# UNIT – II

## GEOMETRICAL APPLICATIONS OF DIFFERENTIAL CALCULUS

**Curvature:**

At each point on a curve, with equation y=f(x), the tangent line turns at a certain rate. A measure of this rate of turning is the curvature

$$K = \frac{f''(x)}{(1+[f'(x)])^{3/2}}$$

**Radius of curvature in Cartesian form:**

If the curve is given in Cartesian coordinates as *y(x)*, then the radius of curvature is

$\rho = (1 + [y']^{2})^{(3/2)}/y''$ where $y'=\frac{dy}{dx}$, $y'' = (d^{2} y)/(dx^{2})$ .

**Radius of curvature in Parametric form:**

If the curve is given parametrically by functions *x(t)* and *y(t)*, then the radius of curvature is

$\rho = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{x'y - \square\square}$ , $x' = \frac{dx}{dt}$, $x = \frac{d^2x}{dt^2}$, $y' = \frac{dy}{dt}$, $y = \frac{d^2y}{dt^2}$

**Examples:**

1. Find the radius of the curvature at the point $\left(\frac{1}{\frac{4,1}{4}}\right)$ on the curve $\sqrt{x} + \sqrt{y} = 1$ .

   Solution: $\sqrt{x} + \sqrt{y} = 1$

   Differentiating w. r. t x ,we get

   $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} y' = 0$ .   $y' = -\frac{\sqrt{y}}{\sqrt{x}}$

   At $\left(\frac{1}{\frac{4,1}{4}}\right)$, $y' = -1.$

   $y'' = -[(\sqrt{x}\ 1/(2\sqrt{y})\ y' - \sqrt{y}\ 1/(2\sqrt{x}))/x]$

At $\left(\frac{\frac{1}{4,1}}{4}\right)$, $y'' = -[(1/2\ \ 1/(2\ 1/2)\ (-1) - 1/2\ \ 1/(2\ 1/2))/(1/4)]$ = 4.

$$\rho = \frac{(1+1)^{\frac{3}{2}}}{4} = \frac{1}{\sqrt{2}}$$

2. Show that the radius of the curvature at any point of the curve $y = c\cosh\left(\frac{x}{c}\right)$ is $\frac{y^2}{c}$.

   Solution: $y = c\cosh\left(\frac{x}{c}\right)$

   Differentiating y w. r. t x we get

   $y' = \sinh\left(\frac{x}{c}\right)$

   $y'' = 1/c\ \ \cosh(x/c)$

   $$\rho = \frac{\left[1+\sinh^2\left(\frac{x}{c}\right)\right]^{\frac{3}{2}}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)} = c\cosh^2\left(\frac{x}{c}\right) = \frac{y^2}{c}.$$

3. Find the radius of the curvature of the curve y = $x^2$(x-3) at the points where the tangent is parallel to the x – axis.

   Solution: y = $x^2$(x-3)

   Differentiating y w. r. t x we get

   $y' = 3x^2 - 6x$

   $y'' = 6x - 6$

   The points at which the tangent parallel to the x – axis can be found by equating y' to zero.

   i.e., $3x^2 - 6x = 0 \Rightarrow x = 0, x = 2.$

   At x = 0,y" = -6. At x = 2, y" = 6.

   Therefore at x = 0 and x = 2, $\rho = \frac{1}{6}$.

4. Prove that the radius of the curvature of the curve at any point of the cycloid

   $x = a(t + \sin t), y = a(1 + \cos t)$ is $\frac{4a\cos t}{2}$.

Solution: We have $x = a(t + sint), y = a(1 + cost)$.

Therefore $\frac{dx}{dt} = a(1 + cost)$, $\frac{dy}{dt} = asint.$

Now $$\frac{dy}{dt} = \frac{dy/dt}{dx/dt} = \frac{asint}{a(1+cost)} = \frac{\frac{2\sin t}{2}\cos t}{\frac{2}{2cos^2\frac{t}{2}}} = \frac{\tan t}{2}.$$

Also $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\tan t}{2}\right) = \left\{\frac{d}{dt}\left(\frac{\tan t}{2}\right)\right\}\frac{dt}{dx} = \frac{1}{2}sec^2\frac{t}{2}\frac{1}{a(1+cost)} = \frac{1}{4a}sec^4\frac{t}{2}.$$

Hence $$\rho = \frac{\left(1 + tan^2\frac{t}{2}\right)^{\frac{3}{2}}}{\frac{1}{4a}sec^4\frac{t}{2}} = \frac{4a\cos t}{2}.$$

**Centre and Circle of curvature:**

Let the equation of the curve be y = f(x). let P be the given point (x,y) on this curve and Q the point (x+Δx,y+Δy) in the neighborhood of P. let N be the point of intersection of the normals at P and Q. As Q→P, suppose N→C. Then C is the centre of curvature of P. The circle whose centre C and radius $\rho$ is called the circle of curvature. The co-ordinates of the centre of curvature is denoted as $(\bar{x}, \bar{y})$.

where $\bar{x} = x - (y'(1 + y'^2))/y''$, $\bar{y} = y + ((1 + y'^2))/y''$.

**Equation of the circle of curvature:**

If $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature and $\rho$ be the radius of curvature at any point (x,y) on a curve, then the equation of the circle of curvature at that point is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

**Examples:**

1. Find the centre of curvature of the curve $a^2y = x^3$.

   Solution: $a^2y = x^3$

   $\frac{dy}{dx} = \frac{3x^2}{a^2}$ and $\frac{d^2y}{dx^2} = \frac{6x}{a^2}$

$$\overline{x} = x - \frac{x}{2}\left(1 + \frac{9x^4}{a^4}\right) = \frac{x}{2}\left[1 - \frac{9x^4}{a^4}\right]$$

$$\overline{y} = \frac{x^3}{a^2} + \frac{\left[1 + \frac{9x^4}{a^4}\right]}{\frac{6x}{a^2}} = \frac{5x^3}{2a^2} + \frac{a^2}{6x}.$$

Therefore the required centre of curvature is $\left(\frac{x}{2}\left[1 - \frac{9x^4}{a^4}\right], \frac{5x^3}{2a^2} + \frac{a^2}{6x}\right)$.

2. Find the centre of curvature of y = $x^2$ at $\left(\frac{1}{\frac{2,1}{4}}\right)$.

   Solution: y' = 2x, y" = 2.

   At $\left(\frac{1}{\frac{2,1}{4}}\right)$, y' = 1, y" = 2.

   Therefore $\overline{x} = \frac{1}{2} - \frac{(1+1)}{2} = -\frac{1}{2}, \overline{y} = \frac{1}{4} + 1 = \frac{5}{4}$.

   Therefore the required centre of curvature is $\left(-\frac{1}{\frac{2,5}{4}}\right)$.

3. Find the centre of curvature of the curve xy = $a^2$ at (a,a).

   Solution: $y^{\uparrow\prime} = -a^{\uparrow}2/x^{\uparrow}2 \;\; , y'' = 2a^{\uparrow}2\, x^{\uparrow}(-3)$ . At (a,a) y' = -1, y" = $\frac{2}{a}$

   Therefore $\overline{x} = a + \frac{2}{2/a} = 2a, \overline{y} = a + \frac{2}{2/a} = 2a.$

   The required centre of curvature is (2a, 2a).

4. Find the circle of curvature of the curve $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{\frac{2,3a}{2}}\right)$.

Solution: $x^3 + y^3 = 3axy$

$$3x^2 + 3y^2y' = 3a(xy' + y)$$

$$y' = \frac{ay - x^2}{y^2 - ax}$$

$y'$ at $\left(\frac{3a}{\frac{2,3a}{2}}\right)$ is -1.

$$y'' = ((y^{\uparrow}2 - ax)(ay^{\uparrow\prime} - 2x) - (ay - x^{\uparrow}2)(2yy^{\uparrow\prime} - a))/(y^{\uparrow}2 - ax)^{\uparrow}2$$

$$y'' at (3a/2, 3a/2) = (-32)/3a$$

$$\rho = \frac{2\sqrt{2(3a)}}{32}$$

$$\bar{x} = \frac{3a}{2} - \frac{2}{32/_{3a}} = \frac{21a}{16}$$

$$\bar{y} = \frac{3a}{2} - \frac{2}{32/_{3a}} = \frac{21a}{16}.$$

The circle of curvature is $\left(x - \frac{21a}{16}\right)^2 + \left(y - \frac{21a}{16}\right)^2 = \frac{9a^2}{128}$

5. Find the circle of curvature at the point (2,3) on $\frac{x^2}{4} + \frac{y^2}{9} = 2$.

Solution: $\frac{2x}{4} + \frac{2yy'}{9} = 0 \Rightarrow y' = \frac{-9x}{4y} \Rightarrow y'(2,3) = \frac{-3}{2}$

$y'' = (-9(y - xy^{\uparrow\prime}))/(4y^{\uparrow}2)$ , $y''$ at $(2,3) = (-3)/2$

$\rho = \frac{13^{\frac{3}{2}}}{12}$, $\bar{x} = 2 - \frac{(-3/_2)(1 + 9/_4)}{\frac{-3}{2}} = \frac{-5}{4}$,

$$\bar{y} = 3 + \frac{(1 + 9/_4)}{\frac{-3}{2}} = \frac{5}{6}$$

The circle of curvature is $\left(x + \frac{5}{4}\right)^2 + \left(y - \frac{5}{6}\right)^2 = \frac{13^3}{12^2}$

**Evolute and Involute**

**Evolute:** Evolute of the curve is defined as the locus of the centre of curvature for that curve.

**Involute :** If C' is the evolute of the curve C then C is called the involute of the curve C'.

**Procedure to find the evolute:**

Let the given curve be f(x,y,a,b) = 0. (1)

Find y' and y" at the point P.

Find the centre of curvature $(\bar{x}, \bar{y})$. Using $\bar{x} = x - (y'(1 + y'^2))/y''$, $\bar{y} = y + ((1 + y'^2))/y''$. (2)

Eliminate x, y from (1), (2) we get f$(\bar{x}, \bar{y}, a, b) = 0$ . (3)

Equation (3) is the required evolute.

**Examples**:

**1.** Show that the evolute of the cycloid $x = a(\theta + sin\theta), y = a(1 - cos\theta)$ is another cycloid given by $x = a(\theta - sin\theta), y - 2a = a(1 + cos\theta)$.

Solution: $\frac{dx}{d\theta} = a(1 + cos\theta), \frac{dy}{d\theta} = asin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{asin\theta}{a(1 + cos\theta)} = \frac{\tan\theta}{2}$$

$$y'' = d/d\theta\,(\tan\theta/2)\ (d\theta)/dx = (sec^4\,\theta/2)/4a$$

$$\bar{x} = a(\theta + sin\theta) - \frac{\frac{\tan\theta}{2\left(1 + tan^2\frac{\theta}{2}\right)}}{sec^4\frac{\theta}{2}/4a} = a(\theta + sin\theta) - 2asin\theta = a(\theta - sin\theta)$$ ,

$$\bar{y} = a(1 - cos\theta) + \frac{\left(1 + tan^2\frac{\theta}{2}\right)}{sec^4\frac{\theta}{2}/4a} = a(1 - cos\theta) + 4acos^2\frac{\theta}{2} = a(1 + cos\theta) + 2a.$$

$\bar{x} = a(\theta - sin\theta), \bar{y} - 2a = a(1 + cos\theta)$.

The locus of $\bar{x}$ and $\bar{y}$ is $x = a(\theta - sin\theta), y - 2a = a(1 + cos\theta)$.

**2.** Prove that the evolute of the curve $x = a(cos\theta + \theta sin\theta), y = a(sin\theta - \theta cos\theta)$ is a circle $x^2 + y^2 = a^2$.

Solution: $\frac{dx}{d\theta} = a(-sin\theta + sin\theta + \theta cos\theta) = a\theta cos\theta,\ \frac{dy}{d\theta} = a\theta sin\theta.$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta cos\theta}{a\theta sin\theta} = tan\theta$$ .

$y'' = 1/(a\theta\, cos^3\,\theta)$ ,

$$\overline{x} = a(cos\theta + \theta sin\theta) - \frac{tan\theta\left(1 + tan^2\theta\right)}{1/_{a\theta cos^3}\theta} = acos\theta,$$

$$\overline{y} = a(sin\theta - \theta cos\theta) + \frac{(1 + tan^2\theta)}{1/_{a\theta cos^3}\theta} = asin\theta.$$

Eliminating , $\overline{x}$ *and* $\overline{y}$ we get $\overline{x}^2 + \overline{y}^2 = a^2$ .

The evolute of the given curve is $x^2 + y^2 = a^2$ .