 60^\circ\text{F, } 1 \text{ atm, } \rho = P/R T = 2116/(1716 \cdot 520) = 0.00237 \text{ slug/ft}^3 = 1.22 \text{ kg/m}^3$$

$$M_{\text{air}} = 28.97$$

$$k = 1.4$$

$$C_v = 4293 \text{ ft}^2/(\text{s}^2 \text{ } ^\circ\text{R}) = 718 \text{ m}^2/(\text{s}^2 \text{ } ^\circ\text{K})$$

$$C_p = 6009 \text{ ft}^2/(\text{s}^2 \text{ } ^\circ\text{R}) = 1005 \text{ m}^2/(\text{s}^2 \text{ } ^\circ\text{K})$$

I.7 Transport Properties

Certain transport properties are important as they relate to the diffusion of momentum due to shear stresses. Specifically:

μ coefficient of viscosity (dynamic viscosity) $\{M / L \ t\}$

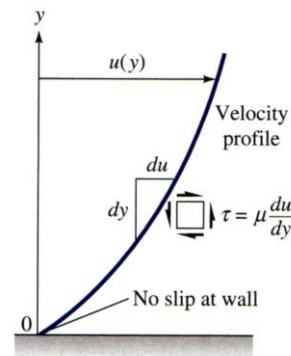
ν kinematic viscosity $= \mu / \rho \{L^2 / t\}$

This gives rise to the definition of a Newtonian fluid.

Newtonian fluid: A fluid which has a **linear** relationship between shear stress and velocity gradient.



The linearity coefficient in the equation is the coefficient of viscosity μ



Flows constrained by solid surfaces can typically be divided into two regimes:

- a. Flow near a bounding surface with
 1. significant velocity gradients
 2. significant shear stresses

This flow region is referred to as a "**boundary layer**."

b. Flows far from bounding surface with

1. negligible velocity gradients
2. negligible shear stresses
3. significant inertia effects

This flow region is referred to as "free stream" or "inviscid flow region."

An important parameter in identifying the characteristics of these flows is the

$$\text{Reynolds number} = \text{Re}$$

This physically represents the ratio of inertia forces in the flow to viscous forces. For most flows of engineering significance, both the characteristics of the flow and the important effects due to the flow, e.g., drag, pressure drop, aerodynamic loads, etc., are dependent on this parameter.

Surface Tension

Surface tension, γ , is a property important to the description of the interface between two fluids. The dimensions of γ are F/L with units typically expressed as newtons/meter or pounds-force/foot. Two common interfaces are water-air and mercury-air. These interfaces have the following values for surface tension for clean surfaces at 20°C (68°F):

Contact Angle

For the case of a liquid interface intersecting a solid surface, the contact angle, θ , is a second important parameter. For $\theta < 90^\circ$, the liquid is said to 'wet' the surface; for $\theta > 90^\circ$, this liquid is 'nonwetting.' For example, water does not wet a waxed car surface and instead 'beads' the surface. However, water is extremely wetting to a clean glass surface and is said to 'sheet' the surface.

Liquid Rise in a Capillary Tube

The effect of surface tension, γ , and contact angle, θ , can result in a liquid either rising or falling in a capillary tube. This effect is shown schematically in the Fig. E 1.9 on the following page.

A force balance at the liquid-tube-air interface requires that the weight of the vertical column, h , must equal the vertical component of the surface tension force. Thus:

$$\rho \pi R^2 h = 2 \pi R Y \cos \theta$$

Solving for h we obtain

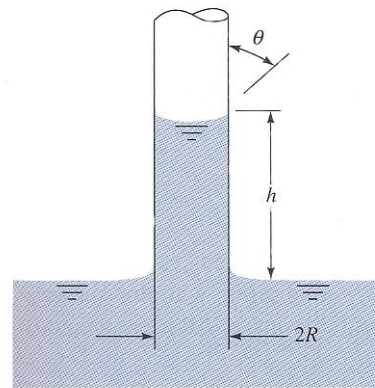


Fig. E1.9

Fig. E 1.9 Capillary Tube Schematic

Thus, the capillary height increases directly with surface tension, Y , and inversely with tube radius, R . The increase, h , is positive for $\theta < 90^\circ$ (wetting liquid) and negative (capillary depression) for $\theta > 90^\circ$ (non-wetting liquid).

Example:

Given a water-air-glass interface ($\theta = 0^\circ$, $Y = 0.073 \text{ N/m}$, and $\rho = 1000 \text{ kg/m}^3$), $R = 1 \text{ mm}$, determine the capillary height, h .

$$h = \frac{2(0.073 \text{ N/m})\cos 0^\circ}{(1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(0.001 \text{ m})} = 1.5 \text{ cm}$$

For a mercury-air-glass interface with $\theta = 130^\circ$, $Y = 0.48 \text{ N/m}$ and $\rho = 13,600 \text{ kg/m}^3$, the capillary rise will be

$$h = \frac{2(0.48 \text{ N/m})\cos 130^\circ}{(13,600 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(0.001 \text{ m})} = -0.46 \text{ cm}$$

GOVERNING EQUATIONS OF FLUID DYNAMICS

1. GOVERNING EQUATIONS

1.1 Introduction

We are all familiar with fluids and we have all been fascinated at some time by fluid behavior: pouring, flowing, splashing and so on. At some level we have all built an intuitive understanding on how fluids behave.

Understanding fluid dynamics has been one of the major advances of physics, applied mathematics and engineering over the last hundred years. Starting with the explanations of aerofoil theory (i.e., why aircraft wings work), the study of fluids continues today with looking at how internal and surface waves, shock waves, turbulent fluid flow and the occurrence of chaos can be described mathematically. At the same time, it is important to realize how much of engineering depends on a proper understanding of fluids: from flow of water through pipes, to studying effluent discharge into the sea; from motions of the atmosphere, to the flow of lubricants in a car engine.

Fluid dynamics is also the key to our understanding of some of the most important phenomena in our physical world: ocean currents and weather systems, convection currents such as the motions of molten rock inside the Earth and the motions in the outer layers of the Sun, the explosions of supernovae and the swirling of gases in galaxies.

Like many fascinating subjects, understanding is not always easy. In particular, for fluid dynamics there are many terms and mathematical methods which will probably be unfamiliar. Although the basic concepts of velocity, mass, linear momentum, forces, etc., are the building blocks, the slippery nature of fluids means that applying those basic concepts sometimes takes some work.

Fluids can fascinate us exactly because they sometimes do unexpected things, which means we have to work harder at their mathematical explanation.

1.2 The continuity assumption

There are basically two ways of deriving the equations which governs the motion of a fluid. One of these methods approaches the question from the molecular point of view. The alternative method which is used to derive the equations which govern the motion of a fluid uses the **continuum concept**.

The continuity assumption considers fluids to be continuous. That is, properties such as density, pressure, temperature, and velocity are taken to be well-defined at infinitely small points, and are assumed to vary continuously from one point to another. The discrete, molecular nature of a fluid is ignored.

Those problems for which the continuity assumption does not give answers of desired accuracy are solved using statistical mechanics. In order to determine whether to use conventional fluid dynamics (a subdiscipline of continuum mechanics) or statistical mechanics, the Knudsen number:

$$Kn = \frac{\lambda}{L} = \frac{k_B T}{\sqrt{2\pi\sigma^2 PL}}$$

where

T, temperature (K)

k_B , Boltzmann's constant

P, total pressure (Pa)

is evaluated for the problem. Problems with Knudsen numbers at or above unity must be evaluated using statistical mechanics for reliable solutions.

The vast majority of phenomena encountered in fluid mechanics fall well within the continuum domain and may involve liquids as well as gases. The continuum method is generally used to describe fluid dynamics.

1.3 Modeling fluids

When it comes to deducing the equations that govern fluid motion, there are two fundamentally different approaches: the **Eulerian description** and the **Lagrangian description**. The two

viewpoints have been named in honor of the Swiss mathematician Leonhard Euler (1707–1783) and the French mathematician and mathematical physicist Joseph-Louis Lagrange (1736-1813), respectively.

1.3.1 Eulerian description

In the **Eulerian description** a **fixed reference frame** is employed relative to which a fluid is in motion. Time and spatial position in this reference frame, $\{t, r\}$ are used as **independent variables**. The fluid variables such as mass, density, pressure and flow velocity which describe the physical state of the fluid flow in question are **dependent variables** as they are functions of the independent variables. Thus their derivatives are partial with respect to $\{t, r\}$. For example, the flow velocity at a spatial position and time is given by: $u(r, t)$ and the corresponding acceleration at this position and time is then:

$$a = \frac{\partial u(t, r)}{\partial t} \quad r = \text{constant}$$

1.3.2 Lagrangian description

In the **Lagrangian description** the fluid is described in terms of its constituent **fluid elements**. Different fluid elements have different “labels”, e.g., their spatial positions at a certain fixed time t say q . The **independent variables** are thus $\{t, q\}$ and the particle position $r(t, q)$ is a dependent variable. One can then ask about the rate of change in time in a reference frame *commoving* with the fluid element, and this then depends on time and particle label, which particular fluid element is being followed.

For example, if a fluid element has some velocity $u(t, q)$ then the acceleration

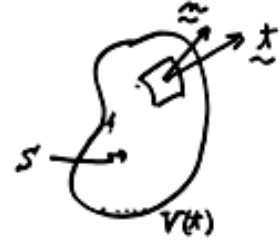
it feels will be:

$$a = \frac{\partial u(t, q)}{\partial t} \quad q = \text{constant}$$

1.3.3 Reynold's Transport Theorem

The method used to derive the basic equations of motion from the conservation laws is to use the continuum concept and to follow an arbitrarily shaped control volume in a lagrangian frame of reference. The combination of the arbitrary control volume and the lagrangian coordinate system means that material derivative of volume integrals will be encountered. Though it is necessary to transform such terms into equivalent expressions involving volume integrals of eulerian derivatives. The theorem which permits such transformation is called **Reynolds' transport theorem**.

Consider a specific mass of fluid and follow it for a short period of time dt as it flows. Let α be any property of the fluid such as mass, momentum in some direction or its energy. Since a specific mass of fluid is being considered and since x , y , z and t are the independent variables in the lagrangian framework, the quantity α will be a function of t only as the control volume moves with the fluid. That gives the preferred form of **Reynolds' transport theorem**:



$$\frac{D}{Dt} \int_V \alpha dV = \int_V \left[\frac{\partial \alpha}{\partial t} + \nabla \cdot (\alpha \mathbf{u}) \right] dV$$

Typical Volume

It is conventional in fluid mechanics to use D instead of d in time derivatives if it is the derivative of a Lagrangian quantity.

Having established the method to be used to derive the basic conservation equations it remains actually to invoke the various conservation principles.

1.4 Governing equations

The mathematical modeling of flows is described by the theory of **continuum mechanics**. The **governing equations** consist of **conservation equations** and **constitutive equations**. The **conservation equations** are derived from the principle of conservation of mass, the principle of balance of linear momentum, and the principle of balance of energy.

Constitutive equations are relating the stresses in the fluid to the deformation history.

Whereas **conservation equations** apply whatever the material studied, the **constitutive equations** depend from the material. The conservation and constitutive equations are used to calculate flows in complex geometries. For such problems, it is generally not possible to obtain an analytical solution of the governing equations.

1.4.1 Conservation of mass

Consider a specific mass of fluid whose volume V is arbitrarily chosen. If this given mass is followed as it flows, its size and shape will be observed to change but the mass will remain unchanged. This is the principle of mass conservation which applies to fluids in which no nuclear reactions are taking place.

The equation which express conservation of mass is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k} (\rho u_k) = 0$$

The above equation express more than the fact that mass is conserved. Since it is a partial differential equation, the implication is that the velocity is continuous. For this reason is usually called the **continuity equation**.

In many practical cases of fluid flow the variation of density of the fluid may be ignored, for example in most cases of the flow of liquids. In such cases the fluid is said to be incompressible, which means that as a given mass of fluid is followed, not only will its mass be observed to remain constant but it's volume and hence its density.

Mathematically this special simplification of the continuity equation can be written as:

$$\frac{D\rho}{Dt} = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \rho \frac{\partial u_k}{\partial x_k} = 0$$

Equation of continuity, either in the general form or the incompressible form is the first condition which has to be satisfied by the velocity and the density of the fluid.

1.4.2 Conservation of momentum

The principle of conservation of momentum is in fact an application of Newton's second law of motion to an element of fluid. That is, when considering a given mass of fluid in a lagrangian frame of reference, it is stated that the rate at which the momentum of the fluid mass is changing is equal to the net external force acting on the mass. Thus the mathematical equation which results from imposing the physical law of conservation of momentum is:

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_i$$

where the left-hand side represents the rate of change of momentum of a unit the volume of fluid. The first term is the familiar temporal acceleration term, while the second term is a convective acceleration and allows for local accelerations (around obstacles) even when the flow is steady. This second term is also nonlinear since the velocity appears quadratically. On the right hand side are the forces which are causing the acceleration.

Fluid Forces

There are two types:

- surface forces (proportional to area)
- body forces(proportional to volume)

Surface forces are usually expressed in terms of *stress* σ (= force per unit area):

$force = stress \times area$

The major surface forces are:

- pressure p : always acts normal to a surface
- viscous stresses σ_{ij} : frictional forces arising from relative motion of fluid layers.

Body forces

The main body forces are:

- *gravity*: the force per unit volume is $\rho g = \rho(0, 0, -g)$

(For constant-density fluids the effects of pressure and weight can be combined

in the governing equations as a piezometric pressure $p^* = p + \rho g z$)

- *Coriolis forces* (in rotating reference frames).

Separating surface forces (determined by a stress tensor) and body forces (\mathbf{f} per unit

volume), the integral equation for the i component of momentum may be written:

$$\frac{d}{dt} \int_V \rho u_i dV + \oint_{\partial V} \rho u_i \cdot d\mathbf{A} = \oint_{\partial V} \sigma_{ij} dA_j + \int_V f_i dV$$

surface forces

body forces

1.4.3 Conservation of energy

The principle of conservation energy amounts to an application of the first law of thermodynamics to a fluid element as it flows. When applying the first law of thermodynamics to a flowing fluid the instantaneous energy of the fluid is considered to be the sum of the internal energy per unit mass and the kinetic energy per unit mass. In this way the modified form of the first law of thermodynamics which will be applied to an element of fluid states that the rate of change in the total energy (intrinsic plus kinetic) of the fluid as it flows is equal to the sum of the rate at which work is being done on the fluid by external forces and the rate on which heat is being added by conduction. In this way the mathematical expression becomes:

$$\frac{D}{Dt} \int_V (\rho e + \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u}) dV = \int_S u P dS + \int_V u \rho f dV - \int_S \mathbf{q} \cdot \mathbf{n} dS$$

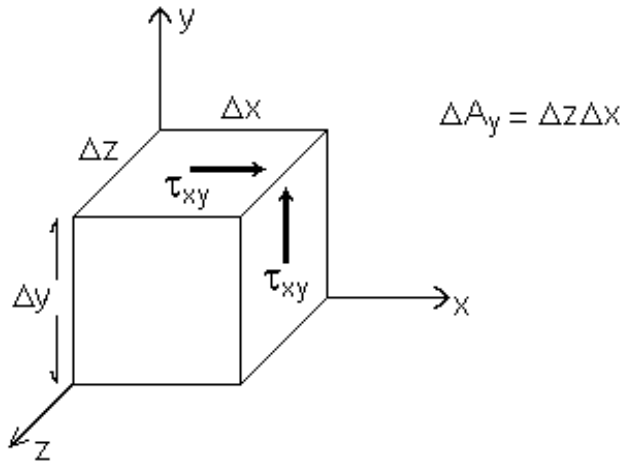
1.4.4 Constitutive equations

The basic conservation laws discussed in the previous section represent five scalar equations which the fluid properties must satisfy as the fluid flows. The continuity and the energy equation are scalar equations while the momentum equation is a vector which represents three scalar equations. But our basic conservation laws have introduced seventeen unknowns: the scalars ρ and e , the density and internal energy respectively, the vectors u_j and q_j , the velocity and heat flux respectively, each vector having three components and the stress tensor σ_{ij} , which has in general nine independent components.

In order to obtain a complete set of equations the stress tensor σ_{ij} and the heat-flux vector q_j must be further specified. This leads to so-called “constitutive equations” in which the stress tensor is related to the deformation tensor and the heat-flux vector is related to the temperature gradients.

In order to achieve this end the postulates for a **newtonian fluid** are used. It should be pointed out that some fluid do not behave in a newtonian manner and their special characteristic are among the topics of current research. One example is the class of fluids called viscoelastic fluids whose properties may be used to reduce the drag of a body.

The stress tensor is supposed to satisfy the following condition: when the fluid is at rest the stress is hydrostatic and the pressure exerted by the fluid is the thermodynamic pressure.



The above figure represents stress acting on a small cube, where τ_{ij} are the components of the *shear-stress tensor* which depends upon the motion of the fluid only.

Using that condition the constitutive relation for stress in a newtonian fluid becomes:

$$\sigma_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

The nine elements of the stress tensor have now been expressed in terms of the pressure and the velocity gradients and two coefficients λ and μ . These coefficients cannot be determined analytically and must be determined empirically. They are the *viscosity coefficients* of the fluid.

The second constitutive relation is Fourier's Law for heat conduction:

$$q_j = -k \frac{\partial T}{\partial x_j}$$

where q_j is the heat-flux vector, k is the thermal conductivity of the fluid and T is the temperature.

1.4.5 Navier-Stokes Equations

The equation of momentum conservation together with the constitutive relation for a Newtonian fluid yield the famous Navier-Stokes equations, which are the principal conditions to be satisfied by a fluid as it flows:

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j} \left(\lambda \frac{\partial u_k}{\partial x_k} \right) + \frac{\partial}{\partial x_i} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] + \rho f$$

The Navier-Stokes equations for an **incompressible** fluid of **constant density** is:

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = -\frac{\partial p}{\partial x_j} + \mu \frac{\partial^2 u_j}{\partial^2 x_i} + \rho f_i$$

In the special case of negligible viscous effects, Navier-Stokes equations become:

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = -\frac{\partial p}{\partial x_j} + \rho f_i$$

known as Euler equations.

1.4.6 In conclusion

the equations which govern the motion of a Newtonian fluid are:

The continuity equation:

$$\frac{\partial \rho}{\partial t} + u_k \frac{\partial \rho}{\partial x_k} + \rho \frac{\partial u_k}{\partial x_k} = 0$$

The Navier-Stokes equations:

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j} \left(\lambda \frac{\partial u_k}{\partial x_k} \right) + \frac{\partial}{\partial x_i} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] + \rho f$$

The energy equation

$$\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = -p \frac{\partial u_k}{\partial x_k} + \frac{\partial}{\partial x_j} \left(k \frac{\partial T}{\partial x_j} \right) + \lambda \left(\frac{\partial u_k}{\partial x_k} \right)^2 + \mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \frac{\partial u_j}{\partial x_j}$$

The thermal equation of state:

$$p = p(\rho, T)$$

The caloric equation of state:

$$e = e(\rho, T)$$

The above set of equations represents seven equations which are to be satisfied by seven unknowns. Each of the continuity, energy and state equations supplies one scalar equation, while the Navier-Stokes equation supplies three scalar equations. The seven unknowns are: the pressure (p), density (ρ), internal energy (e), temperature (T) and velocity components (u_j). The parameters λ, μ and k are assumed to be known from experimental data, and they may be constants or specified functions of the temperature and pressure.

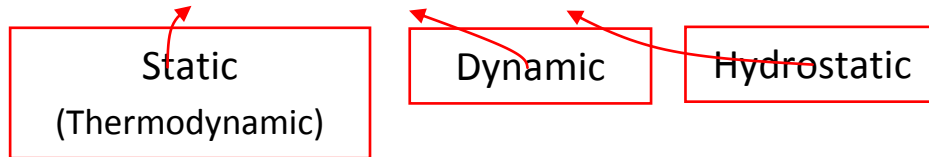
©Useful concepts associated with the Bernoulli equation

- Static, Stagnation, and Dynamic Pressures

Bernoulli eq. **along a streamline**

constant

(Unit of Pressure)



× Physical meaning of each term

i) 1st term (Static pressure, p)

: Only due to the fluid weight

∴

where (h_{4-3} : Piezometer reading)

ii) 2nd term (Dynamic pressure,)

: Pressure increase or decrease due to fluid motion

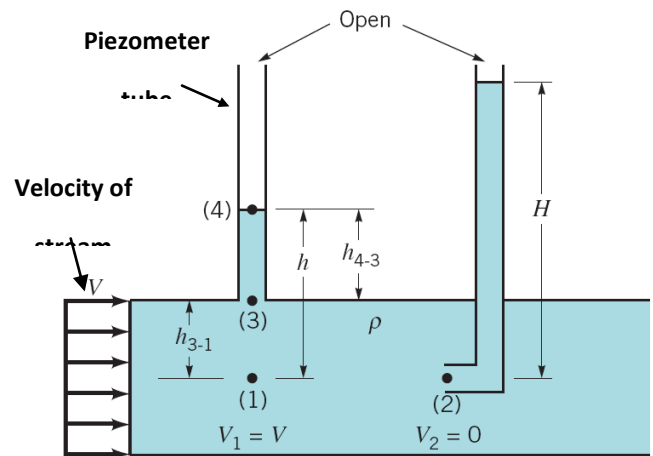
iii) 3rd term (Gravitational potential,)

: Pressure change due to elevation change

Between Point (1) and Point (2) on the same streamline,

- Difference in p between piezometer and tube inserted into a flow

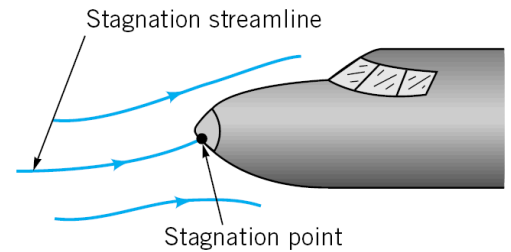
(**Stagnation pressure**)



(Elevation change,: Due to dynamic pressure)

✖ Point (2): **Stagnation point** ()

Line through point (2): **Stagnation streamline**



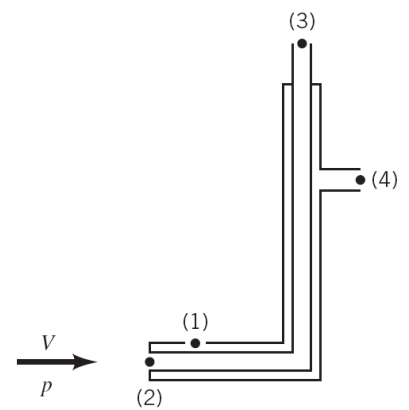
iv) Total pressure = constant (Along a streamline)

⊙ Special application of Static and Stagnation Pressure

- Determination of fluid speed (**Pitot-static tube**)

By measuring pressure at points (3) and (4)

and neglecting the elevation effect (e.g. gases)

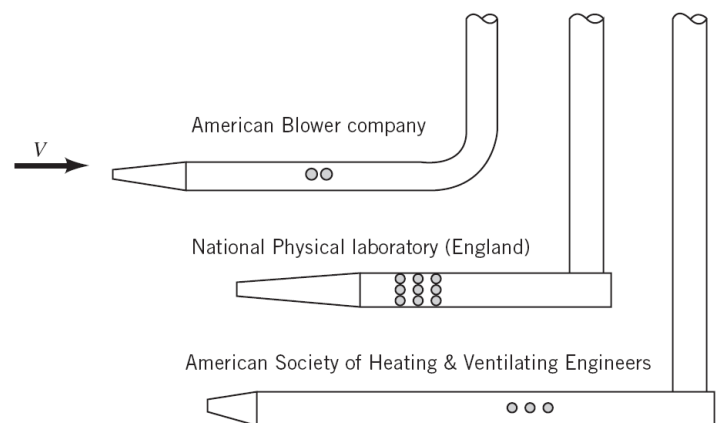
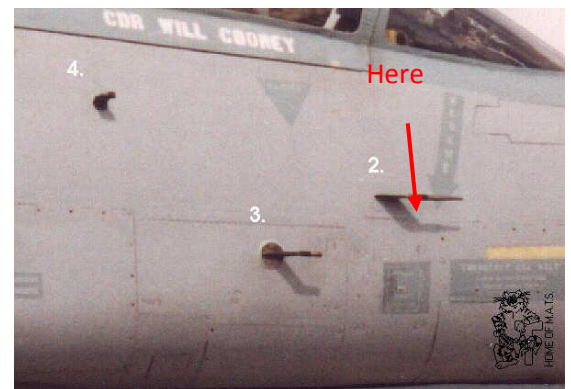
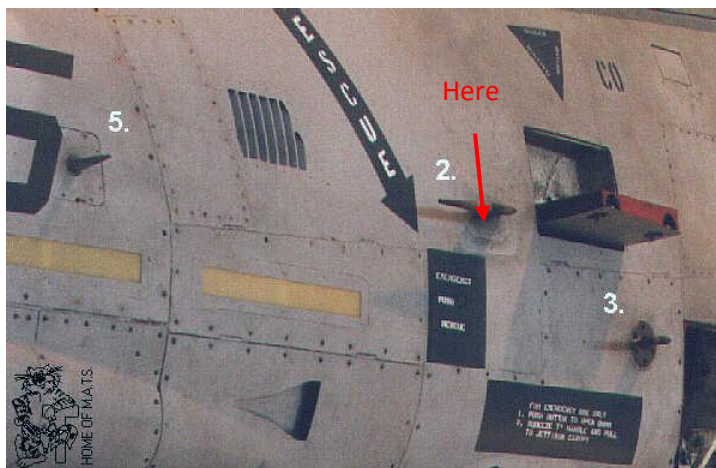
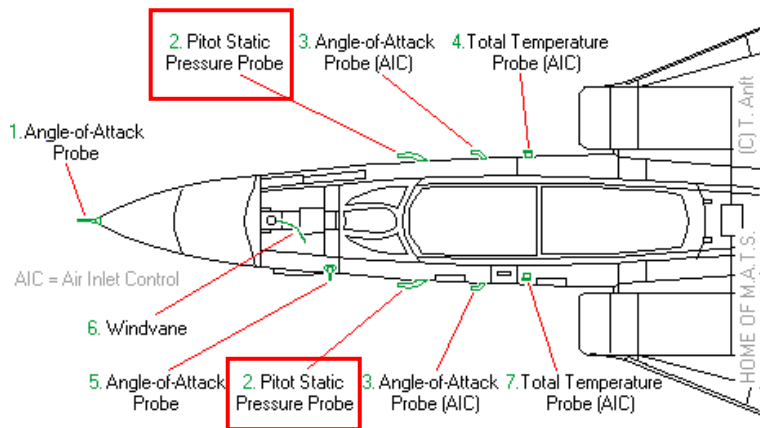


At point (3), (Stagnation Pressure)

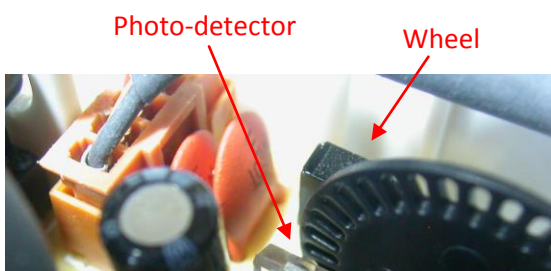
where p and V : Static pressure and fluid velocity at (2)

At point (4) (Small holes), (Static pressure ~ Piezometer)

Then, \therefore **Pitot-static tube**



Typical Pitot-static tube designs.



☉ How to use the Bernoulli equation (Examples)

Choose 2 points (1) and (2) along a streamline

: 6 variables

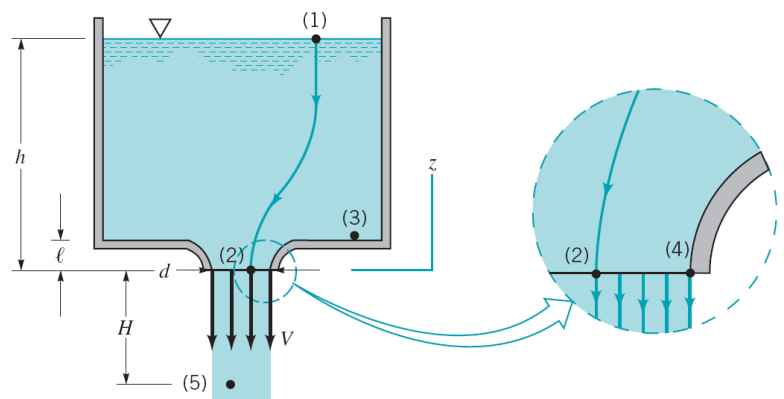
→ 5 more conditions from the problem description

E.g. 1 Free Jets (A jet of liquid from *large* reservoir)

Question: Find V of jet stream from a nozzle of container

Consider a situation shown

Step 1. Select one streamline
between (1) & (2)



Step 2. Apply the Bernoulli eq.

between (1) & (2)



= 0 (Large container → Negligible change of fluid level)

$\rho_1 = \rho_2$ (Why?) $[p_1 = p_2 \text{ (Normal Bernoulli eq.)}]$

Then,

(If we choose z_1 as h and z_2 as 0)

or

$$\therefore V = \sqrt{\frac{2\gamma h}{\rho}} = \sqrt{2gh} \quad \text{: Velocity at the exit plane of nozzle}$$

※ Velocity at any point [e.g. (5) in the figure] outside the nozzle,

∴ (H : Distance from the nozzle)

: Conversion of P. E. to K.E. without viscosity (friction)

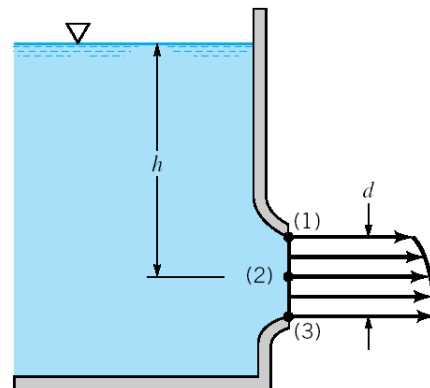
cf. Free body falling from rest without air resistance.

• In case of horizontal nozzle shown,

(Elevation difference)

By assuming $d \ll h$,

: Average velocity at the nozzle



E.g. 2 Confined Flows

- Fluid flowing **within** a container connected with nozzles and pipes

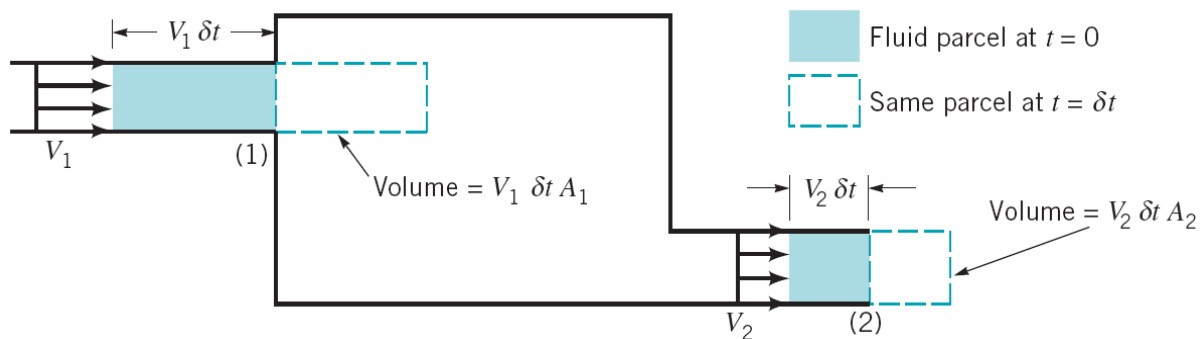
※ What to know

- We can't use the atmospheric p as a standard
- But, we have one more useful concept of conservation of mass
 - : No change of fluid mass in fixed volume (**continuity equation**)

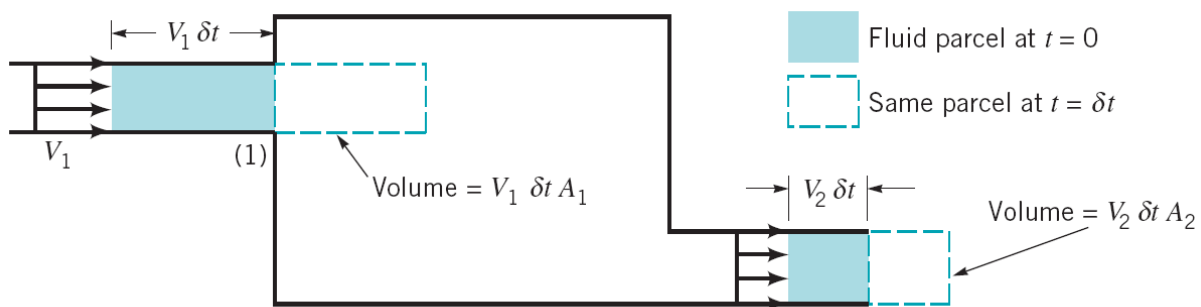
- Mass flow rate, (kg/s or slug/s) for a steady flow

Mass of fluid entering the container across inlet per unit time

= Mass of fluid leaving the container across outlet per unit time,



(Q : Volume flowrate, m^3/s)



For an incompressible fluid,

$$\therefore \text{ or } Q_1 = Q_2$$

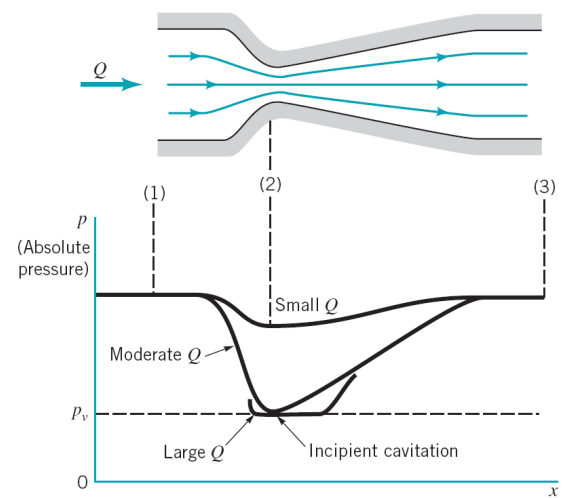
✖ Change in area

→ Change in fluid velocity ($A_1 V_1 = A_2 V_2$)

→ Change in pressure

✖ Increase in velocity → Decrease in pressure

: Blow off the roof by hurricane, Cavitation damage, etc.



E.g. 3 Flowrate Measurement

Case 1: Determine Q in pipes and conduits (Closed container)

Consider three typical devices (Depend on the type of restriction)

Step 1. At section (1)

: Low V , High p

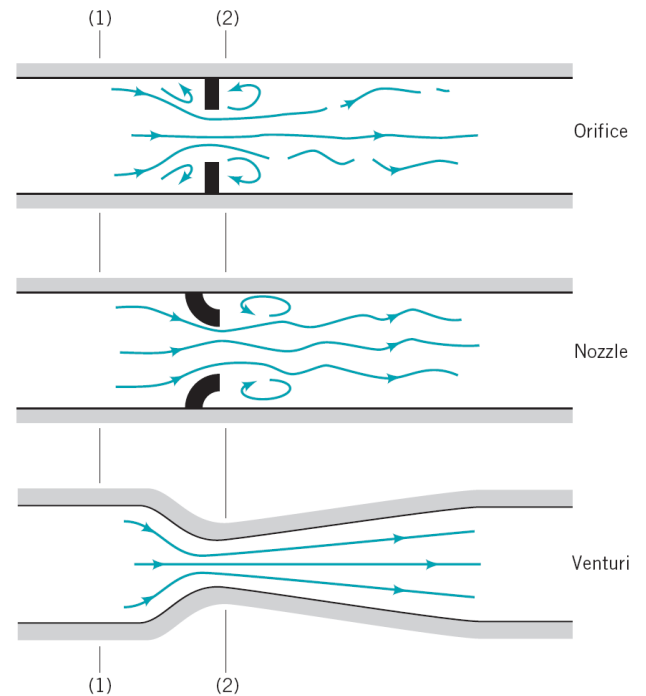
At section (2)

: High V , Low p

Step 2. Apply the Bernoulli Eq.

For a horizontal flow ($z_1 = z_2$)

Step 3. Use the continuity Eq.



Case 2: Determine Q in flumes and irrigation ditches (Open channel)

Typical devices: Sluice gate (Open bottom)

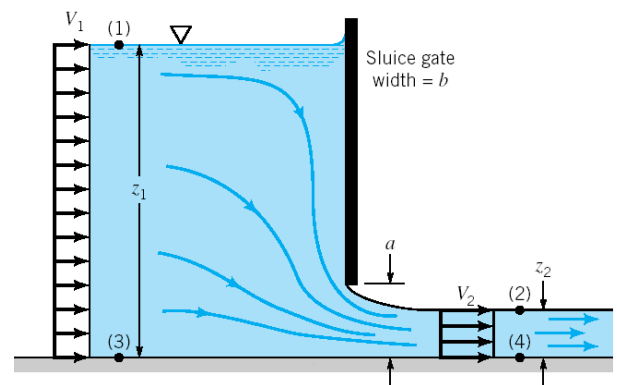
& Sharp-crested weir (Open top)

- Consider a sluice gate shown

Bernoulli Eq. & Continuity Eq. [(1) \rightarrow (2)]

$$(p_1 = p_2 = 0)$$

and

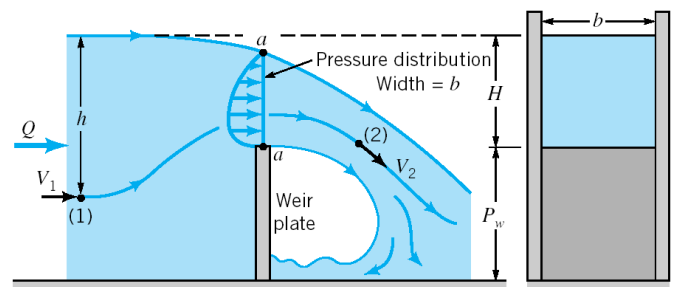


Finally, the flowrate,

✖ In case of or or

\therefore or

- The sharp-crested weir (Open top)
(If similar to horizontal free stream)



Then, Average velocity across the top of the weir $\propto \sqrt{2gH}$

Flow area for the weir $\propto Hb$

$$\therefore Q = C_1 H b \sqrt{2gH} = C_1 b H^{3/2} \sqrt{2g}$$

● Energy Line (EL) and the Hydraulic Grade Line (HGL)

- Geometrical Interpretation of the Bernoulli Eq.

Bernoulli eq.: Conserved total energy along streamline, i.e.

constant on a streamline = H (**Unit of Length**)

※ Heads: : Pressure head, : Velocity head, z : Elevation head

H : Total head

Energy line (EL): Representation of Energy versus Heads

