SUBJECT:**ENGINEERING MATHEMATICS-I**

SUBJECT CODE :**SMT1101**

UNIT –IV ORDINARY DIFFERENTIAL EQUATIONS

Exact differential equation.

A first order differential equation of type $M(x,y)dx + N(x,y)dy = 0$ is called an *exact differential equation* if there exists a function of two variables *u*(*x*,*y*) with continuous partial derivatives such that $du(x,y) = M(x,y)dx + N(x,y)dy$

The general solution of an exact equation is given by $u(x,y) + \int f(y)dy = c$,where *c* is an arbitrary constant

Test for Exactness

Let functions $M(x,y)$ and $N(x,y)$ have continuous partial derivatives in a certain domain *D*.

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is an exact equation if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Algorithm for Solving an Exact Differential Equation

1. First it's necessary to make sure that the differential equation is *exact* using the *test for exactness*:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

2. Integrate M with respect to x keeping y constant ie $\int Mdx$
3. Integrate those terms in N not containing x with respect to y.ie $\int \left[N - \frac{\partial}{\partial y}\int Mdx \right] dy$
4. The general solution of the exact differential equation is given by $\int Mdx + \int \left[N - \frac{\partial}{\partial y}\int Mdx \right] dy = c$

Example1. Solve $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

$M = 5x^4 + 3x^2y^2 - 2xy^3 \qquad N = 2x^3y - 3x^2y^2 - 5y^4$

$\Rightarrow \frac{\partial M}{\partial y} = 6x^2y - 6xy^2 \qquad$ *and* $\quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2 \qquad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \therefore$ *the given equation is exact.*

The required solution is given by $\int M dx + \int [terms\ of\ N\ not\ containing\ x] dy = c$

$$\int (5x^4 + 3x^2y^2 - 2xy^3)dx \quad + \quad \int (-5y^4)dy = c$$

$$x^5 + x^3y^2 - x^2y^3 - y^5 = c$$

Equations Reducible to Exact equations.

Rule1. $If\ \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ is function of x alone, say f(x) $then\ I.F = e^{\int f(x)dx}$

Rule2. $If\ \frac{-1}{M}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ is function of y alone, say f(y) $then\ I.F = e^{\int f(y)dy}$

Rule3. $If\ M\ is\ of\ the\ form\ M = yf_1(xy)\ N\ is\ of\ the\ form\ N = xf_2(xy), then\ I.F = \frac{1}{Mx - Ny}$

Rule4. $If Mdx + Ndy = 0\ is\ a\ homogeneous\ equation\ in\ x\ and\ y\ then\ I.F = \frac{1}{Mx + Ny}$

Example2. Solve $(2x\log x - xy)dy + 2ydx = 0.$

Solution . Given $(2x\log x - xy)dy + 2ydx = 0.$ (1)

Here M = 2y, N = 2xlogx-xy.

$$\Rightarrow \frac{\partial M}{\partial y} = 2 \quad and \quad \frac{\partial N}{\partial x} = 2(1 + \log x) - y \Rightarrow$$

$$\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = \frac{-2\log x + y}{2x\log x - xy} = -\frac{1}{x} = f(x).$$

$$I.F = e^{\int f(x)dx} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = x^{-1} = \frac{1}{x}$$

$$(1) I.F \Rightarrow \frac{2y}{x}dx + (2\log x - y)dy = 0 \Rightarrow mdx + ndy = 0\ which\ is\ exact.$$

The required solution is given by $\int m dx + \int [terms\ of\ n\ not\ containing\ x] dy = c$

$\Rightarrow$ The required solution is given by $\int \frac{2y}{x}dx + \int (-y)dy = 0$.

$\Rightarrow$ The required solution is given by $2y\log x - \frac{y^2}{2} = 0$.

Example3. Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$

Solution . Given $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$ (1)

Here $M = y^4 + 2y \quad N = xy^3 + 2y^4 - 4x$

$$\Rightarrow \frac{\partial M}{\partial y} = 4y^3 + 2 \quad and \quad \frac{\partial N}{\partial x} = y^3 - 4 \Rightarrow \quad \frac{-1}{M}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = -\frac{(4y^3 + 2) - (y^3 - 4)}{y^4 + 2y} = -\frac{3}{y} = f(y).$$

$$I.F = e^{\int f(y)dy} = e^{\int \frac{-3}{y}dy} = e^{-3\log y} = y^{-3} = \frac{1}{y^3}$$

$$(1) I.F \Rightarrow (y + \frac{2}{y^2})dx + (x + 2y - \frac{4x}{y^3})dy = 0 \Rightarrow mdx + ndy = 0\ which\ is\ exact.$$

The required solution is given by $\int m dx + \int [terms\ of\ n\ not\ containing\ x] dy = c$

$\Rightarrow$ The required solution is given by $\int (y+\frac{2}{y^2})dx+\int (2y)dy=c$.

The required solution is given by $x(y+\frac{2}{y^2})+y^2=c$

Example4. Solve $y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0.$

Solution . Given $y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0.$ (1)

$$\Rightarrow y(1+2xy)dx+x(1-xy)dy=0$$

$M=y(1+2xy)=yf_1(xy)$ and $N=x(1-xy)=xf_2(xy)$,

Then I.F= $\frac{1}{Mx-Ny}==\frac{1}{y(1+2xy)x-x(1-xy)y}=\frac{1}{3x^2y^2}$

$$(1)I.F\Rightarrow(\frac{1}{3x^2y}+\frac{2}{3x})dx+(\frac{1}{3xy^2}-\frac{1}{3y})dy=0\Rightarrow mdx+ndy=0\; which\ is\ exact.$$

The required solution is given by $\int mdx+\int [terms\,of\ n\ not\,containing\ x]dy=c$

$\Rightarrow$ The required solution is given by $\int (\frac{1}{3x^2y}+\frac{2}{3x})dx+\int (-\frac{1}{3y})dy=c$.

$\Rightarrow$ The required solution is given by $-\frac{1}{3xy}+\frac{2\log x}{3}-\frac{\log y}{3}=c.$

Example5. Solve $\frac{dy}{dx}=\frac{x^3+y^3}{xy^2}.$

Solution . Given $(x^3+y^3)dx+(xy^2)dy=0.$ (1)

$$Here\ M=(x^3+y^3)\,and\ N=-(xy^2)\,which\,are\,\text{hom}\,ogeneous\,in\,x\,and\ y.$$

$$then\,I.F=\frac{1}{Mx+Ny}=\frac{1}{(x^3+y^3)x+(-xy^2)y}=\frac{1}{x^4}$$

$$(1)I.F\Rightarrow(\frac{1}{x}+\frac{y^3}{x^4})dx-(\frac{y^2}{x^3})dy=0\Rightarrow mdx+ndy=0\; which\ is\ exact.$$

The required solution is given by $\int mdx+\int [terms\,of\ n\ not\,containing\ x]dy=c$

$\Rightarrow$ The required solution is given by $\int (\frac{1}{x}+\frac{y^3}{x^4})dx+\int (0)dy=c$.

$\Rightarrow$ The required solution is given by $\log x-\frac{y^3}{3x^3}=c.$

Example6. *Solve* $(y^3-2x^2y)dx+(2xy^2-x^3)dy=0$

Solution . Given $(y^3-2x^2y)dx+(2xy^2-x^3)dy=0.$

$$Here\frac{dy}{dx}=f(x,y)=-\frac{y^3-2x^2y}{2xy^2-x^3}$$ (1) which are homogeneous in x and y.

$$put\ y=vx\ \ in\ \ (1), v+x\frac{dv}{dx}=\frac{-v^3x^3+2x^2vx}{2xv^2x^2-x^3}=\frac{2v-v^3}{2v^2-1}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{2v-v^3}{2v^2-1}-v=\frac{3v-3v^3}{2v^2-1}$$

$$\Rightarrow \frac{2v^2-1}{-3v(v^2-1)}dv=xdx \Rightarrow \frac{2v^2-1}{v(v^2-1)}dv=\left[\frac{A}{v}+\frac{B}{v+1}+\frac{C}{v-1}\right]dv=-3\frac{dx}{x}$$

$$\Rightarrow\int\left[\frac{1}{v}+\frac{1/2}{v+1}+\frac{1/2}{v-1}\right]dv=\int -3\frac{dx}{x}+c$$

$$\Rightarrow\log(v\sqrt{v^2-1})=-\log x^3+\log c$$

$$\Rightarrow x^3(v\sqrt{v^2-1})=c$$

$$\Rightarrow x^2y^2(x^2-y^2)=c$$

LINEAR EQUATIONS OF HIGHER ORDER

A linear equation of n[th] order with constant coefficients is of the form

$$\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+.....a_ny=X \qquad (1)$$

where a_1, a_2,…….a_n are constants and X is a function of x. This equation can also be written in the form $\left(D^n+a_1D^{n-1}+a_2D^{n-2}+.....+a_n\right)y=X$ where $D=\frac{d}{dx}, D^2=\frac{d^2}{dx^2}$,…….. $D^n=\frac{d^n}{dx^n}$

Consider $\left(D^n+a_1D^{n-1}+a_2D^{n-2}+.....+a_n\right)y=0$ (2)

The general solution of equation (2) is given by $Y=c_1y_1+c_2y_2+......c_ny_n$

where $y_1,y_2,......y_n$ are n independent solutions and $c_1,c_2,......c_n$ are arbitrary constants.

Y is called the complementary function (C.F) of equation (1).

Suppose u is a particular solution (particular integral) of equation (1)

Then the general solution of equation (1) is of the form y=Y+u where Y is the complementary function and u is a particular integral (P.I).

Thus y=C.F + P.I

To find Complementary functions

Case (1)

Roots of the A.E are real and distinct say m_1 and m_2

y=$c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case (2)

Roots of the A.E are imaginary then

y=$e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$

Case (3)

Roots of the A.E are real and equal say $m_1 = m_2$ then

$y = e^{m_1 x}\left(c_1 x + c_2\right)$

1.Solve $\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + 3y = 0$

Put $\frac{d}{dx} = D$

$\left(D^2 y - 2Dy + 3y\right) = 0$

$\left(D^2 - 2D + 3\right)y = 0$

The auxiliary equation is $m^2 - 2m + 3 = 0$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - (4)(1)(3)}}{(2)(1)}$$

$$m = 2 \pm \frac{\sqrt{-8}}{2}$$

$$m = \frac{2 \pm i2\sqrt{2}}{2}$$

$$m = 1 \pm i\sqrt{2}$$

C.F = $e^{x}\left[c_1\cos\left(\sqrt{2}x\right)+c_2\sin\left(\sqrt{2}x\right)\right]$

The general solution is y = C.F+P.I

$$y=e^{x}\left[c_1\cos\left(\sqrt{2}x\right)+c_2\sin\left(\sqrt{2}x\right)\right]+0$$

To find Particular integral

When the R.H.S of the given differential equation is a function of x , we have to find particular Integral.

Case (i)

If $f(x)=e^{ax}$, then $P.I=\frac{1}{F(D)}e^{ax}$. Replace D by a in F(D), provided F(D)$\neq$ 0.

If F(a) = 0 then $P.I=\frac{x}{F'(D)}e^{ax}$ provided $F'(a)\neq 0$

If F'(a) = 0 then $P.I=\frac{x}{F''(D)}e^{ax}$ provided $F''(a)\neq 0$ and so on

Case (ii)

If f(x) = sinax or cosax then $P.I=\frac{1}{F(D)}\sin ax or \cos ax$

Replace $D^2 by -a^2$ in F(D), provided F(D)$\neq$ 0.

If F(D) = 0, when we replace D^2 by $-a^2$ then proceed as case (i)

Case (iii)

If f(x) = a^n then $P.I=\frac{1}{F(D)}x^n$

$P.I=\left[F(D)\right]^{-1}x^n$, Expand $\left[F(D)\right]^{-1}$ by using binomial theorem and then operate on x^n.

Case (iv)

If f(x)=e^{ax}x, where X is sinax (or) cosax (or) x then

$$P.I=\frac{1}{F(D)}e^{ax}X=e^{ax}\frac{1}{F(D+a)}X$$

Here $\frac{1}{F(D+a)}X$ can be evaluated by using anyone of the first three types.

Problems

1.Solve $\left(D^2+6D+9\right)y=5e^{3x}$

$m^2+6m+9=0$

$(m+3)^2=0$

m=-3,-3

$C.F=(c_1x+c_2)e^{-3x}$

P.I $= \left(\frac{1}{(D^2+6D+9)}\right)5e^{3x}$

$= \left(\frac{1}{(3)^2+6(3)+9}\right)5e^{3x}$

$= \frac{5}{36}e^{3x}$

The general solution is y = C.F + P.I

y= $(c_1x+c_2)e^{-3x}+\frac{5}{36}e^{3x}$

2. Solve $\left(D^2+6D+5\right)y=e^{-x}$

$m^2+6m+5=0$

$(m+5)(m+1)=0$

m=-1,-5

C.F = $c_1e^{-x}+c_2e^{-5x}$

P.I $= \left(\frac{1}{(D^2+6D+5)}\right)e^{-x}$

$= \left(\frac{1}{(-1)^2+6(-1)+5}\right)e^{-x}$

$$= \frac{x}{2D+6}e^{-x} = \frac{x}{2(-1)+6}e^{-x}$$

$$= \frac{x}{4}e^{-x}$$

The general solution is y = C.F + P.I

y= $c_1e^{-x} + c_2e^{-5x} + \frac{x}{4}e^{-x}$

2.Solve $(D^2 + D + 1)y = \sin 2x$

Solution:

The auxiliary equation is $m^2 + m + 1 = 0$

$$m = \frac{-1 \pm i\sqrt{3}}{2}$$

C.F= $e^{\frac{-x}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right]$

P.I $= \left(\frac{1}{(D^2 + D + 1)}\right)\sin 2x$

$$= \left(\frac{1}{(-4 + D + 1)}\right)\sin 2x$$

$$= \left(\frac{1}{D-3}\right)\sin 2x$$

$$= \left(\frac{D+3}{D^2-9}\right)\sin 2x$$

$$= \left(\frac{D+3}{-13}\right)\sin 2x$$

$$=- \frac{2\cos 2x}{13} - \frac{3\sin 2x}{13}$$

The general solution is y = C.F + P.I

$$y = e^{\frac{-x}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}x}{2}\right)\right] - \frac{2\cos 2x}{13} - \frac{3\sin 2x}{13}$$

3.Solve $(D^2 + 3D + 2)y = x^2$

Solution:

The auxiliary equation is $m^2 + 3m + 2 = 0$

(m + 2) (m + 1) = 0

Hence m = -2, -1

C.F = $c_1 e^{-2x} + c_2 e^{-x}$

$$\text{P.I} \quad = \left(\frac{1}{(D^2 + 3D + 2)}\right) x^2$$

$$= \frac{1}{2}\left(1 + \frac{3D + D^2}{2}\right)^{-1} x^2$$

$$= \frac{1}{2}\left(1 - \left(\frac{3D + D^2}{2}\right) + \left(\frac{3D + D^2}{2}\right)^2\right) x^2$$

$$= \frac{1}{2}\left(1 - \frac{3D}{2} + \frac{7D^2}{4}\right) x^2$$

$$= \frac{1}{2}\left(x^2 - 3x + \frac{7}{2}\right)$$

The general solution is y = C.F + P.I

$$y = c_1 e^{-2x} + c_2 e^{-x} + \frac{1}{2}\left(x^2 - 3x + \frac{7}{2}\right)$$

4. Solve $(D^2 - 4D + 3)y = e^x \cos 2x$

Solution:

The auxiliary equation is $m^2 - 4m + 3 = 0$

(m -1) (m -3) = 0

Hence m = 1, 3

C.F = $c_1 e^x + c_2 e^{3x}$

P.I = $\left(\frac{1}{(D^2 - 4D + 3)}\right) e^x \cos 2x$

$$= \left(\frac{e^x}{(D+1)^2 - 4(D+1) + 3}\right)\cos 2x$$

$$= \left(\frac{e^x}{D^2 - 2D}\right)\cos 2x$$

$$= \left(\frac{e^x}{-4 - 2D}\right)\cos 2x$$

$$= -\frac{1}{2}\left(\frac{e^x}{D+2}\right)\cos 2x$$

$$= -\frac{e^x}{2}\left(\frac{D-2}{D^2-4}\right)\cos 2x$$

$$= -\frac{e^x}{2}\left[\frac{(D-2)\cos 2x}{-8}\right]$$

$$= \frac{e^x}{16}(-2\sin 2x - 2\cos 2x)$$

$$= -\frac{e^x}{8}(\sin 2x + \cos 2x)$$

The general solution is y =C.F+ P.I

y = $c_1 e^x + c_2 e^{3x} - \frac{e^x}{8}(\sin 2x + \cos 2x)$

5. Solve $(D^2 - 2D + 2)y = e^x \sin x$

The auxiliary equation is $m^2 - 2m + 2 = 0$

$m = 1 \pm i$

C.F = $e^x [c_1 \cos x + c_2 \sin x]$

P.I $= \left(\frac{1}{(D^2-2D+2)}\right)e^x \sin x$

$= \left[\frac{e^x}{(D+1)^2-2(D+1)+2}\right]\sin x$

$= \left[\frac{e^x}{D^2+1}\right]\sin x$

$= \left[\frac{e^x}{(D+i)(D-i)}\right]\sin x$

$= e^x$ Imaginary part of $\left[\frac{1}{(D+i)(D-i)}\right]e^{ix}$

$= e^x$ Imaginary part of $\left[\frac{1}{2i}xe^{ix}\right]$

$= e^x$ Imaginary part of $\left[-\frac{1}{2}ix(\cos x + i\sin x)\right]$

$= -\frac{1}{2}xe^x \cos x$

The general solution is y = C.F + P.I

y = $e^x[c_1\cos x + c_2\sin x] \ -\frac{1}{2}xe^x\cos x$

6. Solve $(D^3-3D^2+3D-1)y = x^2e^x$

The auxiliary equation is $m^3-3m^2+3m-1=0$

$(m-1)^3 = 0$

m=1 (thrice)

C.F $= e^x(c_1+c_2x+c_3x^2)$

P.I $= \frac{1}{D^3-3D^2+3D-1}x^2e^x$

$= \left[\frac{e^x}{(D+1)^3-3(D+1)^2+3(D+1)-1}\right]x^2$

= $e^x\left(\frac{1}{D^3}\right)x^2$

= $\frac{e^x x^5}{60}$ (By integrating x^2 thrice with respect to x)

The general solution is y = C.F + P.I

$$y=e^x\left(c_1+c_2x+c_3x^2\right)+\frac{e^x x^5}{60}$$

Linear Differential Equations with variable coefficients

An equation of the form

$$a_0x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+.....a_ny=X$$

Where a_0 , a_1 ,…….a_n are constants and X is a function of x is called Euler's homogeneous linear differential equation.

Equation can be reduced to constant coefficient by means of transformation z = log x.Then

$xD=\theta$, $x^2D^2=\theta(\theta-1)$, $x^3D^3=\theta(\theta-1)(\theta-2)$ where $\theta=\frac{d}{dz}$.

1.Solve $x^2y''-xy'+4y=\cos(\log x)+x\sin(\log x)$

Solution:

Put z = log x and $\theta=\frac{d}{dz}$

The given equation reduces to

$$[\theta(\theta-1)-\theta+4]y=\cos z+e^z\sin z$$

$$[\theta^2-2\theta+4]y=\cos z+e^z\sin z$$

The auxiliary equation is m^2 -2m + 4=0

$$m=1\pm i\sqrt{3}$$

Hence $C.F=e^z\left(c_1\cos\sqrt{3}z+c_2\sin\sqrt{3}z\right)$

$$= x\left[c_1 \cos\left(\sqrt{3}\log x\right) + c_2 \sin\left(\sqrt{3}\log x\right)\right]$$

P.I = $\left[\dfrac{1}{\theta^2 - 2\theta + 4}\right]\cos z + \left[\dfrac{1}{\theta^2 - 2\theta + 4}\right]\left(e^z \sin z\right)$

$$= \left[\frac{1}{3-2\theta}\right]\cos z + e^z\left[\frac{1}{(\theta+1)^2 - 2(\theta+1) + 4}\right](\sin z)$$

$$= \left[\frac{1}{3-2\theta}\right]\cos z + e^z\left[\frac{1}{\theta^2 + 3}\right](\sin z)$$

$$= \left[\frac{3+2\theta}{9-4\theta^2}\right]\cos z + \frac{e^z \sin z}{(-1+3)}$$

$$= \left[\frac{3+2\theta}{13}\right]\cos z + \frac{e^z \sin z}{2} \quad = \frac{1}{13}(3\cos z - 2\sin z) + \frac{1}{2}e^z \sin z$$

$$= \frac{1}{13}\left[3\cos(\log x) - 2\sin(\log x)\right] + \frac{1}{2}x\sin(\log x)$$

The solution is y = C.F + P.I

$$y = x\left[c_1 \cos\left(\sqrt{3}\log x\right) + c_2 \sin\left(\sqrt{3}\log x\right)\right] + \frac{1}{13}\left[3\cos(\log x) - 2\sin(\log x)\right] + \frac{1}{2}x\sin(\log x)$$

2.Solve $(x^2D^2 + 2xD + 4)y = x^2 + 2\log x$

Solution:

Put z = log x and $\theta = \dfrac{d}{dz}$

The given equation reduces to

$$\left[\theta(\theta-1) + 2\theta + 4\right]y = e^{2z} + 2z$$

$$\left[\theta^2 + \theta + 4\right]y = e^{2z} + 2z$$

The auxiliary equation is $m^2+m+4=0$

$$m = \frac{-1 \pm \sqrt{1-16}}{2} = \frac{-1 \pm i\sqrt{15}}{2}$$

$$\text{C.F} \quad = \quad e^{-\frac{z}{2}}\left[c_1 \cos\left(\frac{\sqrt{15}}{2}\right)z + c_2 \sin\left(\frac{\sqrt{15}}{2}\right)z\right]$$

$$= x^{-\frac{1}{2}}\left[c_1 \cos\left(\frac{\sqrt{15}}{2}\right)\log x + c_2 \sin\left(\frac{\sqrt{15}}{2}\right)\log z\right]$$

$$\text{P.I} = \left[\frac{1}{\theta^2 + \theta + 4}\right]\left(e^{2z} + 2z\right) = P.I_1 + P.I_2$$

$$P.I_1 = \left[\frac{1}{\theta^2 + \theta + 4}\right]\left(e^{2z}\right)$$

$$= = \frac{e^{2z}}{10} = \frac{x^2}{10}$$

$$P.I_2 = \left[\frac{1}{\theta^2 + \theta + 4}\right](2z)$$

$$= \frac{1}{2}\left[\frac{1}{1 + \left(\frac{\theta + \theta^2}{4}\right)}\right](z)$$

$$= \frac{1}{2}\left[1 + \frac{\theta + \theta^2}{4}\right]^{-1} z \; = \; \frac{1}{2}\left[1 - \frac{\theta}{4}\right]z$$

$$= \frac{1}{2}\left[z - \frac{1}{4}\right]$$

$$= \frac{1}{2}\log x - \frac{1}{8}$$

The general solution is y= C.F+ $P.I_1 + P.I_2$

$$\text{Y} = x^{-\frac{1}{2}}\left[c_1 \cos\left(\frac{\sqrt{15}}{2}\right)\log x + c_2 \sin\left(\frac{\sqrt{15}}{2}\right)\log z\right] + \frac{x^2}{10} + \frac{1}{2}\log x - \frac{1}{8}$$

SIMULTANEOUS FIRST ORDER EQUATIONS

Example 1 Solve the simultaneous equations $\frac{dx}{dt}+2x-3y=5t,\ \frac{dy}{dt}-3x+2y=0$ given that x(0) = 0, and y(0) = -1

<u>Solutions:</u> The given equation can be written as

(D+2)x-3y=5t …(1)

(D+2)y-3x=0 …(2)

(1) x 3 $\Rightarrow$ 3(D+2) x-9y=15t

(2) x (D+2) $\Rightarrow$ 3(D+2)x-(D+2)^{2}y=0

(-) (+)

-9y+(D+2)^{2}y = 15t

$$\left[D^2+4D-5\right]y = 15t$$

A.E is $m^2+4m-5 = 0$ m=-5, m=1

C.F $= Ae^{-5t} + Be^{t}$

$$\text{P.I} = \frac{1}{D^2+4D-5}15t$$

$$= \frac{1}{-5\left(1-\frac{D^2+4D}{5}\right)}15t$$

$$= -3\left(1-\frac{D^2+4D}{5}\right)^{-1} t$$

$$= -3\left(1+\frac{4}{5}D+\frac{21}{25}D^2+..\right)t$$

$$= -3t-\frac{12}{5}$$

$$y(t) = Ae^{-5t}+Be^{t}-3t-\frac{12}{5} \qquad (4)$$

To find x(t) sub(4) in (2)

$$3x = \frac{dy}{dt}+2y$$

$$x(t) = -Ae^{-5t}+Be^{t}-2t-\frac{13}{5} \qquad (5)$$

$$x(0)=0 \Rightarrow -A+B=\frac{13}{5} \qquad (6)$$

$y(0) = -1 \Rightarrow A + B = \frac{7}{5}$ (7)

from (6) & (7)

$$A = \frac{-3}{5}, B = 2$$

$$\therefore x(t) = \frac{3}{5}e^{-5t} + 2e^{t} - 2t - \frac{13}{5}$$

$$y(t) = \frac{-3}{5}e^{-5t} + 2e^{t} - 3t - \frac{12}{5}$$

Example :2

Solve $\frac{dx}{dt} + 2y = 5e^{t}; \frac{dy}{dt} - 2x = 5e^{t}$ given that x(0)=-1 and y(0)=3.

Solution

Given: $\frac{dx}{dt} + 2y = 5e^{t}; \frac{dy}{dt} - 2x = 5e^{t}$

i.e $Dx+2y=5e^{t}$ …(1)

$-2x+Dy=5e^{t}$ …(2)

(1) x –2 $\Rightarrow$ $-2Dx-4y=-10e^{t}$ …(3)

(2) x D $\Rightarrow$ $-2Dx+D^{2}y=5e^{t}$ …(4)

(+) (-) (-)

from (3) –(4) $(D^{2}+4)\,y=15e^{t}$

A.E is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

C.F = Acos2t + Bsin2t.

P.I = $\frac{1}{D^2+4}15e^{t} = \frac{1}{-1^2+4}15e^{t} = 5e^{t}$.

y(t)= Acos2t + Bsin2t +5e^{t} (5)

To find X(t),Sub (5) in (2) $2x = \frac{dy}{dt} - 5e^{t} \Rightarrow x(t) = -A\sin 2t + B\cos 2t$ (6)

Given that x(0)=-1 $\Rightarrow x(0) = -1 = B \Rightarrow B = -1$ and y(0)=3 $\Rightarrow y(0) = 3 = A + 5 \Rightarrow A = -2$

$x(t) = 2\sin 2t - \cos 2t$ and $y(t) = -2\cos 2t - \sin 2t + 5e^{t}$

Example :3

Solve the simultaneous equations $(D+5)x + y = e^{t}$; $(D+3)y - x = e^{2t}$

Solution: Given that

$(D+5)x+y=e^t$ …(1)

$-x+(D+3)y=e^{2t}$ …(2)

(1) $\Rightarrow$ $(D+5)x+y=e^t$

(2) x (D+5) $\Rightarrow$ $-(D+5)x+(D+3)(D+5)\,y=2e^{2t}+5e^{2t}$ …(3)

(1) + (3) $\Rightarrow$ $((D+3)\,(D+5)+1)\,y=e^t+7e^{2t}$

ie., $(D^2+8D+16)y=e^t+7e^{2t}$

A.E is $m^2+8m+16=0$

$m=-4,-4$

C.F is $y(t)=(At+B)e^{-4t}$

$$\text{P. I} = \frac{1}{D^2+8D+16}[e^t+7e^{2t}] = \frac{e^t}{25}+\frac{7e^{2t}}{36}$$

$\therefore$ Complete solution is

$$y(t)=(At+B)e^{-4t}+\frac{e^t}{25}+\frac{7e^{2t}}{36}$$

$$\frac{dy}{dt}=-4(At+B)e^{-4t}+Ae^{-4t}+\frac{e^t}{25}+\frac{14e^{2t}}{36}$$

$$=-4(At+B)e^{-4t}+Ae^{-4t}+\frac{e^t}{25}+\frac{7e^{2t}}{18}$$

To find x(t) sub (4) in (2)

$$-x-4(At+B)e^{-4t}+Ae^{-4t}+\frac{e^t}{25}+\frac{7e^{2t}}{18}$$

$$+3(At+B)e^{-4t}+\frac{3e^t}{25}+\frac{21e^{2t}}{36}=e^{2t}$$

$$-x+(1-t)Ae^{-4t}-Be^{-4t}+\frac{4e^t}{25}+\frac{35e^{2t}}{36}=e^{2t}$$

$$x(t)=(1-t)Ae^{-4t}-Be^{-4t}+\frac{4e^t}{25}-\frac{e^{2t}}{36}$$

$$y(t)=(At+B)e^{-4t}+\frac{e^t}{25}+\frac{7e^{2t}}{36}$$

METHOD OF VARIATION OF PARAMETERS

Example:1

Use the method of variation of parameter to solve $(D^2+4)y = \cot 2x$.

Solution:

A.E is $m^2+4=0$; $m=\pm 2i$

The C. F = $e^{ox}[A\cos 2x+B\sin 2x]$

Now,

$$f_1 = \cos 2x \qquad f_2 = \sin 2x$$
$$f_1' = -2\sin 2x \qquad f_2' = 2\cos 2x$$
$$f_1 f_2' - f_1' f_2 = 2(\cos^2 2x + \sin^2 2x) = 2$$

P.I = $P\, f_1 + Q\, f_2$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\, dx$$

$$= -\int \frac{\sin 2x\ \cot 2x}{2}\, dx$$

$$P = -\frac{1}{2}\int \cos 2x dx$$

$$= -\frac{1}{4}\sin 2x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\, dx$$

$$= \int \frac{\cos 2x\ \cot 2x}{2}\, dx$$

$$= \frac{1}{2}\int \frac{\cos^2 2x}{\sin 2x}\, dx$$

$$= \frac{1}{2}\int \frac{1-\sin^2 2x}{\sin 2x}\, dx$$

$$= \frac{1}{2}\int (\operatorname{cosec} 2x - \sin 2x) dx$$

$$= \frac{1}{2}\left\{ -\frac{1}{2}\log(\operatorname{cosec} 2x + \cot 2x) + \frac{1}{2}\cos 2x \right\}$$

$\therefore$ P.I=Pf_1+Qf_2

$$= \frac{1}{4}\sin 2x[\cos 2x - \log(\cos ec2x + \cot 2x)] - \frac{1}{4}\cos 2x \sin 2x$$

$$=- -\frac{1}{4}\sin 2x \log(\cos ec2x + \cot 2x)$$

$\therefore$ The complete solution is

$$y = (A\cos 2x + B\sin 2x) - \frac{1}{4}\sin 2x \log(\cos ec2x + \cot 2x)$$

Examples :2

Solve $(D^2+a^2)y$ = secax by the method of variation of parameters.

Solution:

Given $(D^2+ a^2)y$ = secax

A.E is $m^2+ a^2 = 0$

$m = \pm ai$

$\therefore$ C.F = A cosax + Bsinax

$$f_1 = \cos ax \qquad f_2 = \sin ax$$

$$f_1' = -a\sin ax \qquad f_2' = a\cos ax$$

$$f_1 f_2' - f_1' f_2 = a\cos^2 ax + a\sin^2 ax = a$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}dx$$

$$=-\int \frac{\sin ax \sec ax}{a}dx$$

$$= -\frac{1}{a}\int \sin ax \frac{1}{\cos ax}dx$$

$$= -\frac{1}{a}\int \frac{\sin ax}{\cos ax}dx = \frac{1}{a^2}\log[\cos ax]$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}dx$$

$$= \int \frac{\cos ax \sec ax}{a}dx$$

$$= \frac{1}{a}\int \cos ax \frac{1}{\cos ax}dx = \frac{1}{a}x$$

$$\therefore P.I = P\ f_1 + Qf_2 = \frac{1}{a^2}\log(\cos ax)\cos ax + \frac{1}{a}x\sin ax$$

$\therefore$ Complete solution y=C.F+P.I.

Example :3

Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x\ \log x$ by using method of variation of parameter.

Solution:

A.E is $m^2-2m+1=0$

C.F is $(Ax+B)e^x$

Where $f_1 = xe^x$ $\qquad$ $f_2 = e^x$

$$f_1^{'} = xe^x + e^x \qquad f_2^{'} = e^x$$

$$f_1 f_2^{'} - f_1^{'} f_2 = xe^{2x} - (xe^x + e^x)e^x = -e^{2x}$$

P.I=Pf_1+Qf_2 Where

$$P = -\int \frac{f_2 X}{f_1 f_2^{'} - f_1^{'} f_2} dx$$

$$= -\int \frac{e^x e^x \log\ x}{-e^{2x}} dx$$

$$= \int \log\ x\ \ dx$$

$$= x\ \log x - x$$

$$Q = \int \frac{f_1 X}{f_1 f_2^{'} - f_1^{'} f_2}$$

$$= \int \frac{xe^x.e^x \log x}{-e^{2x}} dx = -\int x\ \log x\ dx$$

$$= -\int \log\ x d\left(\frac{x^2}{2}\right)$$

$$= -\frac{x^2}{2}\log\ x + \frac{x^2}{4}$$

$$\therefore P.I = Pf_1 + Qf_2$$

$$= (x \; log \; x - x)xe^x + \left(\frac{-x^2 \; log \; x}{2} + \frac{x^2}{4}\right)e^x$$

s

$$= x^2e^x \; log \; x - x^2e^x - \frac{x^2e^x \; log \; x}{2} + \frac{x^2e^x}{4}$$

$$= \frac{x^2e^x \; log \; x}{2} - \frac{3x^2e^x}{4} = \frac{1}{4}x^2e^x(2 \; log \; x - 3)$$

The complete solution is

$$y = (Ax + B)e^x + \frac{x^2e^x}{4}(2 \; log \; x - 3)$$

Example:4

Use the method of variation of parameter to solve $\left(D^2 + a^2\right)y = \cot ax$.

Solution:

A.E is $m^2 + a^2 = 0 \quad m = \pm \; ai$

Then $C.F = e^{ox}\left[A\cos ax + B\sin ax\right]$

Now,

$$f_1 = \cos ax \qquad\qquad f_2 = \sin ax$$

$$f_1^{'} = -a \; \sin ax \qquad\qquad f_2^{'} = a\cos ax$$

$$f_1 f_2^{'} - f_1^{'} f_2 = a(\cos^2 ax + \sin^2 ax) = a$$

P.I = $P \; f_1 + Q \; f_2$

P = $-\int \frac{f_2 X}{f_1 f_2^{'} - f_1^{'} f_2} dx$

$$= -\int \frac{\sin ax \; \cot ax}{a} dx$$

P $= -\frac{1}{a}\int \cos ax dx$

$$= -\frac{1}{a^2}\sin ax$$

$$Q = \int \frac{f_1 X}{f_1 f_2^{'} - f_1^{'} f_2} dx$$

$$= \int \frac{\cos ax \;\; \cot ax}{a} dx$$

$$=\frac{1}{a}\int\frac{\cos^2 ax}{\sin ax}dx$$

$$=\frac{1}{a}\int\frac{1-\sin^2 ax}{\sin ax}dx$$

$$=\frac{1}{a}\int(\cos ecax-\sin ax)dx$$

$$=\frac{1}{a}\left\{-\frac{1}{a}\log(\cos ecax+\cot ax)+\frac{1}{a}\cos ax\right\}$$

$\therefore$ P.I=Pf_1+Qf_2

$$=\frac{1}{a^2}\sin ax[\cos ax-\log(\cos ecax+\cot ax)]-\frac{1}{a^2}\cos ax\sin ax$$

$$=--\frac{1}{a^2}\sin ax\log(\cos ecax+\cot ax)$$

$\therefore$ The complete solution is

$$y=(A\cos ax+B\sin ax)-\frac{1}{a^2}\sin ax\log(\cos ecax+\cot ax)$$

Example:5

Solve $(D^2-1)y=\frac{1}{1+e^x}$ by using method of variation of parameter.

Solution:

A.E is $m^2-1=0$

C.F is Ae^x+Be^{-x}

Where $f_1=e^x$ $\quad f_2=e^{-x}$

$f_1'=e^x$ $\quad f_2'=-e^{-x}$

$f_1f_2'-f_1'f_2=-e^xe^{-x}-e^{-x}e^x=-2$

P.I=Pf_1+Qf_2 Where

$$P=-\int\frac{f_2X}{f_1f_2'-f_1'f_2}dx$$

$$=-\int\frac{e^{-x}}{-2(1+e^x)}dx$$

$put\ e^x=t\Rightarrow e^xdx=dt$

$$=\frac{1}{2}\int\frac{1}{t^2(1+t)}dt$$

$$= \frac{1}{2}\int \left(\frac{-1}{t} + \frac{1}{t^2} + \frac{1}{1+t}\right)dt$$

$$= \frac{1}{2}\left[-\log t - \frac{1}{t} + \log(1+t)\right]$$

$$= \frac{1}{2}\left[-x - e^{-x} + \log(1+e^x)\right]$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}$$

$$= \int \frac{e^x}{-2(1+e^x)}dx$$

$$put\ 1+e^x = t \Rightarrow e^x dx = dt$$

$$= -\frac{1}{2}\int \frac{1}{t}dt$$

$$= -\frac{1}{2}\log(1+e^x)$$

P.I = $Pf_1 + Qf_2$ $= \frac{e^x}{2}\left[-x - e^{-x} + \log(1+e^x)\right] - \frac{e^{-x}}{2}\log(1+e^x)$

The complete solution is $y(x) = Ae^x + Be^{-x} + \frac{e^x}{2}\left[-x - e^{-x} + \log(1+e^x)\right] - \frac{e^{-x}}{2}\log(1+e^x)$