ENVELOPE

A curve which touches each member of a given family of curves is called envelope of that family.

Procedure to find envelope for the given family of curves:

Case 1: Envelope of one parameter family of curves

Let us consider y = f(x,α) to be the given family of curves with 'α' as the parameter.

Step 1: Differentiate w.r.t to the parameter α partially, and find the value of the parameter

Step 2: By Substituting the value of parameter α in the given family of curves, we get the required envelope.

Special Case: If the given equation of curve is quadratic in terms of parameter,i.e. $A\alpha^2+B\alpha+c=0$, then envelope is given by **discriminant = 0** i.e. $B^2- 4AC=0$

Case 2: Envelope of two parameter family of curves.

Let us consider y = f(x,α, β) to be the given family of curves, and a relation connecting the two parameters α and β, g(α, β) = 0

Step 1: Consider α as independent variable and β depends α . Differentiate y = f(x,α, β) and g(α, β) = 0, w.r. to the parameter α partially.

Step 2: Eliminating the parameters α, β from the equations resulting from step 1 and g(α, β) = 0, we get the required envelope.

Problems on envelope of one parameter family of curves :

1. Find the envelope of $y = mx + am^p$ where m is the parameter and a, p are constants

Solution : Differentiate $y = mx + am^p$ (1)

with respect to the parameter m, we get,

$$0 = x + pam^{p-1}$$

$$\Rightarrow \quad m = \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}} \quad (2)$$

Using (2) eliminate m from (1)

$$y = \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}} x + a\left(\frac{-x}{pa}\right)^{\frac{p}{p-1}}$$

$$\Rightarrow \quad y^{p-1} = \left(\frac{-x}{pa}\right) x^{p-1} + a^{p-1}\left(\frac{-x}{pa}\right)^{p}$$

i.e. $ap^{p} y^{p-1} = -x^{p} p^{p-1} + (-x)^{p}$

which is the required equation of envelope of (1)

2. Determine the envelope of $x\sin\theta - y\cos\theta = a\theta$, where θ being the parameter.

Solution : Differentiate ,

$$x\sin\theta - y\cos\theta = a\theta \qquad (1)$$

with respect to θ , we get,

$$x\cos\theta + y\sin\theta = a \qquad (2)$$

As θ cannot be eliminated between (1) and (2) ,we solve (1) and (2) for x and y in terms of θ.

For this, multiply (2) by sinθ and (1) by cosθ and then subtracting, we get,

$y = a(\sin\theta - \theta\cos\theta)$. Using similar simplification, we get, $x = a(\theta\sin\theta + \cos\theta)$.

3. (Leibnitz's problem) Calculate the envelope of family of circles whose centres lie on the x-axis

and radii are proportional to the abscissa of the centre.

Solution : Let (a,0) be the centre of any one of the member of family of curves with a as the parameter. Then the equation of family of circles with centres on x-axis and radius proportional to the abscissa of the centre is

$$(x-a)^2 + y^2 = ka^2 \qquad (1)$$

where k is the proportionality constant. Differentiating (1) with respect to a, we get,

$$-2(x-a) = 2ka$$

i.e. $a = \dfrac{x}{1-k}$.

From (1) , $\left(x - \dfrac{x}{1-k}\right)^2 + y^2 = \dfrac{k}{(1-k)^2} x^2$

i.e. $\left(k^2 - k\right)x^2 + (1-k)^2 y^2 = 0, \qquad k \neq 1$

4. Find the envelope of $x\sec^2\theta + y\cos ec^2\theta = a$, where θ is the parameter.

Solution : The given equation is rewritten as , $x\left(1+\tan^2\theta\right) + y\left(1+\cot^2\theta\right) = a$

i.e. $x\tan^4\theta + (x+y-a)\tan^2\theta + y = 0$,

which is a quadratic equation in $t = \tan^2\theta$. Therefore the required envelope is given by the discriminant equation : B^2-4AC = 0

i.e. $(x+y-a)^2 - 4xy = 0$

i.e. $x^2 + y^2 - 2xy - 2ax - 2ay + a^2 = 0$.

Envelope of Two parameter family of curves :

1. Find the envelope of family of straight lines ax+by=1, where a and b are parametersconnected by the relation ab = 1

Solution :

$$ax + by = 1 \qquad (1)$$

$$ab = 1 \qquad (2)$$

Differentiating (1) with respect to a (considering 'a' as independent variable and 'b' depends on a).

$$x + \frac{db}{da} y = 0$$

i.e. $$\frac{db}{da} = \frac{-x}{y} \qquad (3)$$

Differentiating (2) with respect to a

$$b + a\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-b}{a} \qquad (4)$$

From (3) and (4), we have

$$\frac{x}{y} = \frac{b}{a}$$

i.e. $$\frac{ax}{1} = \frac{by}{1} = \frac{ax+by}{2} = \frac{1}{2}$$

$\therefore$ $a = \frac{1}{2x}$ and $b = \frac{1}{2y}$ (5)

Using (5) in (2), we get the envelope as 4xy = 1

2. Find the envelope of family of straight lines $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$, where a and b are parameters connected by the relation $\sqrt{a} + \sqrt{b} = 1$

Solution :

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1 \qquad (1)$$

$$\sqrt{a} + \sqrt{b} = 1 \qquad (2)$$

Differentiating (1) with respect to a

$$\frac{\sqrt{x}}{-2a^{3/2}} + \frac{\sqrt{y}}{-2b^{3/2}}\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-\sqrt{x}}{\sqrt{y}}\frac{b^{3/2}}{a^{3/2}} \qquad (3)$$

Differentiating (2) with respect to a

$$\frac{1}{2\sqrt{a}} + \frac{1}{2\sqrt{b}}\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-\sqrt{b}}{\sqrt{a}} \qquad (4)$$

From (3) and (4), we have

$$\frac{\sqrt{x}}{\sqrt{y}}\frac{b}{a} = 1$$

i.e. $$\frac{\sqrt{\frac{x}{a}}}{\sqrt{a}} = \frac{\sqrt{\frac{y}{b}}}{\sqrt{b}} = \frac{\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}}}{\sqrt{a} + \sqrt{b}} = \frac{1}{1}$$

$\therefore \quad a = \sqrt{x}$ and $b = \sqrt{y}$ (5)

Using (5) in (2), we get the envelope as $x^{1/4} + y^{1/4} = 1$

3. Find the envelope of family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are parameters connected by the relation $a^2b^3 = c^5$

$$\frac{x}{a}+\frac{y}{b}=1 \qquad (1)$$

$$a^2b^3 = c^5 \qquad (2)$$

Differentiating (1) with respect to a,

$$\frac{-x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0$$

i.e. $$\frac{db}{da}=\frac{-b^2x}{a^2y} \qquad (3)$$

Differentiating (2) with respect to a

$$2ab^3+3a^2b^2\frac{db}{da}=0$$

i.e. $$\frac{db}{da}=\frac{-2b}{3a} \qquad (4)$$

From (3) and (4), we have

$$\frac{3x}{a}=\frac{2y}{b}$$

i.e. $$\frac{\frac{x}{a}}{3}=\frac{\frac{y}{b}}{2}=\frac{\frac{x}{a}+\frac{y}{b}}{5}=\frac{1}{5}$$

$\therefore \quad a=\dfrac{5x}{3}$ and $\quad b=\dfrac{5y}{2}$ (5)

Using (5) in (2), we get the envelope as $$x^2y^3=\frac{72}{3125}c^5$$

4. Find the envelope of the family of circles whose centres lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and which pass through its centre.

Solution: Let (α,β) be the centre of arbitrary member of family of circles which lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose centre is (0,0). Therefore, equation of the circles passing through origin and having centreat (α,β) is

$$x^2+y^2-2\alpha x-2\beta y=0 \quad (1)$$

with

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1 \quad (2)$$

Differentiating (1) with respect to α ('α' as independent variable and 'β' depends on α),

$$x+\frac{d\beta}{d\alpha}y=0$$

i.e. $\frac{d\beta}{d\alpha}=\frac{-x}{y}$ (3)

Differentiating (2) with respect to α

$$\frac{2\alpha}{a^2}+\frac{2\beta}{b^2}\frac{d\beta}{d\alpha}=0$$

i.e. $\frac{d\beta}{d\alpha}=\frac{-b^2\alpha}{a^2\beta}$ (4)

From (3) and (4), we have

$$\frac{x}{y}=\frac{b^2\alpha}{a^2\beta}$$

i.e. $\frac{\alpha x}{\frac{\alpha^2}{a^2}}=\frac{\beta y}{\frac{\beta^2}{b^2}}=\frac{\alpha x+\beta y}{\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}}=\frac{k}{1}$, where k = αx+βy

$\therefore \qquad \alpha = \dfrac{a^2 x}{k}$ and $\quad \beta = \dfrac{b^2 y}{k}$ (5)

From (1), we have , $x^2 + y^2 = 2k$ (6)

Using (5) and (6) in (2), we get the envelope as $\left(x^2 + y^2\right)^2 = 4\left(a^2 x^2 + b^2 y^2\right)$

5. Determine the equation of the envelope of family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where the parameters a and b are connected by the relation $\frac{a^2}{l^2} + \frac{b^2}{m^2} = 1$, *l* and m are non-zero constants.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

$$\frac{a^2}{l^2} + \frac{b^2}{m^2} = 1 \qquad (2)$$

Differentiating (1) with respect to a,

$$\frac{-2x^2}{a^3} - \frac{2y^2}{b^3}\frac{db}{da} = 0$$

i.e. $\quad \dfrac{db}{da} = \dfrac{-b^3 x^2}{a^3 y^2}$ (3)

Differentiating (2) with respect to a

$$\frac{2a}{l^2} + \frac{2b}{m^2}\frac{db}{da} = 0$$

i.e. $\quad \dfrac{db}{da} = \dfrac{-m^2 a}{l^2 b}$ (4)

From (3) and (4), we have

$$\frac{b^4x^2}{a^4y^2}=\frac{m^2}{l^2}$$

i.e. $$\frac{\frac{x^2}{a^2}}{\frac{a^2}{l^2}}=\frac{\frac{y^2}{b^2}}{\frac{b^2}{m^2}}=\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{\frac{a^2}{l^2}+\frac{b^2}{m^2}}=\frac{1}{1}$$

$\Rightarrow \quad a^4=l^2x^2$ and $b^4=m^2y^2$

i.e. $a^2=lx$ and $b^2=my$ (5)

Using (5) in (2), we get the envelope as $\frac{x}{l}+\frac{y}{m}=1$

Problems on Evolute as envelope of its normals :

1. Determine the evolute of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ by considering it as an envelope of its normal

Solution : Let P (a cosht, b sinht) be any point on the given hyperbola. Then

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{b\cosh t}{a\sinh t}=\frac{b}{a}\coth t$$

Equation of normal line to the hyperbola is

$$(y-b\sinh t)=\frac{-a}{b\cosh t}(x-a\cosh t) \tag{1}$$

$$\Rightarrow \quad \frac{by}{\sinh t}+\frac{ax}{\cosh t}=a^2+b^2 \tag{2}$$

Differentiating (2) partially with respect to t, we have,

$$\frac{-by}{(\sinh t)^2}\cosh t - \frac{ax}{(\cosh t)^2}\sinh t = 0$$

$$\Rightarrow \quad \tanh t = -\left(\frac{by}{ax}\right)^{1/3}$$

$$\Rightarrow \quad \sinh t = \mp\left(\frac{by}{h}\right)^{1/3} \text{and} \cosh t = \pm\left(\frac{ax}{h}\right)^{1/3} \qquad (3)$$

Where $\qquad h = \sqrt{(ax)^{2/3} - (by)^{2/3}}$

Using (3) in (2) , we get,

$$\frac{by}{-(by)^{1/3}}h + \frac{ax}{(ax)^{1/3}}h = a^2 + b^2$$

i.e. $\left((ax)^{2/3} - (by)^{2/3}\right)\left((ax)^{2/3} - (by)^{2/3}\right)^{1/2} = a^2 + b^2$

i.e. $(ax)^{2/3} - (by)^{2/3} = \left(a^2 + b^2\right)^{2/3}$

2. By considering the evolute of a curve as the envelope of its normal, find the evolute of

$x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$

Solution :

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\theta\sin\theta}{\theta\cos\theta} = \tan\theta \, .$$

Equation of normal line to the hyperbola is

$$(y - (\sin\theta - \theta\cos\theta)) = \frac{-1}{\tan\theta}(x - (\cos\theta + \theta\sin\theta))$$

$$\Rightarrow \quad y\sin\theta - \sin^2\theta + \theta\sin\theta\cos\theta = -x\cos\theta + \cos^2\theta + \theta\sin\theta\cos\theta$$

i.e. $$y\sin\theta + x\cos\theta = 1 \qquad (1)$$

Differentiating (1) with respect to the parameter θ, we have

$$y\cos\theta - x\sin\theta = 0 \qquad (2)$$

Multiplying (1) by cosθ and (2) by sinθ and then subtracting, we have,

$$x = \cos\theta \qquad (3)$$

Similarly we get,

$$y = \sin\theta \qquad (4)$$

Eliminating θ between (3) and (4) we get the required evolute as $x^2 + y^2 = 1$