CONCEPT OF STABILITY & ROOT LOCUS TECHNIQUES

Stability

A system is stable if its output is bounded (finite) for any bounded (finite) input.

Characteristic equation

The denominator Polynomial of $C(s)/R(s)$ is the characteristics of the system.

Response of a system

Let the Closed loop T.F $= \frac{C(s)}{R(s)} = M(s)$

$\therefore$ output / Response in s domain $C(s) = M(s)R(s)$

$$c(t) = L^{-1}[C(s)]$$
$$r(t) = L^{-1}[R(s)]$$

For an impulse input, $r(t) = \delta(t)$

$$\therefore R(s) = L[\delta(t)] = 1$$

Impulse response of the system $= L^{-1}[C(s)]$

$$= L^{-1}[M(s)R(s)]$$
$$= L^{-1}[M(s)] = M(t)$$

Impulse response is the inverse Laplace transform of the system T.F

By Convolution theorem

$$L^{-1}[M(s)R(s)] = \int_{-\infty}^{\infty} m(\tau)\, r(t-\tau)\, d\tau$$

Response of the system in time domain,

$$c(t) = L^{-1}[M(s)R(s)] = \int_{-\infty}^{\infty} m(\tau)\, r(t-\tau)\, d\tau$$

Bounded-Input Bounded-output (BIBO) Stability

A linear relaxed system is said to have BIBO stability if every bounded (finite) I/P results in a bounded (finite) output.

Consider the response in time domain

$$c(t) = \int_0^\infty m(\tau)\, r(t-\tau)\, d\tau$$

If the input $r(t)$ is bounded then there exists a constant A_1, such that $|r(t)| \leq A_1 < \infty$

$$|c(t)| \leq \int_0^\infty |m(\tau)\, r(t-\tau)|\, d\tau$$

$$\leq \int_0^\infty |m(\tau)|\,|r(t-\tau)|\, d\tau$$

$$\leq \int_0^\infty |m(\tau)|\, A_1\, d\tau \qquad \because \text{for bounded I/P } |r(t-\tau)| \leq A_1$$

$$\leq A_1 \int_0^\infty |m(\tau)|\, d\tau$$

If the output $c(t)$ is bounded then, there exists a constant A_2, such that $|c(t)| \leq A_2 \leq \infty$

$$\therefore A_1 \int_0^\infty |m(\tau)|\, d\tau \leq A_2 < \infty$$

The above condition is satisfied if

$$\int_0^\infty |m(\tau)|\, d\tau < \infty$$

$\therefore$ The bounded O/P $\int_0^\infty |m(\tau)|\, d\tau < \infty$

Conclusion

A system with the impulse response $m(\tau)$ is BIBO stable if and only if the Impulse response is absolutely integrable i.e $\int_0^\infty |m(\tau)|\,d\tau$ is finite.

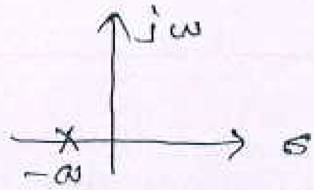
i.e Area under the curve $m(\tau)$ is evaluated from $t=0$ to $t=\infty$ must be of finite value.

Location of Roots on the splane for stability

Roots of the characteristic equation are the poles of the closed loop T.F $M(s) = \dfrac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2}}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots}$

Roots on the splane

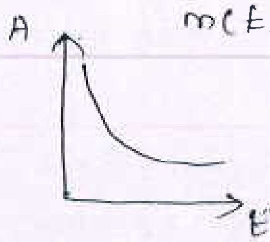
$M(s) = \dfrac{A}{s+a}$



Roots on negative real axis

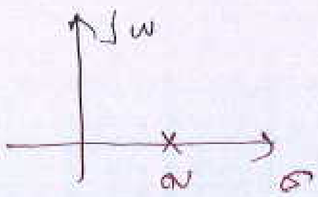
Impulse Response

$m(t) = L^{-1}\left[\dfrac{A}{s+a}\right] = Ae^{-at}$



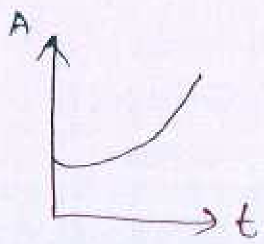
Response is exponentially decaying

$M(s) = \dfrac{A}{s-a}$



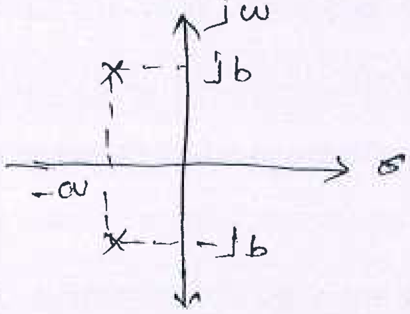
Roots on +ve Real axis

$m(t) = Ae^{at}$



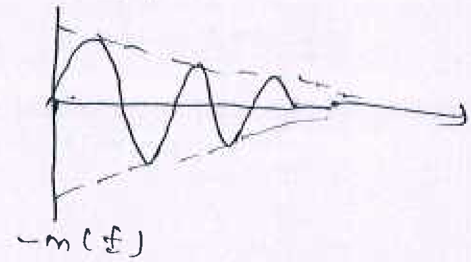
Response is exponentially Increasing

$$M(s) = \frac{A}{s+a+jb} + \frac{A}{s+a-jb}$$



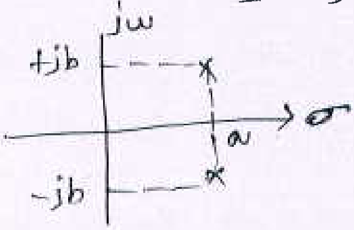
Complex Conjugate roots on left half of splane

$$m(t) = A e^{-(a+jb)t} + A e^{-(a-jb)t}$$



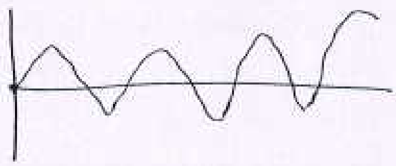
The response is damped sinusoidal.

$$M(s) = \frac{A}{s-a+jb} + \frac{A}{s-a-jb}$$



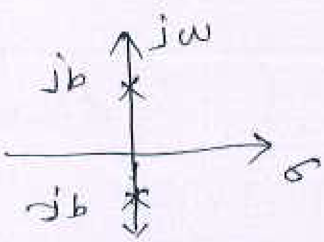
Complex conjugate roots on right half of splane

$$m(t) = A e^{-(-a+jb)t} + A e^{-(-a-jb)t}$$



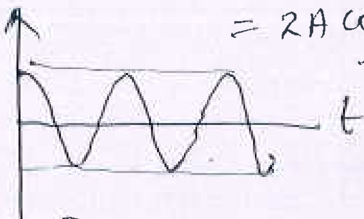
Response is exponentially Increasing sinusoidal

$$M(s) = \frac{A}{s+jb} + \frac{A}{s-jb}$$



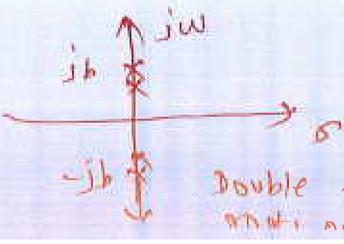
Roots on imaginary axis

$$m(t) = A e^{-jbt} + A e^{jbt}$$
$$= 2A\cos bt$$
$$= 2A\sin(bt+90)$$



Response is oscillatory

$$M(s) = \frac{A}{(s+jb)^2} + \frac{A}{(s-jb)^2}$$



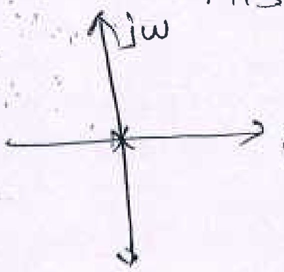
Double pair

$$m(t) = L^{-1}\left[\frac{A}{(s+jb)^2} + \frac{A}{(s-jb)^2}\right]$$
$$= Ate^{-jb} + Ate^{jb}$$
$$= 2At\cos bt$$
$$= 2At\sin(bt+90)$$

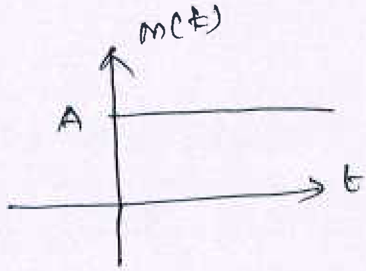
Response is linearly

③

$M(s) = A/s$

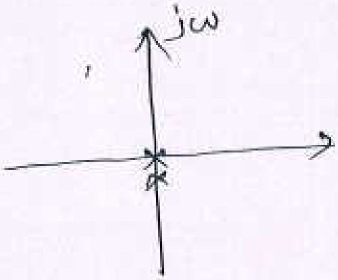


Single roots at origin



$m(t) = A$

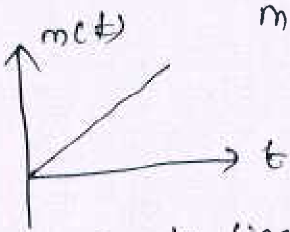
Response is constant



$M(s) = \frac{A}{s^2}$

Double roots at origin

$m(t) = At$



Response is linearly increasing with time.

The characteristic polynomial of a system is $s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$

s^7	1	24	24	23
s^6	9	24	24	15
s^6	3	8	8	5
s^5	1	1	1	
s^4	1	1	1	
s^3	0	0		
s^3	2	1		
s^2	0.5	1		
s^1	-3			
s^0	1			

$A = s^4 + s^2 + 1$

$\frac{dA}{ds} = 4s^3 + 2s$

$s^3 : 4 \quad 2$

$\quad\; 2 \quad 1$

$s^4 + s^2 + 1$

$s^2 = x$

$s = \pm\sqrt{x}$

$x^2 + x + 1 = 0$

$x = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

The system is unstable

Two roots are lying on right half of s plane and five roots are lying on left half of s plane

The characteristic Polynomial of a system is $s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36=0$

$$\begin{array}{c|cccc}
s^7 & 1 & 9 & 4 & 36 \\
s^6 & 5 & 9 & 20 & 36 \\
s^5 & 1 & 0 & 4 & \\
s^4 & 1 & 0 & 4 & \\
s^3 & 0 & 0 & & \\
s^3 & 1 & 9 & & \\
s^2 & & & & \\
s^0 & & & &
\end{array}$$

$\rightarrow 1 \quad 1.8 \quad 4 \quad 7.2$

s^4+4

$4s^3$

$s^4+4=0$

$s^2=x$

$x^2+4=0$

$x=\pm j2$

$s=\pm\sqrt{x}$

$=\pm\sqrt{j2}$ $\quad 2\angle 90$ or

$=\pm\sqrt{2}\angle 45$ $\quad 2\angle -90$

$=\pm(1+j1)$

or

$\pm(1+j1)$

Construct Routh array and determine the stability of the System represented by the characteristic equation $s^5+s^4+2s^3+2s^2+3s+5=0$ Comment on the location of the roots of characteristic equation.

$$\begin{array}{c|ccc}
s^5 & 1 & 2 & 3 \\
s^4 & 1 & 2 & 5 \\
s^3 & \varepsilon & -2 & \\
s^2 & \frac{2\varepsilon+2}{\varepsilon} & & \\
s^1 & \frac{-(5\varepsilon^2+4\varepsilon+4)}{2\varepsilon+2} & & \\
s^0 & 5 & &
\end{array}$$

$\varepsilon\to 0$

$$\begin{array}{c|ccc}
s^5 & 1 & 2 & 3 \\
s^4 & 1 & 2 & 5 \\
s^3 & 0 & -2 & \\
s^2 & \infty & 5 & \\
s^1 & -2 & & \\
s^0 & 5 & &
\end{array}$$

Unstable

STABILITY OF CONTROL SYSTEM

Routh Hurwitz Criterion

It is an algebraic method to determine the absolute stability of a linear time invariant system. It determine the nature of the roots in the s-plane directly without analysing the complete system.

The Routh stability criterion is based on ordering the coefficients of characteristic equation. $a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + --- a_{n-1} s^1 + a_n s^0 = 0$ where $a_0 > 0$ into a schedule called the Routh array.

$$\begin{array}{c|cccc}
s^n & a_0 & a_2 & a_4 & a_6 \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 \\
s^{n-2} & b_1 & b_2 & b_3 & \\
s^{n-3} & c_1 & c_2 & & \\
s^2 & e_1 & & & \\
s^1 & f_1 & & & \\
s^0 & a_n & & &
\end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

1. Normal Routh array (Non-zero elements in the first column of Routh array)

1. Using Routh criterion, determine the stability of the system represented by the characteristic equation $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$. Comment on the location of the roots of characteristic equation.

$$\begin{array}{lll}
s^4 : & 1 & 18 \quad 5 \\
s^3 : & 8 & 16 \\
s^3 : & 1 & 2 \\
s^2 : & 16 & 5
\end{array}$$

$$s^2 : \frac{1 \times 18 - 1 \times 2}{1} \qquad \frac{1 \times 5 - 0}{1}$$

$$s^2 : 16 \qquad 5$$

$$s^1 : \frac{16 \times 2 - 5 \times 1}{16} = 1.7$$

$$s^0 : 5$$

s^1 : 1.7

s^0 : 5

In the first column of routh array all the elements are positive.

∴ (i) The system is stable

(ii) All the four roots are lying on the left half of s-plane.

Conditions

i) If there is no sign change in first column & if there is no row with all zeros, then all the roots are lying on left half of s-plane and the system is stable.

ii) If there are sign changes in first column and there is no row with all zeros, then some of the roots are lying on the right half side of splane & the system is unstable.

no. of sign changes = no of roots lying on the right half of s-plane.

(iii) If there is a row of all zeros, and there is no sign change in the first column then there is a possibility of roots on imaginary axis and system is limitedly or marginally stable.

2. Determine the stability of a closed loop system whose open loop T.F is given by $\frac{10}{(s+1)(5s+1)}$ using Routh criterion.

$$G(s)\,H(s) = \frac{10}{(s+1)(5s+1)}$$

The characteristic equation is given by $1+G(s)H(s)=0$

$$1+\frac{10}{(s+1)(5s+1)}=0$$

$$(s+1)(5s+1)+10=0$$

$5s^2+s+5s+1+10=0$

$5s^2+6s+11=0$ (5)

$s^2 : 5 \quad 11$

$s^1 : 6$

$s^0 : 11$

$s^2 \;\Big|\; \dfrac{6\times 11-5\times 0}{6}=11$

In the first coloumn there is no sign change, so the system is stable and the two roots are lying on the left hand of s plane.

Case II : A Row of all zeros

Construct Routh array and determine the stability of the system whose characteristic equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. Also determine the no of roots lying on right half of s-plane, left half of s-plane and on imaginary axis

The characteristic eqn of the system is

$$s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$$

s^6	1	8	20	16
s^5	2	12	16	
s^5	1	6	8	
s^4	1	6	8	
s^3	0	0		
s^3	1	3		
s^2	3	8		
s^1	0.33			
s^0	8			

The auxillary equation $A=s^4+6s^2+8$ on differentiating A w.r.t s

$$\frac{dA}{ds}=4s^3+12s$$

s^3	4	12
s^3	1	3

In first column there is no sign change. The row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.

The auxillary Polynomial is

$s^4 + 6s^2 + 8 = 0$ Let $s^2 = x$ $\therefore x^2 + 6x + 8 = 0$

$$= \frac{-6 \pm \sqrt{6^2 - 4 \times 8}}{2} = \frac{-6 \pm \sqrt{4}}{2} = -3 \pm 1$$
$$= -2 \text{ or } -4$$

$s^2 = x$

$s = \sqrt{x}$ The roots are $s = \pm\sqrt{x} = \pm\sqrt{-2}$ & $\pm\sqrt{-4}$

$= \pm j\sqrt{2}, -j\sqrt{2}, +j2, -j2$

The roots of auxillary eqn are also roots of characteristic eqn. Hence 4 roots are in the imaginary axis and remaining 2 roots are lying on the left half of s-plane.

Type - III (First element of a row is zero)

Construct Routh array and determine the stability of the system represented by the characteristic equation $9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$.

s^5	9	10	-9
s^4	-20	-1	-10
s^3	9.55	-13.5	
s^2	-29.3	-10	
s^1	-16.8		
s^0	-10		

The system is unstable

Three roots are lying on right half of splane and two roots are lying on left half of splane.

1. Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation $s^5+4s^4+8s^3+8s^2+7s+4=0$

s^5	1	8	7
s^4	~~4~~	~~8~~	~~4~~
s^4	1	2	1
s^3	1	1	0
s^2	1	1	
s^1	ε		
s^0	1		

$\varepsilon \to 0$

$s^2+1=0 \qquad s^2=-1$

$s=\pm\sqrt{-1}=\pm j1$

System is limitedly or marginally stable

Two roots are lying on imaginary axis & three roots are lying on left half of s-plane.

Determine the range of K for stability of unity feed back system whose open loop T/f is $G(s)=\dfrac{K}{s(s+1)(s+2)}$

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\dfrac{K}{s(s+1)(s+2)}}{1+\dfrac{K}{s(s+1)(s+2)}}=\frac{K}{s^3+3s^2+2s+K=0}$$

s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	
s^0	K	

$s^0 \to K>0$

$s^1 \to \dfrac{6-K}{3}>0$

$0<K<6$

The open loop transfer function of a unity feed back control system is given by $G(s)=\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}$ By applying the routh criterion, discuss the stability of the ... as a fn of K ...

oscillations in the closed loop system. what are the corresponding oscillating frequencies.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{(s+2)(s^2+6s+25)(s+4)}}{1+\frac{K}{(s+2)(s+4)(s^2+6s+25)}}$$

$$= \frac{K}{(s+2)(s+4)(s^2+6s+25)+K}$$

The Characteristic equation is $s^4+12s^3+69s^2+198s+200+K=0$

s^4	1	69	200+K
s^3	12	198	
s^3	1	16.5	
s^2	52.5	200+K	
s^1	$\frac{666.25-K}{52.5}$		
s^0	200+K		

$s^1 \rightarrow (666.25-K) > 0$

K less than 666.25

$s^0 \rightarrow (200+K) > 0$

$K > -200$ K starts at 0

$0 < K < 666.25$

$K = 666.25$ for s to be zero.

$$52.5s^2+200+K=0$$

$$s^2 = \frac{-200-666.25}{52.5} = -16.5$$

$$s = \pm\sqrt{-16.5} = \pm j\sqrt{16.5} = \pm j4.06$$

$$\omega = 4.06 \text{ rad/sec}$$

A feed back system has open loop transfer function of $G(s) = \frac{Ke^{-s}}{s(s^2+5s+9)}$. Determine the maximum value of K for stability of closed loop system.

$$G(s) = \frac{Ke^{-s}}{s(s^2+5s+9)} = \frac{K(1-s)}{s(s^2+5s+9)}$$

$$\frac{C(s)}{R(s)} = \frac{K(1-s)}{s(s^2+5s+9)+K(1-s)}$$

$$s(s^2+5s+9)+K(1-s)=0$$

$$s^3+5s^2+(9-K)s+K=0$$

$$\begin{array}{c|cc} s^3 & 1 & 9-K \\ s^2 & 5 & K \\ s^1 & 9-1.2K & \\ s^0 & K & \end{array}$$

$(9-1.2K)>0$

$1.2K<9$

$K<\dfrac{9}{1.2}=7.5$

$s^0 \rightarrow K>0$

$0<K<7.5$

Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation

$$s^6+\cancel{5}s^5+3s^4+3s^3+3s^2+2s+1=0$$

$$\begin{array}{c|cccc} s^6 & 1 & 3 & 3 & 1 \\ s^5 & 1 & 3 & 2 & \\ s^4 & \varepsilon & 1 & 1 & \\ s^3 & \frac{3\varepsilon-1}{\varepsilon} & \frac{2\varepsilon-1}{\varepsilon} & & \\ s^2 & \frac{-2\varepsilon^2+4\varepsilon-1}{3\varepsilon-1} & 1 & & \\ s^1 & \frac{4\varepsilon^2-\varepsilon}{2\varepsilon^2-4\varepsilon+1} & & & \\ s^0 & 1 & & & \end{array}$$

$\varepsilon \rightarrow 0$

$$\begin{array}{c|cccc} s^6 & 1 & 3 & 3 & 1 \\ s^5 & 1 & 3 & 2 & \\ s^4 & 0 & 1 & 1 & \\ s^3 & -\infty & -\infty & & \\ s^2 & 1 & 1 & & \\ s^1 & 0 & & & \\ s^1 & 2 & & & \\ s^0 & 1 & & & \end{array}$$

$s^2+1=0$

$2s+0$

$2s$

$s=\pm j$

System is unstable

Two roots are lying on imaginary axis, two roots are lying on right half of s-plane and two roots are lying on left half of s-Plane

Root Locus

The Root locus technique was introduced by W.R. Evans in 1948 for analysis of Control Systems.

Root locus technique is a powerful tool for adjusting the location of closed loop poles to achieve the desired system performance by varying one or more system Parameters.

Consider the open loop Transfer function

$$G(s) = \frac{K}{s(s+P_1)(s+P_2)}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s(s+P_1)(s+P_2)+K}$$

Characteristics equation $s(s+P_1)(s+P_2)+K=0$

When the gain K, is varied from 0 to ∞, the roots of characteristics equation will take different values.

when $K=0$, roots are given by open loop poles.

when $K \to \infty$, the roots will take the value of open loop zeros.

The path taken by the roots of characteristic equation when open loop gain K is varied from 0 to ∞ are called root loci

Procedure for constructing root locus

Step 1 – Locate the Poles and zeros of $G(s)H(s)$ on the s plane.

The Root locus branch start from open loop poles and terminate at zeros.

Explanation – n- Poles are marked by 'x' → By cross

m- zeros are marked by 'o' → By small circles

Step 2 – Determine the root locus on real axis

Explanation – Take a test point in Real axis. If the total no of poles and zeros on the real axis to the right of this test point is odd number then the test point lies on the root locus. If it is even then the test point does not lies on the root locus.

Step 3. Determine the asymptotes of Root locus branches and meeting point of asymptotes with Real axis.

Expl

$$\text{Angle of asymptotes} = \frac{\pm 180(2q+1)}{n-m}$$

where $q = 0, 1, 2, 3 \text{ ---- } (n-m)$

Centroid (meeting point of asymptote with real axis) $= \dfrac{\text{Sum of Poles} - \text{Sum of Zeros}}{n-m}$

Step 4. Find the Break away & Break in Points.

Exp If there is a root locus on real axis b/w two Poles then there exists a break away point

If there is a root locus on real axis b/w two zeros then there exists a break in point

The break away & break in points is given by

$\frac{dk}{ds} = 0$ with gain k should be +ve & real

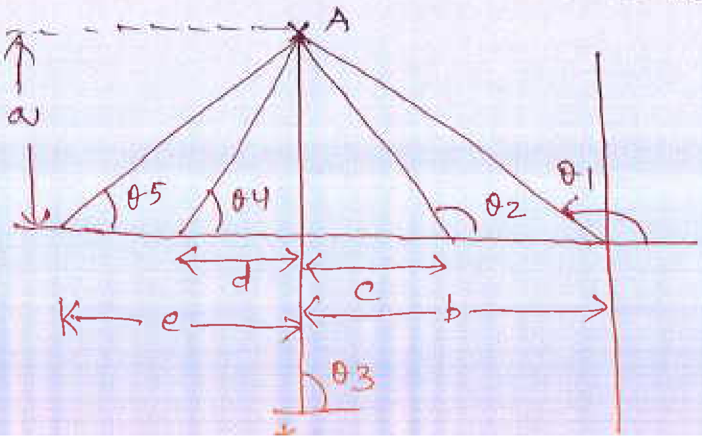
Step 5 If there is a complex pole then determine the angle of departure from the complex pole.

If there is a complex zero then determine the angle of arrival at the complex zero.

Explanation

Angle of Departure and angle of arrival

Angle of Departure = 180 − (sum of angles of vector to the complex pole A from the other poles) + (sum of angles of vector to the complex pole A from zeros)



$\theta_1 = 180 - \tan^{-1}(a/b)$

$\theta_2 = 180 - \tan^{-1}(a/c)$

$\theta_3 = 90^\circ$

$\theta_4 = \tan^{-1}(a/d)$

$\theta_5 = \tan^{-1}(a/e)$

Calculation of angle of departure

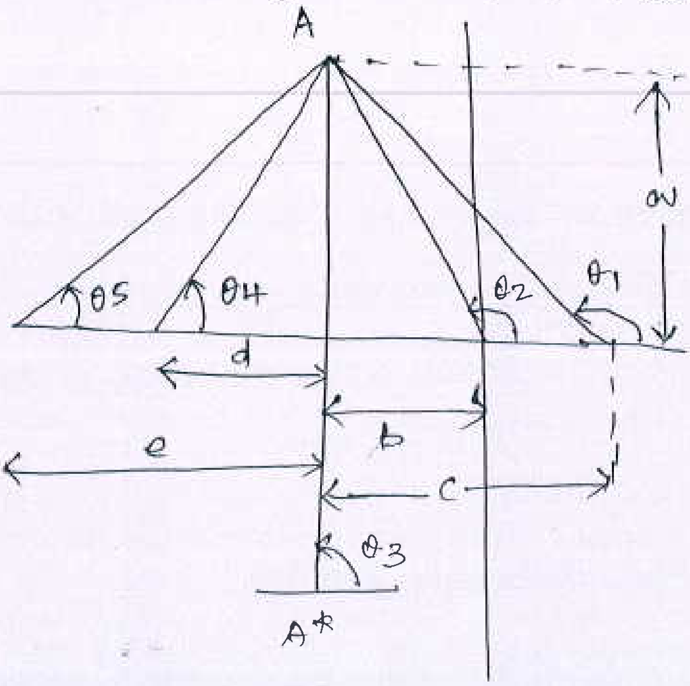
If Poles are complex then they exist only as conjugate Pairs. Consider two complex conjugate Poles A & A*

Angle of departure at Pole A = $180 - (\theta_1+\theta_3+\theta_5)+(\theta_2+\theta_4$

Angle of departure at Pole A* = $-$[Angle of departure at Pole A]

Angle of arrival

Angle of arrival at a complex zero A = 180 − (sum of angles of vectors to the complex zero A from all other zeros) + (sum of angles of vectors to the complex zero A from Poles)



$\theta_1 = 180 - \tan^{-1}(a/c)$

$\theta_2 = 180 - \tan^{-1}(a/b)$

$\theta_3 = 90^\circ$

$\theta_4 = \tan^{-1}(a/d)$

$\theta_5 = \tan^{-1}(a/c)$

If zeros are complex then they exist only as conjugate Pairs. Consider two complex conjugate zeros A & A*

Angle of arrival at zero A = $180 - (\theta_1+\theta_3)+(\theta_2+\theta_4+\theta_5)$

Angle of arrival at zero A* = $-$[Angle of arrival at zero A]

Step 6: Find the Points where the root loci may cross the imaginary axis

Explanation : Let $s = j\omega$ in the characteristics equation seperate the real part and imaginary part

Two equations are obtained one by equating the real part to zero and other by equating the imaginary part to zero. Solve the two equations for ω & k

Step 7:

Take a series of test point in the broad neighbourhood of the origin of the s-plane and adjust the test point to satisfy angle criterion. Sketch the root locus by joining the test point by smooth curve

Step 8: The value of gain k at any point on the locus can be determined from magnitude condition

Magnitude condition is given by

$$\left.\begin{array}{r}\text{Gain } k \text{ at a point}\\ (s=s_a)\end{array}\right\} = \frac{\text{Product of length of vectors from poles to the point } (s=s_a)}{\text{Product of length of vectors from finite zeros to the point } (s=s_a)}$$

Put $\frac{dK}{ds} = 0$ $\quad -(3s^2+8s+13)=0$

$$3s^2+8s+13=0$$

$$s=-1.33 \pm j1.6$$

Check for K

when $s=(-1.33+j1.6)$, the value of K is given by

$$K=-(s^3+4s^2+13s)$$

$$=-\left[(-1.33+j1.6)^3+4(-1.33+j1.6)^2+13(-1.33+j1.6)\right]$$

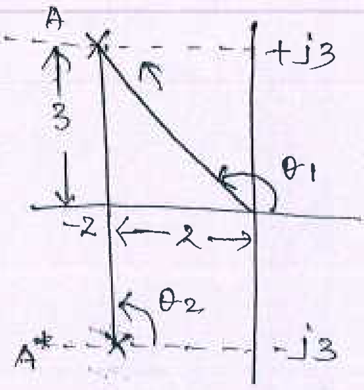
Positive and real

Also when $s=-1.33-j1.6$ the value of K is not equal to real and Positive.

Since the values of K for $s=-(1.33 \pm j1.6)$ are not real and Positive, the Points are not an actual breakaway or breakin Points. The root locus has neither breakaway nor break in Point

Step 5: To find the angle of departure

Let us consider the complex pole A shown. Draw vectors from all other poles to the pole A.



Let the angles of these vectors be θ_1 & θ_2

Here, $\theta_1 = 180 - \tan^{-1}(3/2) = 123.7^\circ$

$\theta_2 = 90^\circ$

Angle of departure from the complex Pole $A = 180-(\theta_1+\theta_2)$

$$=180-(123.7+90^\circ)=-33.7$$

The angle of departure at complex Pole A^* is negative of the angle of departure at complex Pole A.

$\therefore$ Angle of departure at Pole $A^* = +33.7^\circ$

Mark the angles of departure at complex poles using Protractor

Step 6: To find the crossing Point on imaginary axis

The characteristic equation is given by

$$s^3+4s^2+13s+K=0$$

Put $s=j\omega$

$$(j\omega)^3+4(j\omega)^2+13(j\omega)+K=0$$

$$-j\omega^3-4\omega^2+13j\omega+K=0$$

On equating imaginary part to zero, we get

$$-\omega^3+13\omega=0$$

$$-\omega^3=-13\omega$$

$$\omega^2=13$$

$$\omega=\pm 3.6$$

On equating real part to zero, we get

$$-4\omega^2+K=0$$

$$K=4\omega^2=4(13)=52$$

The crossing point of root locus is $\pm j3.6$. The value of K at this crossing point is $K=52$.

The complete root locus sketch is shown. The root locus has three branches one branch start at the pole at origin and travel through negative real axis to meet the zero at infinity. The other two root locus branches starts at complex poles (along the angle of departure), crosses the imaginary axis at $\pm j3.6$ and travel parallel to asymptotes to meet the zeros at infinity.

Sketch the root locus of the system whose open loop transfer function is $G(s)=\dfrac{K}{s(s+2)(s+4)}$. Find the value of K so that the damping ratio of the closed loop system is 0.5.

Step 1: To locate poles & zeros

The poles of open loop t/f are the roots of the equation $s(s+2)(s+4)=0$

$\therefore$ The poles are, $s=0,-2,-4$

Step 2: To find the root locus on real axis

There are three poles on the real axis. Choose a test point on real axis between ...

A unity feed back control system has an open loop transfer function $G(s) = \frac{K}{s(s^2+4s+13)}$. Sketch the root locus.

Solution

Step 1 : To locate Poles and zeros

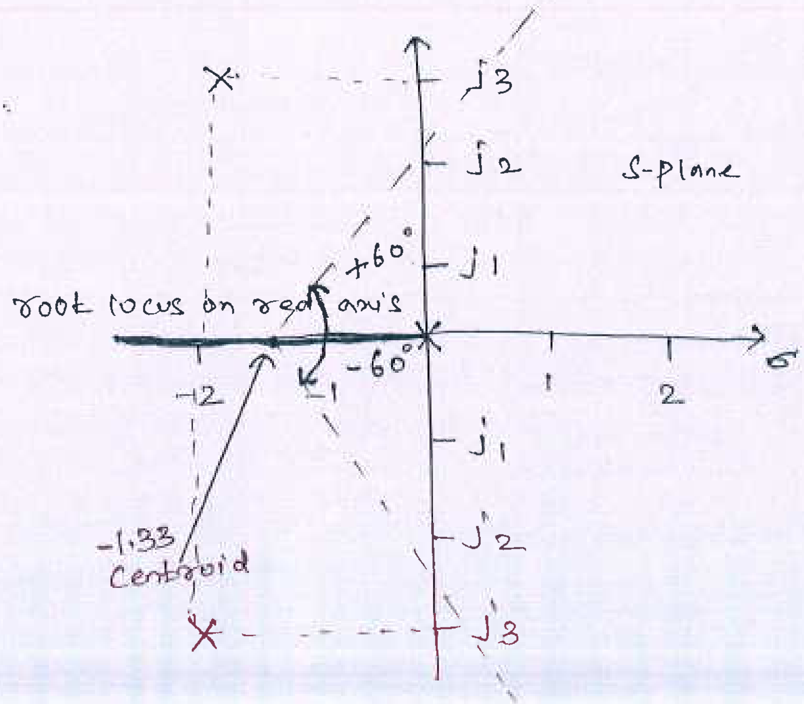
The Poles of open loop t/f function are the roots of the equation, $s(s^2+4s+13) = 0$

The roots of the quadratic are, $s = -2 \pm j3$

$\therefore$ The Poles are $0, -2+j3$ & $-2-j3$

Step 2 : To find the root locus on real axis

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative axis then to the right of that point the total no of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus. The root locus on real axis is shown as a bold line.



Step 3: To find angles of asymptotes and centroid

Since there are 3 poles, the no of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The no of asymptote's required are three.

Angles of asymptotes $= \pm \frac{180(2q+1)}{n-m}$

Here $n = 3$ & $m = 0$

If $q = 0$, Angles $= \pm \frac{180}{3} = \pm 60$

If $q = 1$, Angles $= \pm \frac{180 \times 3}{3} = \pm 180^\circ$

If $q = 2$, Angles $= \pm \frac{180^\circ \times 5}{3} = \pm 300 = \mp 60^\circ$

If $q = 3$, Angles $= \pm \frac{180^\circ \times 7}{3} = \pm 420 = \pm 60^\circ$

$$\text{Centroid} = \frac{\text{sum of Poles} - \text{sum of zeros}}{n-m}$$

$$= \frac{0 - 2 + j3 - 2 - j3 - 0}{3} = -1.33$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a Protractor

Step 4: To find the break away and breakin points

The closed lopp transfer function $\Big\}$ $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(s^2+4s+13)}}{1 + \frac{K}{s(s^2+4s+13)}} = \frac{K}{s(s^2+4s+13)+K}$$

The characteristic equation is $s(s^2+4s+13)+K = 0$

$$s^3 + 4s^2 + 13s + K = 0$$

$$K = -s^3 - 4s^2 - 13s$$

On differentiating the equation of K with respect to s we get

$$\frac{dK}{ds} = -(3s^2 + 8s + 13)$$

of this Point the total no of real Poles and zeros is one, which is an odd number. Hence the real axis between $s=0$ & $s=-2$ will be a part of root locus.

Choose a test Point on real axis b/w $s=-2$ & $s=-4$. To the right of this point, the total no of real Poles and zeros is two which is an even number. Hence the real axis b/w $s=-2$ & $s=-4$ will not be a part of root locus.

choose a test Point on real axis to the left of $s=-4$. To the right of this point, the total no of real poles and zeros is three which is an odd number. Hence the entire negative real axis from $s=-4$ to $-\infty$ will be a part of root locu

Step 3: To find asymptotes and centroid

Since there are three Poles the number of root locus branch are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asympte required are three.

$$\text{Angles of asymptotes} = \frac{\pm 180(2q+1)}{n-m}$$

Here $n=3$ & $m=0$

If $q=0$, $\text{Angles} = \pm \frac{180}{3} = \pm 60°$

If $q=1$, $\text{Angles} = \pm \frac{180 \times 3}{3} = \pm 180°$

If $q=2$, $\text{Angles} = \pm \frac{180 \times 5}{3} = \pm 300 = \mp 60$

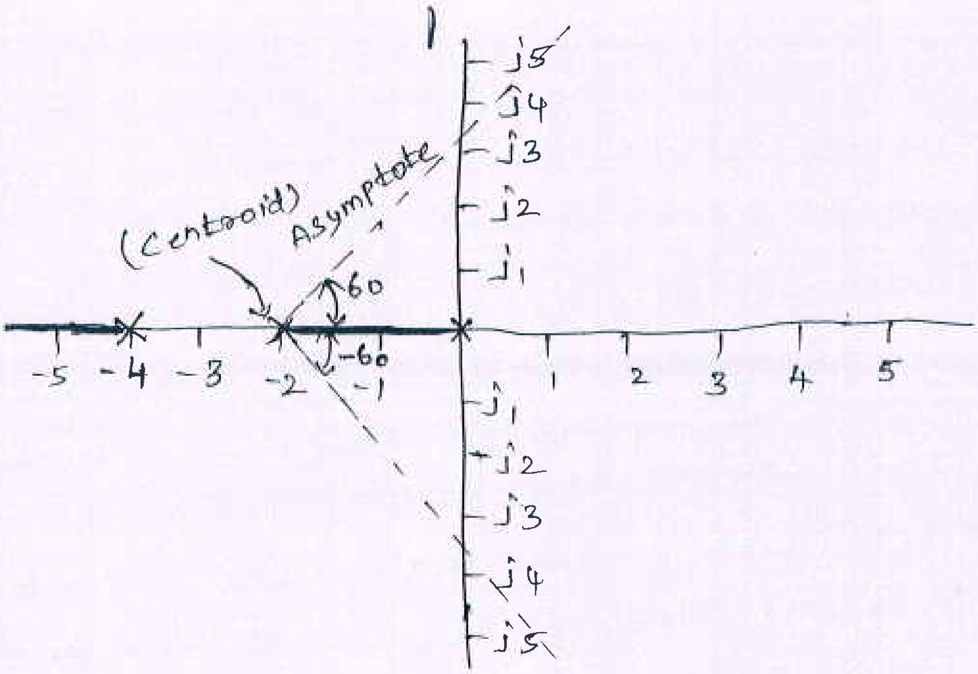
$$\text{Centroid} = \frac{\text{Sum of Poles} - \text{Sum of zeros}}{n-m}$$

$$= \frac{0-2-4-0}{3} = -2$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a Protractor.

Step 4: To find the break away and break in points

The closed loop transfer function } $\frac{C(s)}{R(s)} = \frac{G(s)}{\ldots}$



$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{s(s+2)(s+4)}}{1+\dfrac{K}{s(s+2)(s+4)}} = \frac{K}{s(s+2)(s+4)+K}$$

The characteristic equation is given by

$$s(s+2)(s+4)+K=0$$

$$s(s^2+6s+8)+K=0$$

$$s^3+6s^2+8s+K=0$$

$$K=-s^3-6s^2-8s$$

On differentiating the equation of K with respect to s we get

$$\frac{dK}{ds} = -(3s^2+12s+8)$$

Put $\frac{dK}{ds}=0$

$$-(3s^2+12s+8)=0$$

$$3s^2+12s+8=0$$

$$s=-0.845 \text{ or } -3.154$$

Check for K

When $s=-0.845$, the value of K is given by

$$K=-\left[(-0.845)^3+6(-0.845)^2+8(-0.845)\right]=3.08$$

Since K is positive and real for $s=-0.845$, this point is actual breakaway point.

When $s=-3.154$, the value of K is given by

$$K=-\left[(-3.154)^3+6(-3.154)^2+8(-3.154)\right]=-3.08$$

Since K is negative for $s=-3.154$, this is not a actual break away Point. The break away Point is marked on the negative real axis.

Step 5: To find angle of departure

Since there are no complex pole or zero, we need not find angle of departure or arrival,

Step 6: To find the crossing Point of imaginary axis

The characteristic equation is given by

$$s^3+6s^2+8s+K=0$$

Put $s=j\omega$

$$(j\omega)^3+6(j\omega)^2+8(j\omega)+K=0$$

$$-j\omega^3-6\omega^2+j8\omega+K=0$$

Equating imaginary Part to zero

$$-j\omega^3+j8\omega=0$$

$$-j\omega^3=-j8\omega$$

$$\omega^2=8$$

$$\omega=\pm 2.8$$

Equating real Part to zero

$$-6\omega^2+K=0$$

$$K=6\omega^2=6\times 8=48$$

The crossing Point of root locus is $\pm j2.8$. The Value of K corresponding to this Point is $K=48$.

The complete root locus sketch is shown. The root locus has three branches. One branch starts at the pole at $s=-4$ and travel through negative real axis to meet the zero at infinity. The other two root locus branches starts at $s=0$ & $s=-2$ and travel through negative real axis, break away from real axis at $s=-0.845$, then crosses imaginary axis at $s=\pm j2.8$ and travel Parallel to asymptotes to meet the infinity.

To find the Value of K corresponding to $\zeta=0.5$

Given that $\zeta=0.5$

Let $\alpha=\cos^{-1}\zeta=\cos^{-1}0.5=60$

Draw a line OP, such that the angle b/w line OP and negative

real axis is 60° ($\alpha = 60°$) as shown. The meeting point of the li
OP and root locus gives the dominant pole, Sd.

The Value of K Corresponding to the Point, $s = s_d$ } = Product of length of Vector from all Poles to the Point, S / Product of length of Vector from all zeros to the Point S = S

$$= \frac{l_1 \cdot l_2 \cdot l_3}{1} = 1.3 \times 1.75 \times 3.5$$

$$= 7.96 \approx 8$$

Sketch the root locus for the unity feedback system whose open loop transfer function is $G(s)\,H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$

Step 1: To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s(s+4)(s^2+4s+20) = 0$

The roots of the quadratic are $s = -2 \pm j4$

The Poles are lying at $s = 0, -4, -2+j4$ and $-2-j4$

The zeros are lying at infinity.

Step 2: To find root locus on real axis

There are two Poles on the real axis. choose a test Point on real axis between $s=0$ & $s=-4$. To the right of this point, the total number of real Poles is one which is an odd no. Hence the real axis b/w $s=0$ & $s=-4$ will be a Part of root locus. choose a test Point to the left of $s=-4$, now to the right of this test Point the total number of Poles and zeros is two which is even number. Hence the real axis from $s=-4$ to $s=-\infty$ will not be a Part of root locus. The root locus on real axis is shown as a bold line.

Step 3: To find angles of asymptotes and centroid

Since there are Four Poles, the number of root locus branches are four. There is no finite zero. Hence all the root locus b...

∴ number of asymptotes required is four.

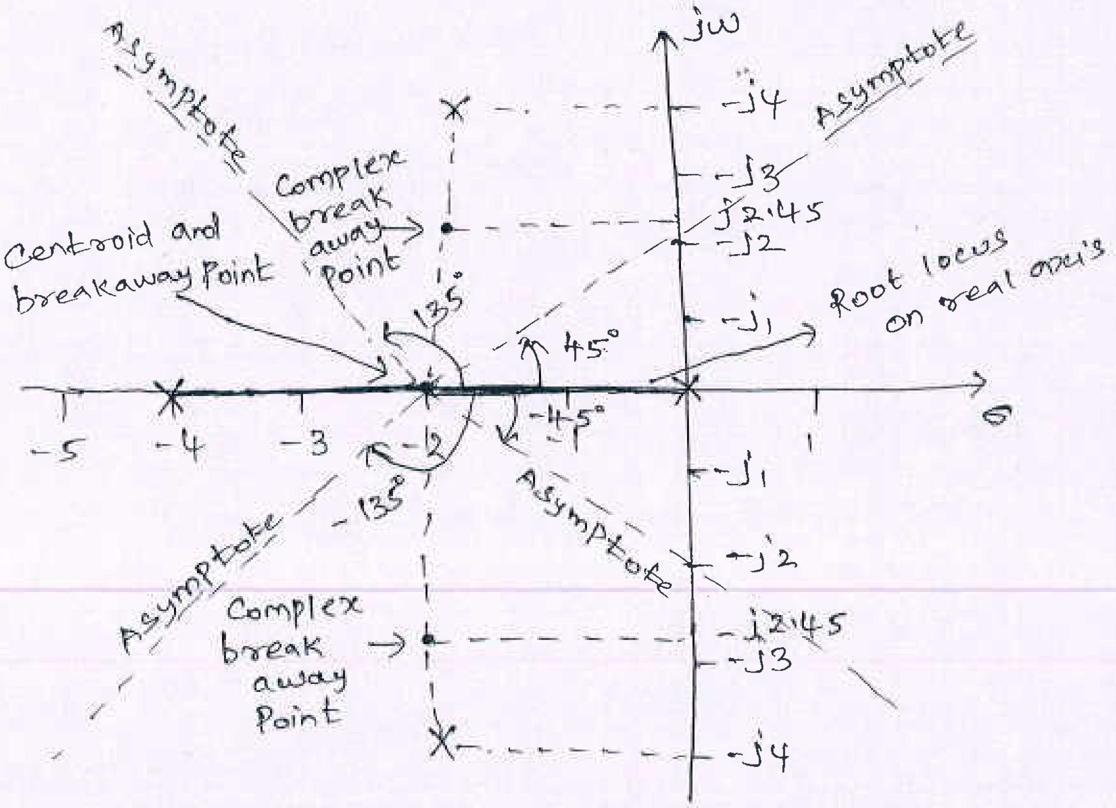
Angle of asymptotes $= \frac{\pm 180(2q+1)}{n-m}$

Here $n=4$ and $m=0$

If $q=0$, Angles $= \pm \frac{180}{4} = 45^\circ$

If $q=1$, Angles $= \pm \frac{180 \times 3}{4} = \pm 135^\circ$

If $q=2$, Angles $= \pm \frac{180 \times 5}{4} = \pm 225 = \mp 135^\circ$



If $q=3$, Angles $= \pm \frac{180 \times 7}{4} = \pm 315 = \mp 45^\circ$

Centroid $= \frac{\text{Sum of Poles} - \text{Sum of Zeros}}{n-m}$

$= \frac{0-4-2+j4-2-j4-0}{4-0} = -2$

Step 4: To find the break away and breakin Point

The closed loop transfer function $\Big\}$ $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+4)(s^2+4s+20)}}{1+\frac{K}{s(s+4)(s^2+4s+20)}}$

$$= \frac{K}{s(s+4)(s^2+4s+20)+K}$$

The characteristic equation is $s(s+4)(s^2+4s+20)+K=0$

$$K = -s(s+4)(s^2+4s+20)$$

$$= -(s^4+8s^3+36s^2+80s)$$

On differentiating the equation of K with respect to s we get

$$\frac{dK}{ds} = -(4s^3+24s^2+72s+80)$$

Put $\frac{dK}{ds}=0$ $\therefore -(4s^3+24s^2+72s+80)=0$

$$4s^3+24s^2+72s+80=0$$

$$s^3+6s^2+18s+20=0$$

The Polynomial $(s^3+6s^2+18s+20)=0$

$$(s+2)(s^2+4s+10)=0$$

$$s=-2\pm j2.45$$

check for K

when $s=-2$, $K=-(s^4+8s^3+36s^2+80s)$

$$=64$$

when $s=-2\pm j2.45$ $\quad K=-(s^4+8s^3+36s^2+80s)$

$=3.16\angle\pm129^\circ$

$$=-(99.7\angle\pm156^\circ+252.4\angle\pm27^\circ+359.5\angle\pm258^\circ+252.8\angle\pm129^\circ]$$

For Positive values of angles,

$$K=-[-91+j40+225+j115-75-j351-159+j196]$$

$$=100$$

For negative values of angles

$$K=-[-91-j40+225-j115-75+j351-159-j196]$$

$$=100$$

For all the roots of the equation $\frac{dK}{ds}=0$, the value of K is positive and real. Hence all the three roots are actual break away points.

Step 5: To find angle of departure

Let us consider the complex pole A. Draw vectors from all other poles to the pole A as shown

Sketch the root locus for the unity feed back system whose open loop transfer function is $G(s) = \frac{K}{s(s^2+6s+10)}$

Step 1 : To locate Poles and zeros

The Poles of open loop transfer function are the roots of the equation $s(s^2+6s+10)=0$

The roots of the quadratic are, $s = \frac{-6 \pm \sqrt{6^2-4\times10}}{2}$

The Poles are $0, -3+j1$ and $-3-j1$

Step 2 : To find the root locus on real axis

There is only one Pole on real axis at the origin. Hence if we choose any test Point on the negative real axis then to the right of that Point the total number of real Poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus. The root locus on real axis is shown as bold line..

Step 3 : To find angles of asymptotes and centroid

Since there are 3 Poles the number of root locus branches are three, There is no finite zero. Hence all the three root locus branches ends at zeros at infinity.

$$\text{Angle of asymptode} = \frac{\pm 180(2q+1)}{n-m}$$

Here $n=3$ & $m=0$

If $q=0$ Angle $= \pm\frac{180}{3} = \pm 60°$

If $q=1$ Angle $= \pm\frac{180\times3}{3} = \pm 180°$

If $q=2$ Angle $= \pm\frac{180\times5}{3} = \pm 300$ or $\mp 60°$

If $q=3$ Angle $= \pm\frac{180\times7}{3} = \pm 420$ or $\pm 60°$

$$\text{Centroid} = \frac{\text{Sum of Poles} - \text{Sum of zeros}}{n-m}$$

$= -3+j1-3-j1 \ldots$

Step 4: To find the breakaway and breakin Points

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s^2+6s+10)}}{1+\frac{K}{s(s^2+6s+10)}} = \frac{K}{s(s^2+6s+10)+K}$$

The characteristic equation is $s(s^2+6s+10)+K=0$

$$\therefore K = -s(s^2+6s+10) = -s^3-6s^2-10s$$

$$\frac{dK}{ds} = -3s^2-12s-10$$

Put $\frac{dK}{ds}=0$ $\quad -3s^2-12s-10=0$

$$3s^2+12s+10=0$$

$$s = \frac{-12 \pm \sqrt{12^2-4\times3\times10}}{2\times3}$$

$$= -1.18 \text{ or } -2.82$$

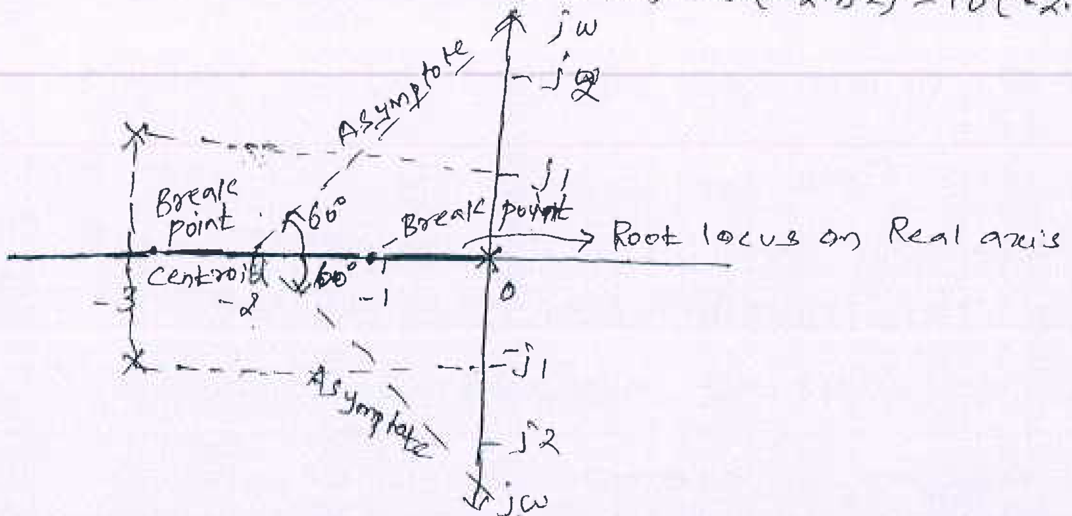
Check for K

When, $s=-1.18$, $K=-s^3-6s^2-10s$

$$= -(-1.18)^3-6(-1.18)^2-10(-1.18) = 5.09$$

When, $s=-2.82$, $K=-s^3-6s^2-10s$

$$= -(-2.82)^3-6(-2.82)^2-10(-2.82) = 2.91$$



Step 5: To find the angle of departure

$$\theta_1 = 180 - \tan^{-1}(1/3) = 161.6°$$

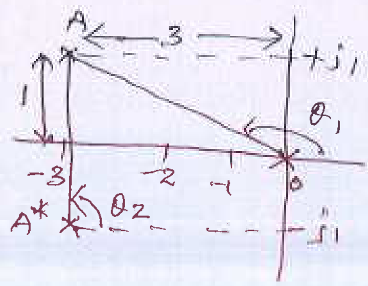
$$\theta_2 = 90°$$

$$A = 180-(\theta_1+\theta_2)$$

$$= 180-(161.6°+90°)$$

$$= -71.6° \approx -72°$$

$$A^* = 72$$



$\omega_{c4} = 4.5$ $A_o = -40 \times \log \frac{4.5}{3} + 5 = -2\,db$

$\omega_{c5} = 6$ $A_o = -20 \times \log \frac{6}{4.5} + (-2) = -4\,db$

$\omega_{c6} = 65$ $A_o = -40 \times \log \frac{65}{6} + (-4) = -45\,db$

$\omega_h = 80$ $A_o = -60 \times \log \frac{80}{65} + (-45) = -50\,db$

Phase plot

$$\phi_o = \angle G_o(j\omega) = \tan^{-1} 2.5\omega + \tan^{-1} 0.22\omega - 90 - \tan^{-1} 35\omega$$
$$- \tan^{-1} 0.0154\omega - \tan^{-1} 0.33\omega - \tan^{-1} 0.167\omega$$

ω	0.01	0.03	0.1	0.4	1	4	10	65
$\angle G_o(j\omega)$ deg	−108	−132	−152	−138	−126	−144	−168	−221 ≈ −220

$\phi_{gco} = -144$ (from bode plot)

$\gamma_o = 180 + \phi_{gco} = 180 - 144 = 36°$

The Phase margin of the compensated system is satisfactory. Hence the design is acceptable.

(2) Consider a unity f/b system with O.L.T.F $G(s) = \dfrac{100}{(s+1)(s+2)(s+5)}$. Design a PI controller, so that the phase margin of the system is $60°$ at a freq of 0.5 rad/sec

$$G(s) = \frac{100}{(s+1)(s+2)(s+5)} = \frac{100}{(1+s)\times 2\left(1+\frac{s}{2}\right)\times 5\left(1+\frac{s}{5}\right)}$$

$$= \frac{10}{(1+s)(1+0.5s)(1+0.2s)}$$

$$G(j\omega) = \frac{10}{(1+j\omega)(1+0.5j\omega)(1+0.2j\omega)}$$

$$= \frac{10}{\sqrt{1+\omega^2}\angle\tan^{-1}\omega \; \sqrt{1+(0.5\omega)^2}\angle\tan^{-1}0.5\omega \; \sqrt{1+(0.2\omega)^2}\angle\tan^{-1}0.2\omega}$$

step 1

$A_1 = |G(j\omega)|$ at $\omega = \omega_1$

$$|G(j\omega)| = \frac{10}{\sqrt{1+\omega^2}\sqrt{1+(0.5\omega)^2}\sqrt{1+(0.2\omega)^2}} = 8.63$$

$\omega = 0.5$

$\phi_1 = \angle G(j\omega)$ at $\omega = \omega_1$

$$\phi_1 = -\tan^{-1}\omega - \tan^{-1}0.5\omega - \tan^{-1}0.2\omega$$

$$= -\tan^{-1}0.5 - \tan^{-1}0.5\times 0.5 - \tan^{-1}0.2\times 0.5$$

step 2:

$$= -46°$$

$\gamma_d = 60$

$$\gamma_u = 180 + \phi_1 = 180 - (-46) = 134°$$

$$\theta = \gamma_d - \gamma_u = 60 - 134 = -74$$

The open loop transfer function of a unity feedback system is given by $G(s) = \frac{K(s+1)}{s^3+as^2+2s+1}$. Determine the value of K and a so that the system oscillates at a frequency of 2 rad/sec.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K(s+1)}{s^3+as^2+2s+1}}{1+\frac{K(s+1)}{s^3+as^2+2s+1}}$$

$$= \frac{K(s+1)}{s^3+as^2+2s+1+K(s+1)}$$

The characteristic equation is

$$s^3+as^2+2s+1+K(s+1)=0$$

$$s^3+as^2+2s+1+Ks+K=0$$

$$s^3+as^2+(2+K)s+1+K=0$$

$$\begin{array}{lll} s^3: & 1 & 2+K \\ s^2: & a & 1+K \\ s^1: & \frac{a(2+K)-(1+K)}{a} & \\ s^0: & 1+K & \end{array}$$

If the elements of s^1 row are all zeros then there exist an even polynomial. If the roots of the auxiliary polynomial are purely imaginary system then the roots are lying on imaginary axis & the system oscillates. The frequency of oscillation is the root of auxiliary polynomial.

From s^1 row

$$\frac{a(2+K)-(1+K)}{a} = 0$$

$a(2+K)-(1+K)=0$

$2a+Ka-1-K=0$

$2a-1+K(a-1)=0$

Put $K=4a-1$

$\therefore 2a-1+(4a-1)(a-1)=0$

$2a-1+4a^2-4a-a+1=0$

$4a^2-3a=0$

(or) $a(4a-3)=0$

since $a \neq 0$

$4a-3=0$

$\therefore a=3/4$

when $a=(3/4)$, $K=4a-1=4*(3/4)-1$

$=2$

The auxiliary Polynomial

$as^2+(1+K)=0$

$as^2=-(1+K)$

$s=\pm j\sqrt{\frac{1+K}{a}}$

Given that $s=\pm j2$

$\sqrt{\frac{1+K}{a}}=2$

$\frac{1+K}{a}=4$

$K=4a-1$

when the system oscillates at a freeqruency of 2 rad/sec, $K=2$ & $a=3/4$

Sketch the root locus for the unity feedback system whose open loop t/f is $G(s)=\frac{K}{s(s^2+6s+10)}$

$0, -3+j1, -3-j1$

$q=\pm 60, \pm 180$

$c=-2$ $K=-s^3-6s^2-10s$

$s=-1.18$ $K=5.09$

$s=-2.82$ $K=2.91$

$\theta_1=180-\tan^{-1}(1/3)=161.6$

$\theta_2=90$

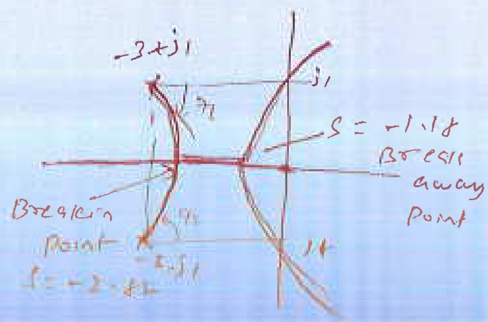
$A=180-(\theta_1+\theta_2)=-72$

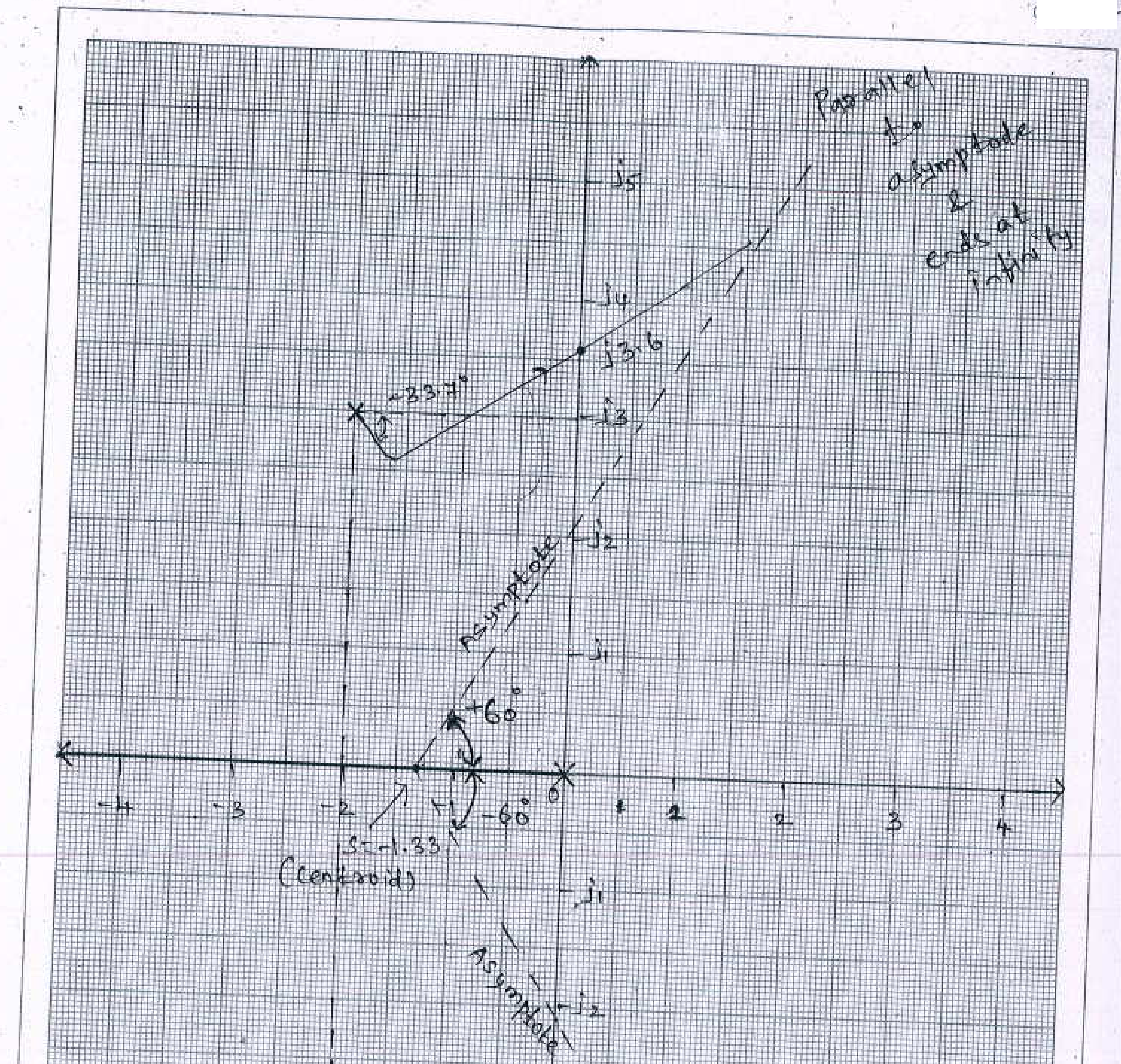
$A^*=72$

$s^3+6s^2+10s+K=0$

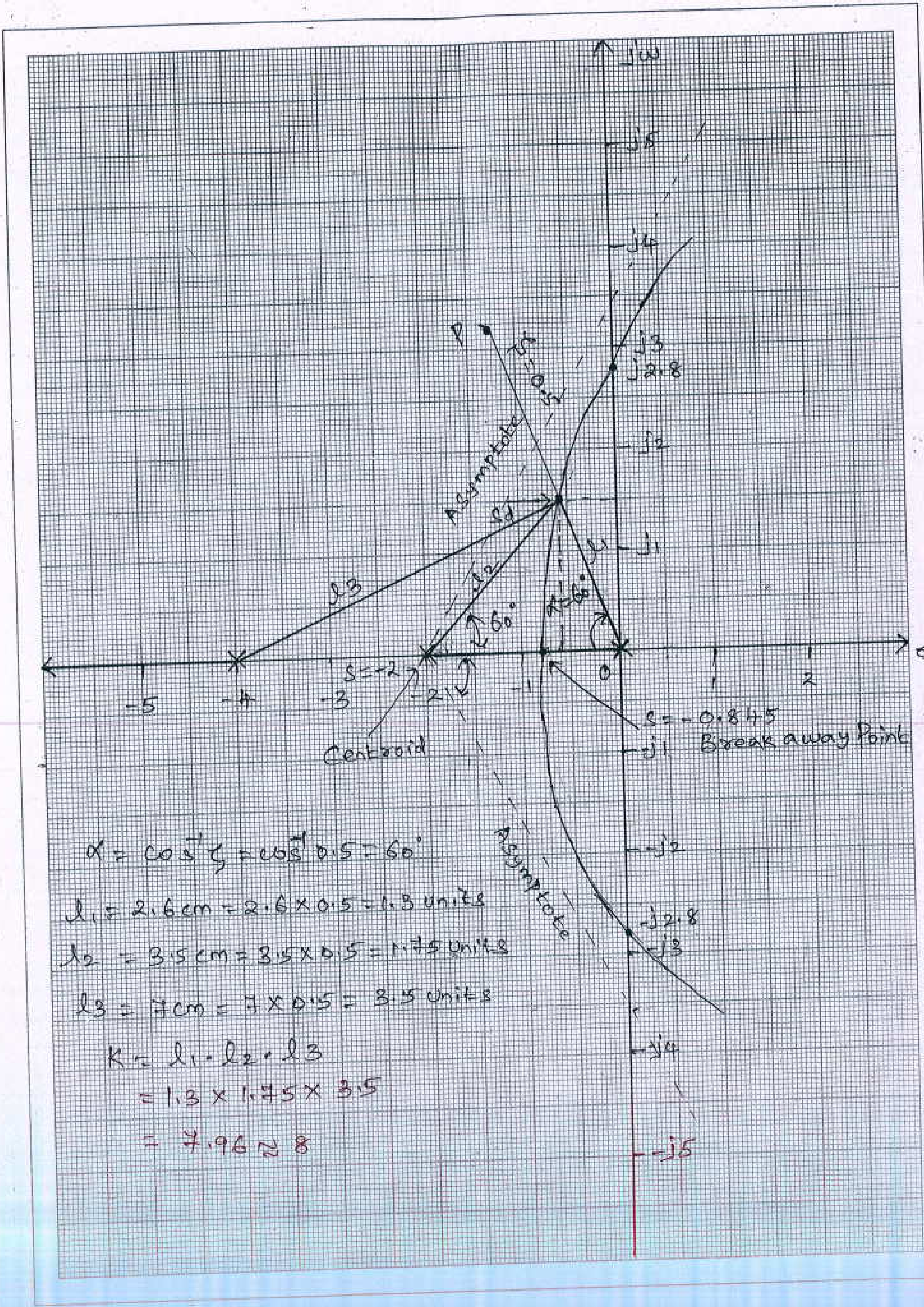
$\omega=\pm 3.2$

$K=60$





Parallel to asymptote & ends at infinity
j5
j4
j3.6
j3
j2
j1
-33.7°
Asymptote
+60°
-60°
-4
-3
-2
-1
0
1
2
3
4
s=-1.33
(centroid)
-j1
-j2
-j3
-j3.6
-j4
-j5
33.7°



$\alpha = \cos^{-1}\zeta = \cos^{-1}0.5 = 60°$

$l_1 = 2.6\,cm = 2.6 \times 0.5 = 1.3$ units

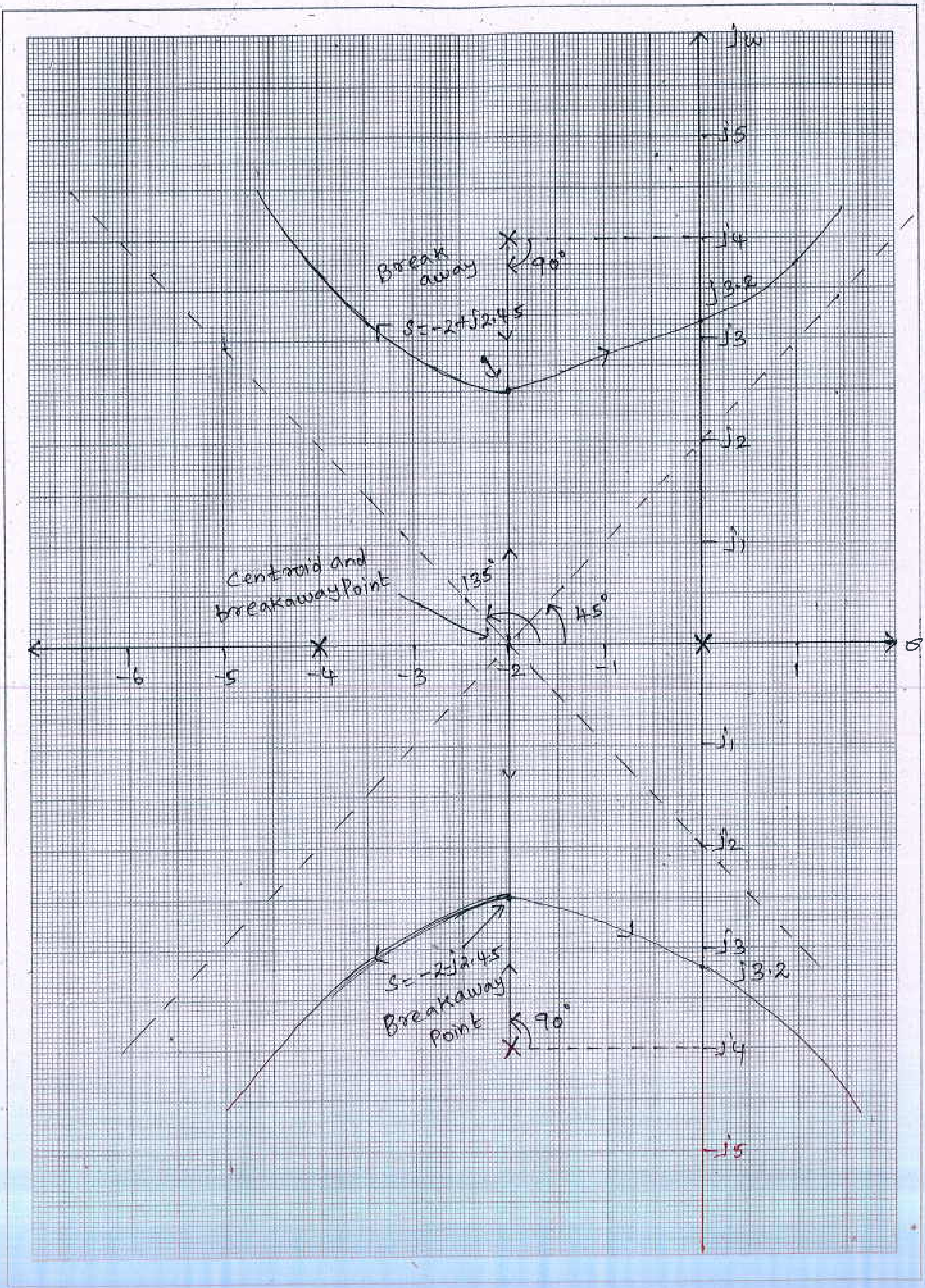
$l_2 = 3.5\,cm = 3.5 \times 0.5 = 1.75$ units

$l_3 = 7\,cm = 7 \times 0.5 = 3.5$ units

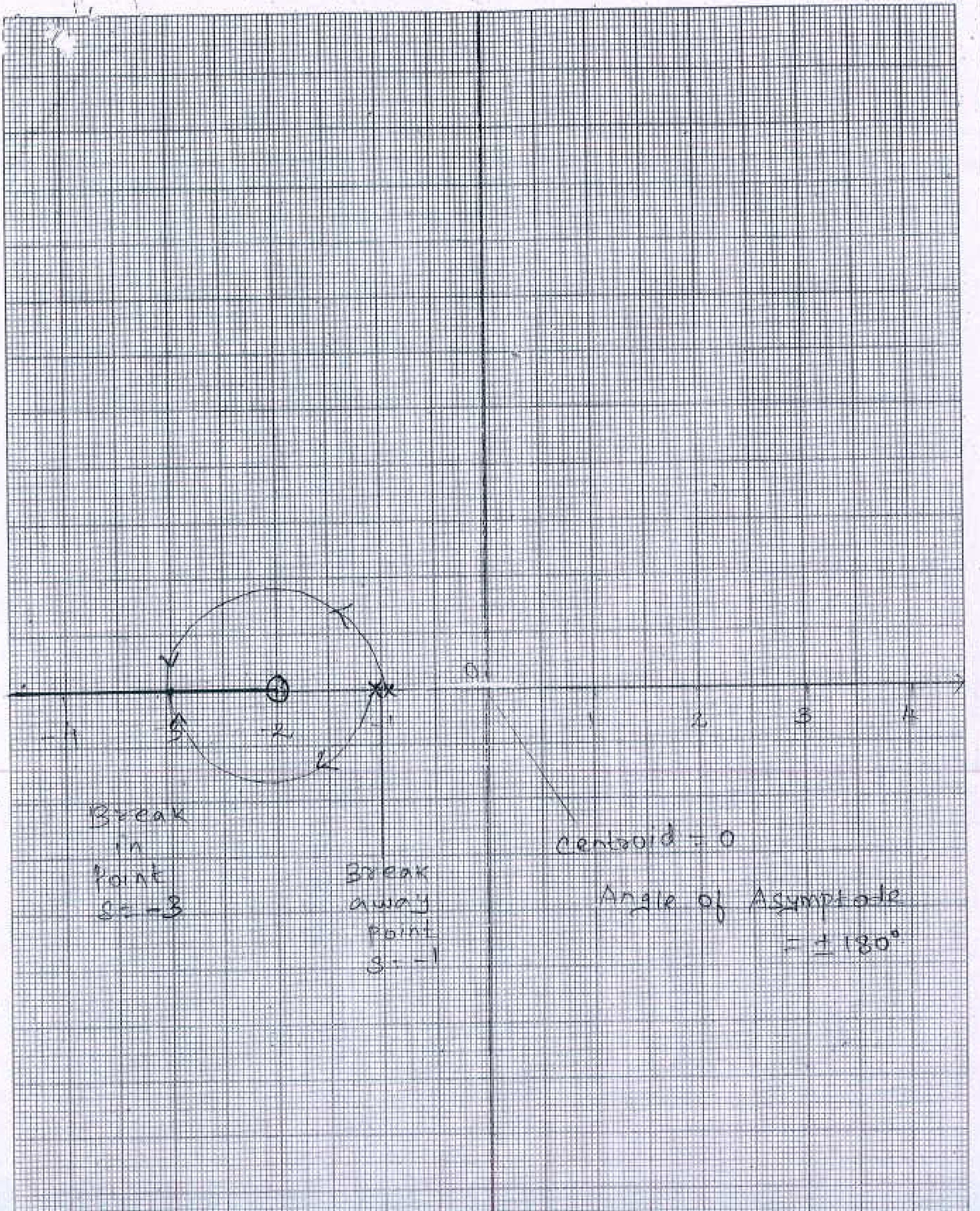
$K = l_1 \cdot l_2 \cdot l_3$

$= 1.3 \times 1.75 \times 3.5$

$= 7.96 \approx 8$


jω
j5
j4
j3.2
j3
j2
j1
σ
-6
-5
-4
-3
-2
-1
1
j1
j2
j3
j3.2
j4
j5
Break away
90°
s = -2+j2.45
Centroid and breakaway Point
135°
45°
s = -2-j2.45
Breakaway Point
90°

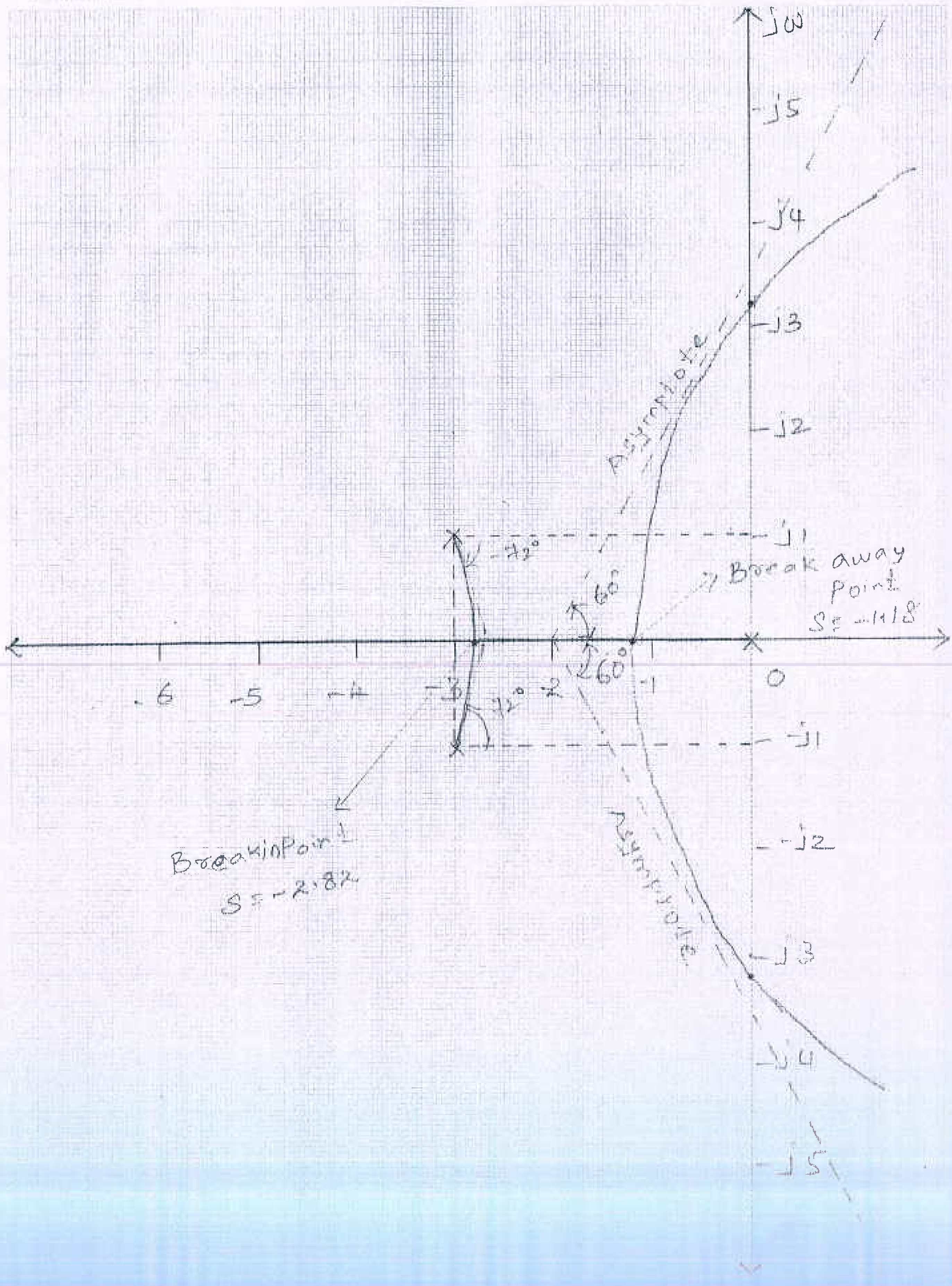
15.G



-4
-3
-2
-1
0
1
2
3
4
Break in Point S=-3
Break away Point S=-1
Centroid = 0
Angle of Asymptote = ±180°

Expt No.

Expt No.



Step 6: To find the crossing point on imaginary axis

The characteristic equation is given by

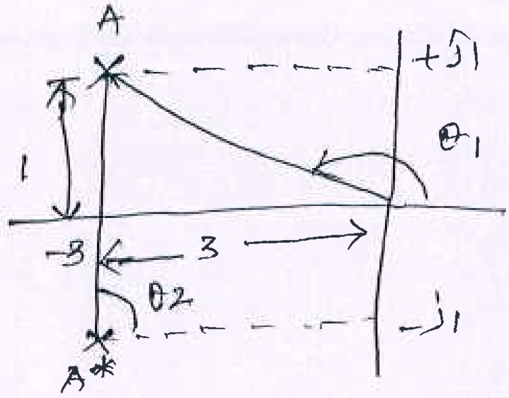
$$s(s^2+6s+10)+K=0$$

$$s^3+6s^2+10s+K=0$$

Put $s=j\omega$

$$(j\omega)^3+6(j\omega)^2+10(j\omega)+K=0$$

$$-j\omega^3-6\omega^2+j10\omega+K=0$$



On equating imaginary part to zero we get

$$-\omega^3+10\,\omega=0$$

$$\omega^3=10\omega$$

$$\omega^2=10$$

$$\omega=\pm\sqrt{10}=\pm 3.16\approx 3.2$$

on equating real part to zero we get

$$-6\omega^2+K=0$$

$$K=6\omega^2$$

$$=6\times 10=60$$

The root locus crosses imaginary axis at $\pm j3.2$ and the gain K corresponding to this point is 60°. This is the limiting value of K for the stability of the system.

The root locus has 3 branches one start at $s=0$ and goes to infinity along negative real axis. The other two root locus branches starts at $s=-3+j1$ and enter the real axis at $s=-2.82$ and then break away from real axis at $s=-1.18$. Finally they travel parallel to asymptotes to meet the