Dynamic Programming

Introduction

Dynamic programming is a mathematical technique of optimized using multistage decision process developed by Richard Bellman.

Bellman's Principle of optimality

An optimal policy has the property what is the initial state and initial decisions, the remaining decisions must constitute an optimal policy for the state resulting for the first decision.

A problem which does not satisfy the principle of optimality cannot be solved by dynamic programming.

Dynamic programming Algorithm

The solution of a multistage problem by dynamic programming involves the following steps.

- Identify the decision variables and specify the objective function to be optimized.
- Decompose the given problem in to a number of a smaller sub problems, identify the state variables
- Write down a general recursive relationship for the optimal policy. Decide either forward or backward is to follow to solve the problem.
- Write the relation giving the optimal decision function for one stage sub problem and solve it.
- Solve the optimal decision function for 2-stage, 3-stage,….(n-1)stage n-stage problem.

Note:

In case of continuous system, the optimal decisions at each stage are obtained by using differentiation.

- If the dynamic programming problem is solved by obtaining the sequence $f_1 \rightarrow f_2 \rightarrow f_3 \rightarrow \dots\dots \rightarrow f_{n-1} \rightarrow f_n$ of optimal solutions then the computation known as forward computation procedure.
- If the dynamic programming problem is solved by obtaining the sequence $f_n \rightarrow f_{n-1} \rightarrow \dots\dots \rightarrow f_2 \rightarrow f_1$ of optimal solutions then the computation known as backward computation procedure.
- The function $y=f(x)$ will attain its maximum if $f'(x)=0$ and $f''(x)<0$.
- The function $y=f(x)$ will attain its minimum if $f'(x)=0$ and $f''(x)>0$

Solution to recursive equation(optimal sub-division problem)

(1)Solve Maximize $Z=y_1.y_2.y_3\dots.y_n$

Subject to $y_1+y_2+y_3+\dots+y_n=c$

and $y_i \geq 0$ (or)

Divide a positive quantity c into n parts in such a way that their product is maximum.

Solution: To develop the recursive equation:

Let $f_n(c)$ be the maximum attainable product $y_1.y_2.y_3.....y_n$. Hence c is divided in to n parts $y_1,y_2,y_3,.....,y_n$. thus $f_n(c)$ becomes a function of n.

For n=1 (one stage problem)

Here c is divided in to only one part, then $y_1=c$ $\therefore$ $f_1(c)=c$ (Trivial solution)-----(1)

For n=2 (Two stage problem)

Here c is divided in to two parts $y_1=x$ and $y_2=c-x$ such that $y_1+y_2=c$

Then $f_2(c) = \underset{0\le x\le c}{Max}\{y_1 y_2\}$

$$= \underset{0\le x\le c}{Max}\{x(c-x)\}$$

$$f_2(c) = \underset{0\le x\le c}{Max}\{xf_1(c-x)\} \text{-----(2)}$$

For n=3 (Three stage problem)

Here c is divided in to three parts. Let $y_1=x$ and $y_2+ y_3=c-x$ such that $y_1+y_2+ y_3=c$

(i.e) c-x is further divided in to two parts whose maximum attainable product $y_2.\ y_3$ is $f_2(c-x)$

Then $f_3(c) = \underset{0\le x\le c}{Max}\{y_1.y_2.y_3\}$

$$f_3(c) = \underset{0\le x\le c}{Max}\{xf_2(c-x)\}$$

In general the recursive equation for the n-stage problem is

$$f_n(c) = \underset{0\le x\le c}{Max}\{xf_{n-1}(c-x)\} \text{-----(3)}$$

To solve the recursive equation

For n=2, equation (3) becomes

$$f_2(c) = \underset{0\le x\le c}{Max}\{xf_1(c-x)\}$$

$$= \underset{0\le x\le c}{Max}\{x(c-x)\}$$

The function x(c-x) will maximum if f'(x)=0 and f''(x)<0

Solving x=c/2

The optimal policy is (c/2,c/2) and $f_2(c) = \left(c/2\right)^2$

For n=3, equation (3) becomes

$$f_3(c) = \underset{0 \le x \le c}{Max}\{x f_2(c-x)\}$$

$$= \underset{0 \le x \le c}{Max}\left\{ x\left(\frac{c-x}{2}\right)^2 \right\}$$

The function $x\left(\frac{c-x}{2}\right)^2$ will attain its maximum if f'(x)=0 and f''(x)<0

Solving x=c/3

The optimal policy is (c/3,c/3, c/3) and $f_3(c) = \left(c/3\right)^3$

Let us assume that the optimal policy for n=m is (c/m,c/m, c/m,…. c/m) and $f_m(c) = \left(c/m\right)^m$

Now for n=m+1 equation (3) becomes

$$f_{m+1}(c) = \underset{0 \le x \le c}{Max}\left\{ x\left(\frac{c-x}{2}\right)^m \right\}$$

Solving x=c/m+1

The optimal policy is (c/m+1, c/m+1, c/m+1,.. c/m+1) and $f_{m+1}(c) = \left(c/m+1\right)^{m+1}$

The results is also true for n=m+1.

Hence by mathematical induction, the optimal policy is (c/n, c/n, c/n,.. c/n) and $f_n(c) = \left(c/n\right)^n$

(2)Solve Minimize $Z=y_1+y_2+y_3+\ldots+y_n$

Subject to $y_1.y_2.y_3\ldots y_n=b$

and $y_i \geq 0$ (or)

Factorize a positive quantity b into n factors in such a way that their sum is minimum.

Solution: To develop the recursive equation:

Let $f_n(b)$ be the minimum attainable sum $y_1+y_2+y_3+\ldots..+y_n$. when the positive quantity b is factorized in to n factors $y_1,y_2,y_3,\ldots..,y_n$. thus $f_n(b)$ becomes a function of n.

For n=1 (one stage problem)

Here c is factorized in to only one factor, then $y_1=b$ $\therefore$ $f_1(b)=b$ (Trivial solution)-----(1)

For n=2 (Two stage problem)

Here b is factorized in to two factors $y_1=x$ and $y_2=b/x$ such that $y_1.y_2=b$

Then $f_2(b) = \underset{0\le x\le b}{Min}\{y_1 + y_2\}$

$$= \underset{0\le x\le b}{Min}\left\{x + b/x\right\}$$

$$f_2(b) = \underset{0\le x\le b}{Min}\left\{x + f_1\left(b/x\right)\right\}\text{-----(2)}$$

For n=3 (Three stage problem)

Here b is factorized in to three factors y_1,y_2 and y_3. Let $y_1=x$ and y_2. $y_3=b/x$ such that $y_1.y_2.y_3=b$

(i.e) b/x is further factorized in to two factors whose minimum attainable sum is $f_2(b/x)$

Then $f_3(b) = \underset{0\le x\le b}{Min}\{y_1 + y_2 + y_3\}$

$$f_3(b) = \underset{0\le x\le b}{Min}\left\{x + f_2\left(b/x\right)\right\}$$

In general the recursive equation for the n-stage problem is

$$f_n(b) = \underset{0\le x\le b}{Min}\left\{x + f_{n-1}\left(b/x\right)\right\}\text{-----(3)}$$

To solve the recursive equation

For n=2, equation (3) becomes

$$f_2(b) = \underset{0\le x\le b}{Min}\left\{x + f_1\left(b/x\right)\right\}$$

$$f_2(b) = \underset{0\le x\le b}{Min}\left\{x + b/x\right\}$$

The function $x + b/x$ will minimum if f'(x)=0 and f''(x)>0

Solving $x = \sqrt{b}$

The optimal policy is $\left(b^{1/2}, b^{1/2}\right)$ and $f_2(b) = 2\sqrt{b} = 2b^{1/2}$

For n=3, equation (3) becomes

$$f_3(b) = \underset{0 \le x \le b}{Min}\left\{x + f_2\left(b/x\right)\right\}$$

$$f_3(b) = \underset{0 \le x \le b}{Min}\left\{x + 2\left(b/x\right)^{1/2}\right\}$$

The function $x + 2\left(b/x\right)^{1/2}$ will attain its minimum if f'(x)=0 and f''(x)>0

Solving $x = b^{1/3}$

The optimal policy is $\left(b^{1/3}, b^{1/3}, b^{1/3}\right)$ and $f_3(b) = 3b^{1/3}$

Let us assume that the optimal policy for n=m is $\left(b^{1/m}, b^{1/m}, b^{1/m}, \ldots, b^{1/m}\right)$ and $f_m(b) = mb^{1/m}$

Now for n=m+1 equation (3) becomes

$$f_{m+1}(b) = \underset{0 \le x \le b}{Min}\left\{x + f_m\left(b/x\right)\right\}$$

$$= \underset{0 \le x \le b}{Min}\left\{x + m\left(b/x\right)^{1/m}\right\}$$

The function $x + m\left(b/x\right)^{1/m}$ will attain its maximum when $x = b^{1/m+1}$

The optimal policy is $\left(b^{1/m+1}, b^{1/m+1}, \ldots., b^{1/m+1}\right)$ and $f_{m+1}(b) = (m+1)b^{\left(1/m+1\right)}$

The results is also true for n=m+1.

Hence by mathematical induction, the optimal policy is $\left(b^{1/n}, b^{1/n}, \ldots., b^{1/n}\right)$ and $f_n(b) = nb^{\left(1/n\right)}$

(3)Solve Minimize Z=$y_1^2+y_2^2+y_3^2+\ldots..+y_n^2$

Subject to $y_1.y_2.y_3\ldots..y_n=b$

and $y_i \geq 0$ (or)

Factorize a positive quantity b into n factors in such a way that their sum their squares is minimum.

Solution: To develop the recursive equation:

Let $f_n(b)$ be the minimum attainable sum $y_1^2+y_2^2+y_3^2+\ldots..+y_n^2$. when the positive quantity b is factorized in to n factors $y_1,y_2,y_3,\ldots..,y_n$. thus $f_n(b)$ becomes a function of n.

For n=1 (one stage problem)

Here c is factorized in to only one factor, then y_1=b

$$f_1(b) = \underset{y_1=b}{Min}\left\{y_1^{\,2}\right\} = b^2$$

$\therefore$ $f_1(b)=b^2$ (Trivial solution)-----(1)

For n=2 (Two stage problem)

Here b is factorized in to two factors y_1=x and y_2=b/x such that $y_1.y_2$=b

Then $f_2(b) = \underset{0\le x\le b}{Min}\left\{y_1^{\,2} + y_2^{\,2}\right\}$

$$= \underset{0\le x\le b}{Min}\left\{x^2 + \left(b/x\right)^2\right\}$$

$$f_2(b) = \underset{0\le x\le b}{Min}\left\{x^2 + f_1\left(b/x\right)\right\} \text{-----(2)}$$

For n=3 (Three stage problem)

Here b is factorized in to three factors y_1,y_2 and y_3. Let y_1=x and y_2. y_3=b/x such that $y_1.y_2.y_3$=b

(i.e) b/x is further factorized in to two factors whose minimum attainable sum is $f_2(b/x)$

Then $f_3(b) = \underset{0\le x\le b}{Min}\left\{y_1^{\,2} + y_2^{\,2} + y_3^{\,2}\right\}$

$$f_3(b) = \underset{0\le x\le b}{Min}\left\{x^2 + f_2\left(b/x\right)\right\}$$

In general the recursive equation for the n-stage problem is

$$f_n(b) = \underset{0\le x\le b}{Min}\left\{x^2 + f_{n-1}\left(b/x\right)\right\} \text{-----(3)}$$

To solve the recursive equation

For n=2, equation (3) becomes

$$f_2(b) = \underset{0\le x\le b}{Min}\left\{x^2 + f_1\left(b/x\right)\right\}$$

$$f_2(b) = \underset{0 \le x \le b}{Min}\left\{x^2 + \left(b/x\right)^2\right\}$$

The function $x^2 + \left(b/x\right)^2$ will minimum if f'(x)=0 and f''(x)>0

Solving $x = \sqrt{b}$

The optimal policy is $\left(b^{1/2}, b^{1/2}\right)$ and $f_2(b) = 2\sqrt{b} = 2\left(b^{1/2}\right)^2$

For n=3, equation (3) becomes

$$f_3(b) = \underset{0 \le x \le b}{Min}\left\{x^2 + f_2\left(b/x\right)\right\}$$

$$f_3(b) = \underset{0 \le x \le b}{Min}\left\{x^2 + 2\left[\left(b/x\right)^{1/2}\right]^2\right\}$$

The function $x + 2\,b/x$ will attain its minimum if f'(x)=0 and f''(x)>0

Solving $x = b^{1/3}$

The optimal policy is $\left(b^{1/3}, b^{1/3}, b^{1/3}\right)$ and $f_3(b) = 3b^{2/3}$

Let us assume that the optimal policy for n=m is $\left(b^{1/m}, b^{1/m}, b^{1/m}, \ldots, b^{1/m}\right)$ and $f_m(b) = mb^{2/m}$

Now for n=m+1 equation (3) becomes

$$f_{m+1}(b) = \underset{0 \le x \le b}{Min}\left\{x^2 + f_m\left(b/x\right)\right\}$$

$$= \underset{0 \le x \le b}{Min}\left\{x^2 + m\left(b/x\right)^{2/m}\right\}$$

The function $x^2 + m\left(b/x\right)^{2/m}$ will attain its maximum when $x = b^{1/m+1}$

The optimal policy is $\left(b^{1/m+1}, b^{1/m+1}, \ldots., b^{1/m+1}\right)$ and $f_{m+1}(b) = (m+1)b^{\left(2/m+1\right)}$

The results is also true for n=m+1.

Hence by mathematical induction, the optimal policy is $\left(b^{1/n}, b^{1/n}, \ldots., b^{1/n}\right)$ and $f_n(b) = nb^{(2/n)}$

(3)Use dynamic programming to show that $p_1 \log p_1 + p_2 \log p_2 + \ldots + p_n \log p_n$

Subject to $p_1 + p_2 + p_3 + \ldots + p_n = 1$ and $p_i \geq 0$ is minimum when

$p_1 = p_2 = p_3 = \ldots = p_n = 1/n$ (or)

Divide unity into n parts so as to minimize the quantity $\sum p_i \log p_i$

Solution: To develop the recursive equation:

Let $f_n(1)$ be the minimum attainable sum $\sum p_i \log p_i$ when the unity 1 is divided n parts $p_1, p_2, p_3, \ldots.., p_n$. thus $f_n(1)$ becomes a function of n.

For n=1 (one stage problem)

Let $p_1=1$ then

$$f_1(1) = \underset{p_1=b}{Min}\{p_1 \log p_1\} = 1\log 1$$

$\therefore\ f_1(1) = 1\log 1$ (Trivial solution)-----(1)

For n=2 (Two stage problem)

Let $p_1=x$ and $p_2=1-x$ such that $p_1+p_2=1$

Then $f_2(1) = \underset{0\leq x\leq 1}{Min}\{p_1 \log p_1 + p_2 \log p_2\}$

$$= \underset{0\leq x\leq 1}{Min}\{x \log x + (1-x)\log(1-x)\}$$

$f_2(1) = \underset{0\leq x\leq 1}{Min}\{x \log x + f_1(1-x)\}$-----(2)

For n=3 (Three stage problem)

Let $p_1=x$ and $p_2+p_3=1-x$ such that $p_1+p_2+p_3=1$

(i.e) 1-x is further divided in to two parts whose maximum attainable sum is $f_2(1-x)$

Then $f_2(1) = \underset{0\leq x\leq 1}{Min}\{p_1 \log p_1 + p_2 \log p_2 + p_3 \log p_3\}$

$$f_3(1) = \underset{0\leq x\leq 1}{Min}\{x \log x + f_2(1-x)\}$$

In general the recursive equation for the n-stage problem is

$$f_n(1) = \underset{0\le x\le 1}{Min}\{x\log x + f_{n-1}(1-x)\}\text{-----(3)}$$

To solve the recursive equation

For n=2, equation (3) becomes

$$f_2(1) = \underset{0\le x\le 1}{Min}\{x\log x + f_1(1-x)\}$$

$$f_2(1) = \underset{0\le x\le 1}{Min}\{x\log x + (1-x)\log(1-x)\}$$

The function $\{x\log x + (1-x)\log(1-x)\}$ will attain its minimum if f'(x)=0 and f''(x)>0

Solving x=1/2

The optimal policy is $\left(1/2, 1/2\right)$ and $f_2(1) = 2\left(1/2 \log 1/2\right)$

For n=3, equation (3) becomes

$$f_3(1) = \underset{0\le x\le 1}{Min}\{x\log x + f_2(1-x)\}$$

$$f_3(1) = \underset{0\le x\le 1}{Min}\left\{x\log x + 2\left[\left(\frac{1-x}{2}\right)\log\left(\frac{1-x}{2}\right)\right]\right\}$$

The function $x\log x + 2\left[\left(\frac{1-x}{2}\right)\log\left(\frac{1-x}{2}\right)\right]$ will attain its minimum if f'(x)=0 and f''(x)>0

Solving x=1/3

The optimal policy is $\left(1/3, 1/3, 1/3\right)$ and $f_3(1) = 3\left(1/3 \log 1/3\right)$

Let us assume that the optimal policy for n=m is $\left(1/m, 1/m, .., 1/m\right)$ and $f_m(1) = m\left(1/m \log 1/m\right)$

Now for n=m+1 equation (3) becomes

$$f_{m+1}(1) = \underset{0\le x\le 1}{Min}\{x\log x + f_m(1-x)\}$$

$$f_{m+1}(1) = \underset{0\le x\le 1}{Min}\left\{x\log x + m\left[\left(\frac{1-x}{m}\right)\log\left(\frac{1-x}{m}\right)\right]\right\}$$

The function $x\log x + m\left[\left(\frac{1-x}{m}\right)\log\left(\frac{1-x}{m}\right)\right]$ will attain its maximum when x=1/m+1

The optimal policy is $\left(1/_{m+1}, 1/_{m+1},\ldots,1/_{m+1}\right)$ and $f_{m+1}(1) = (m+1)\left(1/_{m+1}\log 1/_{m+1}\right)$

The results is also true for n=m+1.

Hence by mathematical induction,

the optimal policy is $\left(1/_n, 1/_n,\ldots,1/_n\right)$ and $f_n(1) = n\left(1/_n \log 1/_n\right)$

(5)Use Bellman's principle of optimality to solve

$Maximize\ Z = b_1x_1 + b_2x_2 + \ldots + b_nx_n$ where $x_1 + x_2 + \ldots + x_n = c$ (positive constant) and $x_1, x_2,\ldots,x_n \geq 0$

Solution: To develop the recursive equation:

Let $f_n(c)$ be the maximum attainable sum $b_1x_1 + b_2x_2 + \ldots + b_nx_n$ where the positive constant c is divided in to n parts $x_1,x_2,x_3,\ldots,x_n$. thus $f_n(c)$ becomes a function of n.

For n=1 (one stage problem)

Let $x_1=c$

$$f_1(c) = \underset{x_1=c}{Max}\{b_1x_1\} = b_1c$$

$\therefore\ f_1(c) = b_1c$ (Trivial solution)-----(1)

For n=2 (Two stage problem)

Here c is divided in to two parts $x_2=z$ and $x_1=c-z$ such that $x_1+x_2=c$

Then $f_2(c) = \underset{0\le z\le c}{Max}\{b_1x_1 + b_2x_2\}$

$$= \underset{0\le z\le c}{Max}\{b_1(c-z) + b_2z\}$$

$f_2(c) = \underset{0\le z\le c}{Max}\{b_2z + f_1(c-z)\}$-----(2)

For n=3 (Three stage problem)

Here c is divided in to three parts. Let $x_3=z$ and $x_1+x_2=c-z$ such that $x_1+x_2+x_3=c$

(i.e) c-z is divided in to two parts whose maximum attainable sum is f_2(c-z)

Then $f_3(c) = \underset{0 \le z \le c}{Max}\{b_1 x_1 + b_2 x_2 + b_3 x_3\}$

$$f_3(c) = \underset{0 \le z \le c}{Max}\{b_3 z + f_2(c - z)\}$$

In general the recursive equation for the n-stage problem is

$$f_n(c) = \underset{0 \le z \le c}{Max}\{b_n z + f_{n-1}(c - z)\} \text{-----(3)}$$

To solve the recursive equation

For n=2, equation (3) becomes

$$f_2(c) = \underset{0 \le z \le c}{Max}\{b_2 z + f_1(c - z)\}$$

$$f_2(c) = \underset{0 \le z \le c}{Max}\{b_2 z + b_1(c - z)\}$$

$$f_2(c) = \underset{0 \le z \le c}{Max}\{(b_2 - b_1)z + b_1 c\}$$

If $(b_2 - b_1)$ positive, then this is maximum for z=c otherwise it will be minimum. $\Rightarrow f_2(c) = b_2 c$

The optimal policy is (0,c) and $\therefore f_2(c) = b_2 c$

For n=3, equation (3) becomes

$$f_3(c) = \underset{0 \le z \le c}{Max}\{b_3 z + f_2(c - z)\}$$

$$f_3(c) = \underset{0 \le z \le c}{Max}\{b_3 z + b_2(c - z)\}$$

$$f_3(c) = \underset{0 \le z \le c}{Max}\{(b_3 - b_2)z + b_2 c\}$$

If $(b_3 - b_2)$ positive, then this is maximum for z=c otherwise it will be minimum. $\Rightarrow f_3(c) = b_3 c$

The optimal policy is (0,0,c) and $\therefore f_3(c) = b_3 c$

Let us assume that the optimal policy for n=m is (0,0,0,….,c) and $f_m(c) = b_m c$

Now for n=m+1 equation (3) becomes

$$f_{m+1}(c) = \underset{0 \le z \le c}{Max}\{b_{m+1} z + f_m(c - z)\}$$

$$f_{m+1}(c) = \underset{0\le z\le c}{Max}\{b_{m+1}z + b_m(c-z)\}$$

$$f_{m+1}(c) = \underset{0\le z\le c}{Max}\{(b_{m+1} - b_m)z + b_m c\}$$

If $(b_{m+1} - b_m)$positive, then this is maximum for z=c otherwise it will be minimum.

$$\Rightarrow f_{m+1}(c) = b_{m+1}c$$

The results is also true for n=m+1.

Hence by mathematical induction, the optimal policy is (0,0,0,….,c) and $f_n(c) = b_n c$

(6)By Dynamic Programming technique, solve the problem

$$Minimize\, Z = x_1^2 + x_2^2 + x_3^2$$

Subject to $x_1 + x_2 + x_3 \ge 15$

And $x_1, x_2, x_3 \ge 0$

Solution: To develop the recursive equation:

It is a three stage problem. The decision variables are $x_1, x_2, x_3$and the state variables are S_1, S_2, S_3 are defined as

$$S_3 = x_1 + x_2 + x_3 \ge 15$$
$$S_2 = x_1 + x_2 = S_3 - x_3$$
$$S_1 = x_1 = S_2 - x_2$$

Let $f_i(S_i)$ be the minimum value of Z at the ith stage where (i=1,2,3)

Now the recursive equations are

$$f_1(S_1) = \underset{0\le x_1\le S_1}{Min}\{x_1^2\} = S_1^2 = (S_2 - x_2)^2$$

$$f_1(S_1) = (S_2 - x_2)^2 \text{-------(1)}$$

$$f_2(S_2) = \underset{0\le x_2\le S_2}{Min}\{x_1^2 + x_2^2\}$$

$$f_2(S_2) = \underset{0\le x_2\le S_2}{Min}\{x_2^2 + f_1(S_1)\} \text{-------(2)}$$

$$f_3(S_3) = \underset{0 \le x_3 \le S_3}{Min}\left\{x_1^{\;2} + x_2^{\;2} + x_3^{\;2}\right\}$$

$$f_3(S_3) = \underset{0 \le x_3 \le S_3}{Min}\left\{x_3^{\;2} + f_2(S_2)\right\} \text{-------(3)}$$

To solve the recursive equation:

From (1) $f_1(S_1) = (S_2 - x_2)^2$

From (2) $f_2(S_2) = \underset{0 \le x_2 \le S_2}{Min}\left\{x_2^{\;2} + f_1(S_1)\right\}$

$$f_2(S_2) = \underset{0 \le x_2 \le S_2}{Min}\left\{x_2^{\;2} + f_1(S_2 - x_2)\right\}$$

$$f_2(S_2) = \underset{0 \le x_2 \le S_2}{Min}\left\{x_2^{\;2} + (S_2 - x_2)^2\right\}$$

The function $x_2^{\;2} + (S_2 - x_2)^2$ will attain its minimum if f'(x)=0 and f''(x)>0

Solving $x_2 = \dfrac{S_2}{2}$

$$f_2(S_2) = \frac{S_2^{\;2}}{2}$$

From (3) $f_3(S_3) = \underset{0 \le x_3 \le S_3}{Min}\left\{x_3^{\;2} + f_2(S_2)\right\}$

$$f_3(S_3) = \underset{0 \le x_3 \le S_3}{Min}\left\{x_3^{\;2} + f_2(S_3 - x_3)\right\}$$

$$f_3(S_3) = \underset{0 \le x_3 \le S_3}{Min}\left\{x_3^{\;2} + \frac{(S_3 - x_3)^2}{2}\right\}$$

The function $x_3^{\;2} + \dfrac{(S_3 - x_3)^2}{2}$ will attain its minimum if f'(x)=0 and f''(x)>0

Solving $x_3 = \dfrac{S_3}{3}$

$$f_3(S_3) = \frac{S_3^{\;2}}{3}$$

But $S_3 \geq 15$ (i.e) Minimum of $S_3 = 15$

Z is minimum when $x_3 = \frac{S_3}{3} \Rightarrow x_3 = 5$

But $S_2 = S_3 - x_3 \Rightarrow S_2 = 10$

$x_2 = \frac{S_2}{2} \Rightarrow x_2 = 5$

Also $S_1 = S_2 - x_2 \Rightarrow S_1 = 5$

$x_1 = S_1 \Rightarrow x_1 = 5$

$\therefore f_3(S_3) = \frac{S_3^{\ 2}}{3} = \frac{15^2}{3} = 75$

The optimal policy is (5,5,5) and min Z=75

(7) Use Dynamic programming solve

Maximize Z=$y_1.y_2.y_3$

Subject to $y_1+y_2+y_3 = 5$

and $y_1, y_2, y_3 \geq 0$

Solution: To develop the recursive equation:

It is a three stage problem. The decision variables are y_1, y_2, y_3 and the state variables are S_1, S_2, S_3 are defined as

$$S_3 = x_1 + x_2 + x_3 = 5$$
$$S_2 = x_1 + x_2 = S_3 - x_3$$
$$S_1 = x_1 = S_2 - x_2$$

Let $f_i(S_i)$ be the maximum value of Z at the ith stage where (i=1,2,3)

Now the recursive equations are

$$f_1(S_1) = \underset{0 \leq y_1 \leq S_1}{Max}\{y_1\} = S_1 = S_2 - y_2$$

$f_1(S_1) = (S_2 - y_2)$-------(1)

$$f_2(S_2) = \underset{0\le y_2 \le S_2}{Max}\{y_1.y_2\}$$

$$f_2(S_2) = \underset{0\le y_2 \le S_2}{Max}\{y_2.f_1(S_1)\}\text{-------(2)}$$

$$f_3(S_3) = \underset{0\le y_3 \le S_3}{Max}\{y_1.y_2.y_3\}$$

$$f_3(S_3) = \underset{0\le y_3 \le S_3}{Max}\{y_3.f_2(S_2)\}\text{-------(3)}$$

To Solve the recursive equation:

From (1) $f_1(S_1) = S_2 - y_2 = S_1 \Rightarrow f_1(S_1) = S_1$

From (2) $f_2(S_2) = \underset{0\le y_2 \le S_2}{Max}\{y_2.f_1(S_1)\}$

$$f_2(S_2) = \underset{0\le y_2 \le S_2}{Max}\{y_2.f_1(S_2 - y_2)\}$$

$$f_2(S_2) = \underset{0\le y_2 \le S_2}{Max}\{y_2.(S_2 - y_2)\}$$

The function $y_2.(S_2 - y_2)$ will attain its maximum if f'(x)=0 and f''(x)<0

Solving $y_2 = \dfrac{S_2}{2}$

$$f_2(S_2) = \left(\frac{S_2}{2}\right)^2$$

From (3) $f_3(S_3) = \underset{0\le y_3 \le S_3}{Max}\{y_3.f_2(S_2)\}$

$$f_3(S_3) = \underset{0\le y_3 \le S_3}{Max}\{y_3.f_2(S_3 - y_3)\}$$

$$f_3(S_3) = \underset{0\le y_3 \le S_3}{Max}\left\{y_3.\left(\frac{S_3 - y_3}{2}\right)^2\right\}$$

The function $y_3.\left(\dfrac{S_3 - y_3}{2}\right)^2$ will attain its maximum if f'(x)=0 and f''(x)<0

Solving $y_3 = \frac{S_3}{3}$

$$\therefore\ f_3(S_3) = \underset{0 \le y_3 \le S_3}{Max}\left\{ y_3 . \left(\frac{S_3 - y_3}{2} \right)^2 \right\} = \left(\frac{S_3}{3} \right)^3$$

But $S_3 = 5 \Rightarrow y_3 = \frac{S_3}{3}$

$$\Rightarrow y_3 = \frac{5}{3}$$

But $S_2 = S_3 - y_3 \Rightarrow S_2 = \frac{10}{3}$

$$y_2 = \frac{S_2}{2} \Rightarrow y_2 = \frac{5}{3}$$

Also $S_1 = S_2 - y_2 \Rightarrow S_1 = \frac{5}{3}$

$$y_1 = S_1 \Rightarrow y_1 = \frac{5}{3}$$

$$\therefore\ f_3(S_3) = \left(\frac{S_3}{3} \right)^3 = \frac{125}{27}$$

The optimal policy is (5/3,5/3,5/3) and max Z=125/27

Solution of L.P.P By D.P.P Technique

(8) Use D.P.P to solve the L.P.P

$$Maximize\ Z = x_1 + 9x_2$$

Subject to $2x_1 + x_2 \le 25$

$$x_2 \le 11$$

And $x_1, x_2 \ge 0$

Solution: The given problem consists of two decision variables and two constraints. Hence the problem has two stages and two state variables.

The state of the equivalent dynamic programming are B_{1j}, B_{2j} (j=1,2)

Using backward computational procedure, we have

$$f_2(B_{12}, B_{22}) = \underset{\substack{0 \le x_2 \le B_{12} \\ 0 \le x_2 \le B_{22}}}{Max}\{9x_2\} = 9 \underset{\substack{0 \le x_2 \le 25 \\ 0 \le x_2 \le 11}}{Max}\{x_2\}$$

Since Max $\{x_2\}$ which satisfies the condition of $0 \le x_2 \le 25$ & $0 \le x_2 \le 11$ is the minimum of (25,11)

$$f_2(B_{12}, B_{22}) = 9Min(25,11) \text{ --------(1)}$$

Now $f_1(B_{11}, B_{21}) = \underset{0 \le x_1 \le \frac{B_{11}}{2}}{Max}\{x_1 + f_2(B_{11} - 2x_1, B_{21})\}$

At this stage $B_{11}=25$, $B_{21}=11$

$$f_1(25,11) = \underset{0 \le x_1 \le \frac{25}{2}}{Max}\{x_1 + 9\min(25 - 2x_1, 11)\}$$

$$\min(25 - 2x_1, 11) = \begin{cases} 11, & if\ 0 \le x_1 \le 7 \\ 25 - 2x_1, & if\ 7 \le x_1 \le 25/2 \end{cases}$$

$$x_1 + 9\min(25 - 2x_1, 11) = \begin{cases} x_1 + 99, & if\ 0 \le x_1 \le 7 \\ 225 - 17x_1, & if\ 7 \le x_1 \le 25/2 \end{cases}$$

Since the maximum of both $x_1 + 99$ and $225 - 17x_1$ occurs only at $x_1 = 7$

$$f_1(25,11) = \underset{0 \le x_1 \le \frac{25}{2}}{Max}\{x_1 + 9\min(25 - 2x_1, 11)\}$$
$$= 7 + 9\min(11,11)$$
$$\therefore f_1(25,11) = 106$$

$$x_2 = \min(25 - 2x_1, 11)$$
$$= \min(11,11)$$
$$\therefore x_2 == 11$$

Hence the optimal solution Max Z=106 at x_1=7 and x_2=11

(9) Solve the following L.P.P Using D.P.P Approach

$$Maximize\ Z = 2x_1 + 5x_2$$

Subject to $2x_1 + x_2 \le 43$

$2x_2 \le 46$ And $x_1, x_2 \ge 0$

Solution: The given problem consists of two decision variables and two constraints. Hence the problem has two stages and two state variables.

The state of the equivalent dynamic programming are B_{1j}, B_{2j} (j=1,2)

Using backward computational procedure, we have

$$f_2(B_{12}, B_{22}) = \underset{\substack{0 \le x_2 \le B_{12} \\ 0 \le 2x_2 \le B_{22}}}{Max} \{5x_2\} = 5 \underset{\substack{0 \le x_2 \le 43 \\ 0 \le x_2 \le 23}}{Max} \{x_2\}$$

Since Max $\{x_2\}$ which satisfies the condition of $0 \le x_2 \le 43$ & $0 \le x_2 \le 23$ is the minimum of (43,23)

$$f_2(B_{12}, B_{22}) = 5Min(43,23) \text{--------(1)}$$

Now $f_1(B_{11}, B_{21}) = \underset{0 \le x_1 \le \frac{B_{11}}{2}}{Max} \{2x_1 + f_2(B_{11} - 2x_1, B_{21})\}$

At this stage $B_{11}=43$, $B_{21}=46$

$$f_1(43,46) = \underset{0 \le x_1 \le \frac{43}{2}}{Max} \{2x_1 + 5\min(43 - 2x_1, 23)\}$$

$$\min(43 - 2x_1, 23) = \begin{cases} 23, & if\ 0 \le x_1 \le 10 \\ 43 - 2x_1, & if\ 10 \le x_1 \le 43/2 \end{cases}$$

$$2x_1 + 5\min(43 - 2x_1, 23) = \begin{cases} 2x_1 + 115, & if\ 0 \le x_1 \le 10 \\ 215 - 18x_1, & if\ 10 \le x_1 \le 43/2 \end{cases}$$

Since the maximum of both $2x_1 + 115$ and $215 - 18x_1$ occurs only at $x_1 = 10$

$$f_1(43,46) = \underset{0 \le x_1 \le \frac{43}{2}}{Max} \{2x_1 + 5\min(43 - 2x_1, 23)\}$$

$$= 20 + 5\min(23,23)$$

$$\therefore f_1(43,46) = 135$$

$$x_2 = \min(43 - 2x_1, 23)$$
$$= \min(23,23)$$
$$\therefore x_2 = 23$$

Hence the optimal solution Max Z=135 at x_1=10 and x_2=23

(10) Solve the following L.P.P Using D.P.P Approach

$$Maximize\ Z = 3x_1 + 5x_2$$

Subject to $x_1 \le 4$; $x_2 \le 6$;

$3x_1 + 2x_2 \le 18$ and $x_1, x_2 \ge 0$

Solution: The given problem consists of two decision variables and three constraints. Hence the problem has two stages and three state variables.

The state of the equivalent dynamic programming are B_{1j},B_{2j} ,B_{3j} (j=1,2,3)

Using backward computational procedure, we have

$$f_2(B_{12}, B_{22}, B_{32}) = \underset{\substack{0 \le x_2 \le B_{22} \\ 0 \le 2x_2 \le B_{32}}}{Max} \{5x_2\} = 5 \underset{\substack{0 \le x_2 \le 6 \\ 0 \le x_2 \le 9}}{Max} \{x_2\}$$

Since Max $\{x_2\}$which satisfies the condition of $0 \le x_2 \le 6$ & $0 \le x_2 \le 9$ is the minimum of (6,9)

$$f_2(B_{12}, B_{22}, B_{32}) = 5Min(6,9) \text{--------(1)}$$

Now $f_1(B_{11}, B_{21}, B_{31}) = \underset{\substack{0 \le x_1 \le B_{11} \\ 0 \le 3x_1 \le B_{31}}}{Max} \left\{ 3x_1 + f_2\left(B_{11} - x_1, B_{21}, \frac{B_{31} - 3x_1}{2} \right) \right\}$

At this stage B_{11}=4, B_{21}=6, B_{31}=18

$$f_1(4,6,18) = \underset{\substack{0 \le x_1 \le 4 \\ 0 \le x_1 \le 6}}{Max} \left\{ 3x_1 + 5\min\left(6, \frac{18 - 3x_1}{2} \right) \right\}$$

$$\min\left(6, \frac{18 - 3x_1}{2} \right) = \begin{cases} 6, & if\ \ 0 \le x_1 \le 2 \\ \frac{18 - 3x_1}{2}, & if\ 2 \le x_1 \le 4 \end{cases}$$

$$3x_1 + 5\min\left(6, \frac{18-3x_1}{2}\right) = \begin{cases} 3x_1 + 30, & if\ \ 0 \le x_1 \le 2 \\ \dfrac{90-9x_1}{2}, if & 2 \le x_1 \le 4 \end{cases}$$

Since the maximum of both $3x_1 + 30$ and $\frac{90-9x_1}{2}$ occurs only at $x_1 = 2$

$$f_1(4,6,18) = \underset{0 \le x_1 \le 4}{Max}\left\{3x_1 + 5\min\left(6, \frac{18-3x_1}{2}\right)\right\}$$
$$= 6 + 5\min(6,6)$$
$$\therefore\ f_1(43,46) = 36$$

$$x_2 = \min\left(6, \frac{18-3x_1}{2}\right)$$
$$= \min(6,6)$$
$$\therefore x_2 == 6$$

Hence the optimal solution Max Z=36 at x_1=2 and x_2=6

(11) Solve the following L.P.P Using D.P.P Approach

$$Maximize\ Z = 4x_1 + 14x_2$$

Subject to $2x_1 + 7x_2 \le 21$

$7x_1 + 2x_2 \le 21$ and $x_1, x_2 \ge 0$

Solution: The given problem consists of two decision variables and two constraints. Hence the problem has two stages and two state variables.

The state of the equivalent dynamic programming are B_{1j},B_{2j} (j=1,2)

Using backward computational procedure, we have

$$f_2(B_{12}, B_{22}) = \underset{\substack{0 \le 7x_2 \le B_{12} \\ 0 \le 2x_2 \le B_{22}}}{Max}\{14x_2\} = 514\underset{\substack{0 \le x_2 \le B_{12}/7 \\ 0 \le x_2 \le B_{22}/2}}{Max}\{x_2\}$$

Since Max {x_2}which satisfies the condition of 0≤7x_2≤43 & 0≤2x_2≤23 is the minimum of (21/7,21/2)

$$f_2(B_{12}, B_{22}) = 14Min\left(21/7, 21/2\right)\text{--------(1)}$$

Now $f_1(B_{11}, B_{21}) = \underset{\substack{0 \le x_1 \le \frac{B_{11}}{2} \\ 0 \le x_1 \le \frac{B_{21}}{7}}}{Max} \{4x_1 + f_2(B_{11} - 2x_1, B_{21} - 7x_1)\}$

At this stage $B_{11}=21$, $B_{21}=21$

$$f_1(21,21) = \underset{\substack{0 \le x_1 \le \frac{21}{2} \\ 0 \le x_1 \le 3}}{Max} \left\{4x_1 + 14\min\left(\frac{21-2x_1}{7}, \frac{21-7x_1}{2}\right)\right\} \text{-----(2)}$$

$$\min\left(\frac{21-2x_1}{7}, \frac{21-7x_1}{2}\right) = \begin{cases} \frac{21-2x_1}{7} & if\ 0 \le x_1 \le 7/3 \\ \frac{21-7x_1}{2} & if\ 7/3 \le x_1 \le 3 \end{cases}$$

(2) implies

$$f_1(21,21) = \underset{\substack{0 \le x_1 \le \frac{21}{2} \\ 0 \le x_1 \le 3}}{Max} \begin{Bmatrix} 42, & , & if\ 0 \le x_1 \le 7/3 \\ 147 - 45x_1, & & if\ 7/3 \le x_1 \le 3 \end{Bmatrix}$$

the maximum value of the above function occurs at each value of $x_1$in (0,7/3) and the maximum value is 42.

Let $x_1 = \lambda$

$$x_2 = \min\left(\frac{21-2x_1}{7}, \frac{21-7x_1}{2}\right)$$

$$= \min\left(\frac{21-2\lambda}{7}, \frac{21-7\lambda}{2}\right)$$

$$\therefore x_2 = \frac{21-2\lambda}{7}\ where\ 0 \le \lambda \le 7/3$$

Hence the optimal solution Max Z=42 at $x_1 = \lambda$ and $x_2 = \frac{21-2\lambda}{7}$