Subject Name :Engineering Mathematics-I

(Common To All Branches Except Bio Groups)

Subject Code: SMT1101

Course Material

UNIT I

MATRICES

CHARACTERISTIC EQUATION:

The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A

Note:

1. Solving $|A - \lambda I| = 0$, we get n roots for λ and these roots are called characteristic roots or eigen values or latent values of the matrix A
2. Corresponding to each value of λ, the equation AX = λX has a non-zero solution vector X

   If X_r be the non-zero vector satisfying AX = λX, when $\lambda = \lambda_r$, X_r is said to be the latent vector or eigen vector of a matrix A corresponding to λ_r

CHARACTERISTIC POLYNOMIAL:

The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A

Working rule to find characteristic equation:

<u>For a 3 x 3 matrix:</u>

<u>Method 1:</u>

The characteristic equation is $|A - \lambda I| = 0$

<u>Method 2:</u>

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where $S_1 = sum\ of\ the\ main\ diagonal\ elements,$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements$,

$S_3 = Determinant of A = \ |A|$

For a 2 x 2 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$

Method 2:

Its characteristic equation can be written as $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = sum\ of\ the\ main\ diagonal\ elements$, $S_2 = Determinant\ of A = \ |A|$

Problems:

1. **Find the characteristic equation of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}$**

Solution: Let A = $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$. Its characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$where $S_1 = sumofthemaindiagonalelements = 1 + 2 = 3,$

$S_2 = Determinant of A = \ |A|$ = 1(2) – 2(0) = 2

Therefore, the characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

2. **Find the characteristic equation of $\begin{pmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{pmatrix}$**

Solution: Its characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where

$S_1 = sumofthemaindiagonalelements$ = 8 + 7 + 3 = 18,

$S_2 = Sumoftheminorsofthemaindiagonalelements = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 5 + 20 + 20 = 45$, $S_3 = Determinant of A = \ |A|$ = 8(5)+6(-10)+2(10) = 40 -60 + 20 = 0

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

3. **Find the characteristic polynomial of $\begin{pmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} \end{pmatrix}$**

Solution: Let A = $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic polynomial of A is $\lambda^2 - S_1\lambda + S_2$ where $S_1 = sum of the main diagonal elements$ = 3 + 2 = 5 and $S_2 = Determinant of A = |A|$ = 3(2) – 1(-1) = 7

Therefore, the characteristic polynomial is $\lambda^2 - 5\lambda + 7$

CAYLEY-HAMILTON THEOREM:

Statement: Every square matrix satisfies its own characteristic equation

Uses of Cayley-Hamilton theorem:

(1) To calculate the positive integral powers of A
(2) To calculate the inverse of a square matrix A

Problems:

1. **Show that the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$ satisfies its own characteristic equation**

Solution:Let A = $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 = 2$

$S_2 = |A| = 1 - (-4) = 5$

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove $A^2 - 2A + 5I = 0$

$A^2 = A(A) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$

$A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Therefore, the given matrix satisfies its own characteristic equation

2. **If A = $\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{5} \end{bmatrix}$ $\boldsymbol{write\ A^2\ interms\ of\ A\ and\ I, using\ Cayley - Hamilton\ theorem}$**

Solution:Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 6$

$S_2 = |A| = 5$

Therefore, the characteristic equation is $\lambda^2 - 6\lambda + 5 = 0$

By Cayley-Hamilton theorem, $A^2 - 6A + 5I = 0$

i.e., $A^2 = 6A - 5I$

3. Verify Cayley-Hamilton theorem, find A^4 and A^{-1} when A = $\begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{bmatrix}$

<u>Solution</u>: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 2 + 2 + 2 = 6$

$S_2 = Sum\ of\ the\ minirs\ of\ the\ main\ diagonal\ elements = 3 + 2 + 3 = 8$

$S_3 = \ |A| = 2(4-1) + 1(-2+1) + 2(1-2) = \ 2(3) - 1 - 2 = 3$

Therefore, the characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

To prove that: $A^3 - 6A^2 + 8A - 3I = 0$------------- (1)

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \ A^2(A) = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$A^3 - 6A^2 + 8A - 3I$$
$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

To find A^4:

$(1) \Rightarrow A^3 - 6A^2 + 8A - 3I = 0 \Rightarrow A^3 = 6A^2 - 8A + 3I$ ------------- (2)

Multiply by A on both sides, $A^4 = 6A^3 - 8A^2 + 3A = 6(6A^2 - 8A + 3I) - \ 8A^2 + 3A$

Therefore, $A^4 = 36A^2 - 48A + 18I - 8A^2 + 3A = 28A^2 - 45A + 18I$

Hence, $A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix} =$$

$$\begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}:

Multiplying (1) by $A^{-1}, A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$$\Rightarrow 3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

4. **Verify that A = $\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{-1} \end{bmatrix}$ satisfies its own characteristic equation and hence find A^4**

Solution:Given A =$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 =$ Sum of the main diagonal elements $= 0$

$S_2 = |A| = -1 - 4 = -5$

Therefore, the characteristic equation is $\lambda^2 - 0\lambda - 5 = 0$ i.e., $\lambda^2 - 5 = 0$

To prove: $A^2 - 5I = 0$---------- (1)

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

To find A^4:

From (1), we get, $A^2 - 5I = 0 \Rightarrow A^2 = 5I$

Multiplying by A^2 on both sides, we get, $A^4 = A^2(5I) = 5A^2 = 5\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$

5. **Find A^{-1} if A = $\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{4} \\ \mathbf{3} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{-1} \end{bmatrix}$, using Cayley-Hamilton theorem**

Solution:The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 2 - 1 = 2$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (-2+1) + (-1-8) + (2+3)$
$= -1 - 9 + 5 = -5$

$S_3 = |A| = 1(-2+1) + 1(-3+2) + 4(3-4) = -1 - 1 - 4 = -6$

The characteristic equation of A is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By Cayley- Hamilton theorem, $A^3 - 2A^2 - 5A + 6I = 0$ ------------ (1)

To find A^{-1}:

Multiplying (1) by A^{-1}, we get, $A^2 - 2A - 5A^{-1}A + 6A^{-1}I = 0 \Rightarrow A^2 - 2A - 5I + 6A^{-1} = 0$

$6A^{-1} = -A^2 + 2A + 5I \Rightarrow A^{-1} = \frac{1}{6}(-A^2 + 2A + 5I)$ ------------- (2)

$$A^2 = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

From (2), $A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$

6. If A = $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $find\ A^n\ in\ terms\ of\ A$

Solution:The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where
$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 2 = 3$

$S_2 = |A| = 2 - 0 = 2$

The characteristic equation of A is $\lambda^2 - 3\lambda + 2 = 0$ i.e., $\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)} = \frac{3 \pm 1}{2} = 2,1$

To find A^n:

When λ^n is divided by $\lambda^2 - 3\lambda + 2$, let the quotient be $Q(\lambda)$ and the remainder be $a\lambda + b$

$\lambda^n = (\lambda^2 - 3\lambda + 2)Q(\lambda) + a\lambda + b$ ---------- (1)

When $\lambda = 1, 1^n = a + b$

When $\lambda = 2, 2^n = 2a + b$

$2a + b = 2^n$ ------------ (2)

$a + b = 1^n$ ------------ (3)

Solving (2) and (3), we get, (2) - (3)⇒ $a = 2^n - 1^n$

(2) – 2 x (3) ⇒ $b = -2^n + 2(1)^n$

i.e., $a = 2^n - 1^n$

$b = 2(1)^n - 2^n$

Since $A^2 - 3A + 2I = 0$ by Cayley-Hamilton theorem, (1) ⇒ $A^n = aA + bI$

$$A^n = (2^n - 1^n)\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix} + [2(1)^n - 2^n]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

7. Use Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix}\mathbf{1} & \mathbf{4}\\\mathbf{2} & \mathbf{3}\end{bmatrix}$ to express as a linear polynomial in A (i) $\boldsymbol{A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I}$ (ii) $\boldsymbol{A^4 - 4A^3 - 5A^2 + A + 2I}$

<u>**Solution**</u>: Given A = $\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$. The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 3 = 4$

$S_2 = |A| = 3 - 8 = -5$

The characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$

By Cayley-Hamilton theorem, we get, $A^2 - 4A - 5I = 0$ ------------ (1)

$\lambda^3 - 2\lambda + 3$

$\lambda^2 - 4\lambda - 5 \,\big|\, \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10$

$\lambda^5 - 4\lambda^4 - 5\lambda^3$

$-2\lambda^3 + 11\lambda^2 - \lambda$

(-)$-2\lambda^3 + 8\lambda^2 + 10\lambda$

$3\lambda^2 - 11\lambda - 10$

$(-)\ 3\lambda^2 - 12\lambda - 15$

$\lambda + 5$

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^2 - 4A - 5I)(A^3 - 2A + 3I) + A + 5I = 0 + A + 5I$$

$= A + 5I$ (by (1)) which is a linear polynomial in A

(i) λ^2

$\lambda^2 - 4\lambda - 5 \,\big|\, \lambda^4 - 4\lambda^3 - 5\lambda^2 + \lambda + 2$

$\lambda^4 - 4\lambda^3 - 5\lambda^2$

(-) $\lambda + 2$

$A^4 - 4A^3 - 5A^2 + A + 2I$ **=** $A^2(A^2 - 4A - 5I) + A + 2I$ = 0 + A + 2I = A +2I (by (1)) which is a linear polynomial in A

8. **Using Cayley-Hamilton theorem, find A^{-1} when A = $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$**

Solution:The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 1 + 1 = 3$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (1-1) + (1-3) + (1-0)$
$= 0 - 2 + 1 = -1$

$S_3 = |A| = 1(1-1) + 0(2+1) + 3(-2-1) = 1(0) + 0 - 9 = -9$

The characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley-Hamilton theorem, $A^3 - 3A^2 - A + 9I = 0$

Pre-multiplying by A^{-1}, we get, $A^2 - 3A - I + 9A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{9}(-A^2 + 3A + I)$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$-A^2 = \begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix}; 3A = \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix}; I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\left(\begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

9. **Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{7} \\ \mathbf{4} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \end{bmatrix}$**

Solution: Given A = $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

The Characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = Sum of the main diagonal elements = 1+2+1 = 4

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = (2-6) + (1-7) + (2-12)$
$= -4 - 6 - 10 = -20$

$S_3 = |A| = 1(2-6) - 3(4-3) + 7(8-2) = -4 - 3 + 42 = 35$

The characteristic equation is $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

To prove that: $A^3 - 4A^2 - 20A - 35I = 0$

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$A^3 - 4A^2 - 20A - 35I = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - \begin{bmatrix} 80 & 92 & 92 \\ 60 & 88 & 148 \\ 40 & 36 & 56 \end{bmatrix} - \begin{bmatrix} 20 & 60 & 140 \\ 80 & 40 & 60 \\ 20 & 40 & 20 \end{bmatrix} - \begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Therefore, Cayley-Hamilton theorem is verified.

10. Verify Cayley-Hamilton theorem for the matrix (i) A = $\begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{5} \end{bmatrix}$ $(\boldsymbol{ii})\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$

Solution:(i) Given A = $\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 3 + 5 = 8$

$S_2 = |A| = 15 - 1 = 14$

The characteristic equation is $\lambda^2 - 8\lambda + 14 = 0$

To prove that: $A^2 - 8A + 14I = 0$

$$A^2 = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 9+1 & -3-5 \\ -3-5 & 1+25 \end{bmatrix} = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}$$

$8A = 8\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 24 & -8 \\ -8 & 40 \end{bmatrix}$

$14I = 14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$

$A^2 - 8A + 14I = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix} - \begin{bmatrix} 24 & -8 \\ -8 & 40 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Hence Cayley-Hamilton theorem is verified.

(ii) Given A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 3 = 4$

$S_2 = |A| = 3 - 8 = -5$

The characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$

To prove that: $\boldsymbol{A^2 - 4A - 5I = 0}$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$4A = 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix}$; $5I = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Hence Cayley-Hamilton theorem is verified.

EIGEN VALUES AND EIGEN VECTORS OF A REAL MATRIX:

Working rule to find eigen values and eigen vectors:

1. Find the characteristic equation $|A - \lambda I| = 0$
2. Solve the characteristic equation to get characteristic roots. They are called eigen values
3. To find the eigen vectors, solve $[A - \lambda I]X = 0$ for different values of λ

Note:

1. Corresponding to n distinct eigen values, we get n independent eigen vectors
2. If 2 or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated eigen values

3. If X_i is a solution for an eigen value λ_i, then cX_i is also a solution, where c is an arbitrary constant. Thus, the eigen vector corresponding to an eigen value is not unique but may be any one of the vectors cX_i
4. Algebraic multiplicity of an eigen value λ is the order of the eigen value as a root of the characteristic polynomial (i.e., if λ is a double root, then algebraic multiplicity is 2)
5. Geometric multiplicity of λ is the number of linearly independent eigen vectors corresponding to λ

Non-symmetric matrix:

If a square matrix A is non-symmetric, then A ≠ A^T

Note:

1. In a non-symmetric matrix, if the eigen values are non-repeated then we get a linearly independent set of eigen vectors
2. In a non-symmetric matrix, if the eigen values are repeated, then it may or may not be possible to get linearly independent eigen vectors.
   If we form a linearly independent set of eigen vectors, then diagonalization is possible through similarity transformation

Symmetric matrix:

If a square matrix A is symmetric, then A = A^T

Note:

1. In a symmetric matrix, if the eigen values are non-repeated, then we get a linearly independent and pair wise orthogonal set of eigen vectors
2. In a symmetric matrix, if the eigen values are repeated, then it may or may not be possible to get linearly independent and pair wise orthogonal set of eigen vectors
   If we form a linearly independent and pair wise orthogonal set of eigen vectors, then diagonalization is possible through orthogonal transformation

Problems:

1. **Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} \end{pmatrix}$**

Solution: Let A = $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where
$S_1 = sum of the main diagonal elements = 1 - 1 = 0,$

$S_2 = Determinant of A = \ |A|$ = 1(-1) – 1(3) = - 4

Therefore, the characteristic equation is $\lambda^2 - 4 = 0$ i.e., $\lambda^2 = 4$ or $\lambda = \pm 2$

Therefore, the eigen values are 2, -2

A is a non-symmetric matrix with non- repeated eigen values

To find the eigen vectors:

$$[A - \lambda I]X = 0$$

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$**--------------- (1)**

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 \\ 3 & -1-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ [**From (1)]**

i.e., $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $3x_1 + x_2 = 0$

$$3x_1 + x_2 = 0$$

i.e., we get only one equation $3x_1 + x_2 = 0 \Rightarrow 3x_1 = -x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{-3}$

Therefore $X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Case 2: If $\lambda = 2$, $\begin{bmatrix} 1-(2) & 1 \\ 3 & -1-(2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ [**From (1)]**

i.e., $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $-x_1 + x_2 = 0 \Rightarrow x_1 - x_2 = 0$

$$3x_1 - 3x_2 = 0 \Rightarrow x_1 - x_2 = 0$$

i.e., we get only one equation $x_1 - x_2 = 0$

$$\Rightarrow x_1 = x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$$

Hence, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. **Find the eigen values and eigen vectors of $\begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \end{bmatrix}$**

Solution: Let A = $\begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \end{bmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2 + 3 + 2 = 7,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 + 3 + 4 = 11,$

$S_3 = Determinant of A = \ |A|$ = 2(4)-2(1)+1(-1) = 5

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

1	1	− 7	11	− 5
	0	1	− 6	5
	1	-6	5	0

$(\lambda - 1)(\lambda^2 - 6\lambda + 5) = 0 \Rightarrow \lambda = 1,$

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)} = \frac{6 \pm \sqrt{16}}{2} = \frac{6 \pm 4}{2} = \frac{6+4}{2}, \frac{6-4}{2} = 5, 1$$

Therefore, the eigen values are 1, 1, and 5

A is a non-symmetric matrix with repeated eigen values

<u>To find the eigen vectors:</u>

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = 5$, $\begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -3x_1 + 2x_2 + x_3 = 0$ --------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------- (2)

$x_1 + 2x_2 - 3x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

2 1 -3 2

-2 1 1 -2

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = 1$, $\begin{bmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

All the three equations are one and the same. Therefore, $x_1 + 2x_2 + x_3 = 0$

Put $x_1 = 0 \Rightarrow 2x_2 + x_3 = 0 \Rightarrow 2x_2 = -x_3 . Taking x_3 = 2 , x_2 = -1$

Therefore, $X_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

Put $x_2 = 0 \Rightarrow x_1 + x_3 = 0 \Rightarrow x_3 = -x_1 . Taking x_1 = 1, x_3 = -1$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

3. **Find the eigen values and eigen vectors of** $\begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$ which is a non-symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 2 + 1 - 1 = 2,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} =$

$-4 - 4 + 4 = -4,$

$S_3 = Determinant of A = |A|$ = 2(-4)+2(-2)+2(2) = - 8 – 4 + 4 = - 8

Therefore, the characteristic equation of A is $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

2	1	-2	-4	8

$0 \quad 2 \quad 0 \quad -8$

1 0 -4 0

$$(\lambda-2)(\lambda^2-4)=0 \Rightarrow \lambda=2, \qquad \lambda = 2,-2$$

Therefore, the eigen values are 2, 2, and -2

A is a non-symmetric matrix with repeated eigen values

To find the eigen vectors:

$$[A-\lambda I]X=0$$

$$\begin{bmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\boldsymbol{\lambda = -2}$, $\begin{bmatrix} 2-(-2) & -2 & 2 \\ 1 & 1-(-2) & 1 \\ 1 & 3 & -1-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0$ --------- (1)

$x_1 + 3x_2 + x_3 = 0$ ------------- (2)

$x_1 + 3x_2 + x_3 = 0$ ------------ (3) . Equations (2) and (3) are one and the same.

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

-1 1 2 -1

3 1 1 3

$$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7} \Rightarrow \frac{x_1}{4} = \frac{x_2}{1} = \frac{x_3}{-7}$$

Therefore, $X_1 = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$

Case 2: If $\boldsymbol{\lambda = 2}$**,** $\begin{bmatrix} 2-2 & -2 & 2 \\ 1 & 1-2 & 1 \\ 1 & 3 & -1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -.3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

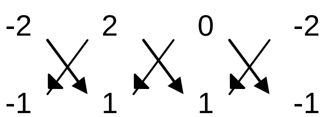
$\Rightarrow 0x_1 - 2x_2 + 2x_3 = 0$---------- (1)

$x_1 - x_2 + x_3 = 0$---------------- (2)

$x_1 + 3x_2 - 3x_3 = 0$------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$



$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{2} = \frac{x_3}{2} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

We get one eigen vector corresponding to the repeated root $\lambda_2 = \lambda_3 = 2$

4. **Find the eigen values and eigen vectors of** $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

<u>**Solution**</u>: Let A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ which is a symmetric matrix

<u>**To find the characteristic equation:**</u>

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum of the main diagonal elements = 1 + 5 + 1 = 7,$

$S_2 = Sum of the minors of the main diagonal elements = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4 - 8 + 4 = 0,$

$S_3 = Determinant of A = |A|$ = 1(4)-1(-2)+3(-14) = - 4 + 2-42 = - 36

Therefore, the characteristic equation of A is $\lambda^3 - 7\lambda^2 + 0\lambda - 36 = 0$

-2	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

$(\lambda - (-2))(\lambda^2 - 9\lambda + 18) = 0 \Rightarrow \lambda = -2,$

$$\lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(18)}}{2(1)} = \frac{9 \pm \sqrt{81 - 72}}{2} = \frac{9 \pm 3}{2} = \frac{9+3}{2}, \frac{9-3}{2} = 6, 3$$

Therefore, the eigen values are -2, 3, and 6

A is a symmetric matrix with non- repeated eigen values

To find the eigen vectors:

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\lambda = -2$, $\begin{bmatrix} 1-(-2) & 1 & 3 \\ 1 & 5-(-2) & 1 \\ 3 & 1 & 1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 + 7x_2 + x_3 = 0$ ------------- (2)

$3x_1 + x_2 + 3x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 \quad x_2 \quad x_3$$

1 3 3 1

7 1 1 7

$$\Rightarrow \frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow \frac{x_1}{-4} = \frac{x_2}{0} = \frac{x_3}{4} = \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Therefore, $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = 3$**,** $\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 + 2x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 \quad x_2 \quad x_3$$

1 3 -2 1

2 1 1 2

$$\Rightarrow \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1} = \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Therefore, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case 3: If $\lambda = 6$**,** $\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -5x_1 + x_2 + 3x_3 = 0$ ---------- (1)

$x_1 - x_2 + x_3 = 0$ ---------------- (2)

$3x_1 + x_2 - 5x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 x_2 x_3$$

1 3 -5 1

-1 1 1 -1

$\Rightarrow \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$

Therefore, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

5. **Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Determine the algebraic and geometric multiplicity**

Solution: Let A = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ which is a symmetric matrix

To find the characteristic equation:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = sum\ of\ the\ main\ diagonal\ elements = 0 + 0 + 0 = 0,$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 - 1 - 1 = -3,$

$S_3 = Determinant\ of\ A = |A|$ = 0 -1(-1)+ 1(1) = 0 + 1 + 1 = 2

Therefore, the characteristic equation of A is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

-1	1	0	-3	-2
	0	-1	1	2
	1	-1	-2	0

$$(\lambda - (-1))(\lambda^2 - \lambda - 2) = 0 \Rightarrow \lambda = -1,$$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \frac{1+3}{2}, \frac{1-3}{2} = 2, -1$$

Therefore, the eigen values are 2, -1, and -1

A is a symmetric matrix with repeated eigen values. The algebraic multiplicity of $\lambda = -1$ is 2

<u>To find the eigen vectors:</u>

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: If $\boldsymbol{\lambda = 2}$, $\begin{bmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -2x_1 + x_2 + x_3 = 0$ --------- (1)

$x_1 - 2x_2 + x_3 = 0$ ------------- (2)

$x_1 + x_2 - 2x_3 = 0$ ------------ (3)

Considering equations (1) and (2) and using method of cross-multiplication, we get,

$$x_1 \quad x_2 \quad x_3$$

1 1 -2 1

-2 1 1 -2

$\Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Therefore, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: If $\boldsymbol{\lambda = -1}$**,** $\begin{bmatrix} 0-(-1) & 1 & 1 \\ 1 & 0-(-1) & 1 \\ 1 & 1 & 0-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + x_2 + x_3 = 0$ ---------- (1)

$x_1 + x_2 + x_3 = 0$ ---------------- (2)

$x_1 + x_2 + x_3 = 0$ ------------ (3). All the three equations are one and the same.

Therefore, $x_1 + x_2 + x_3 = 0$. Put $x_1 = 0 \Rightarrow x_2 + x_3 = 0 \Rightarrow x_3 = -x_2 \Rightarrow \frac{x_2}{1} = \frac{x_3}{-1}$

Therefore, $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Since the given matrix is symmetric and the eigen values are repeated, let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$. X_3 is orthogonal to X_1 and X_2 .

$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l + m + n = 0$ ------------ (1)

$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow 0l + m - n = 0$-------- (2)

Solving (1) and (2) by method of cross-multiplication, we get,

l   m n

1   1   1   1

1   -1   0   1

$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$. Therefore, $X_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Thus, for the repeated eigen value $\lambda = -1,$ there corresponds two linearly independent eigen vectors $X_2\ and\ X_3$. So, the geometric multiplicity of eigen value $\lambda = -1$ is 2

Problems under properties of eigen values and eigen vectors.

1. **Find the sum and product of the eigen values of the matrix** $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

   Solution: Sum of the eigen values = Sum of the main diagonal elements = -3

   Product of the eigen values = |A| = -1 (1 – 1) -1(-1 – 1) + 1(1- (-1)) = 2 + 2 = 4

2. **Product of two eigen values of the matrix A =** $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **is 16. Find the third eigen value**

   Solution: Let the eigen values of the matrix be $\lambda_1, \lambda_2, \lambda_3$.

   Given $\lambda_1\lambda_2 = 16$

   We know that $\lambda_1\lambda_2\lambda_3 = |A|$(Since product of the eigen values is equal to the determinant of the matrix)

   $\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$ = 6(9-1)+2(-6+2) +2(2-6) = 48-8-8 = 32

   $$There\ fore, \lambda_1\lambda_2\lambda_3 = 32 \Rightarrow\ 16\lambda_3 = 32 \Rightarrow \lambda_3 = 2$$

3. **Find the sum and product of the eigen values of the matrix A =** $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **without finding the roots of the characteristic equation**

   Solution:We know that the sum of the eigen values = Trace of A = a + d

   Product of the eigen values = |A| = ad – bc

4. **If 3 and 15 are the two eigen values of A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, find |A|, without expanding the determinant**

Solution:Given $\lambda_1 = 3 \; and \; \lambda_2 = 15, \lambda_3 = ?$

We know that sum of the eigen values = Sum of the main diagonal elements

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$

$$\Rightarrow 3 + 15 + \lambda_3 = 18 \Rightarrow \lambda_3 = 0$$

$We\ know\ that\ the\ product\ of\ the\ eigen\ values = |A|$

$\Rightarrow (3)(15)(0) = |A|$

$\Rightarrow |A| = 0$

5. **If 2, 2, 3 are the eigen values of A = $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$, find the eigen values of A^T**

Solution:By the property "A square matrix A and its transpose A^T have the same eigen values", the eigen values of A^T are 2,2,3

6. **Find the eigen values of A = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$**

Solution:Given A = $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$ .Clearly, A is a lower triangular matrix. Hence, by the property "the characteristic roots of a triangular matrix are just the diagonal elements of the matrix", the eigen values of A are 2, 3, 4

7. **Two of the eigen values of A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the eigen values of A^{-1}**

Solution:Sum of the eigen values = Sum of the main diagonal elements = 3 +5+3 = 11

Given 3,6 are two eigen values of A. Let the third eigen value be k.

Then, 3 + 6 + k = 11 ⇒ $k = 2$

Therefore, the eigen values of A are 3, 6, 2

By the property "If the eigen values of A are $\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

8. Find the eigen values of the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{-5} & \mathbf{4} \end{bmatrix}$. Hence, form the matrix whose eigen values are $\frac{1}{6}$ $\boldsymbol{and -1}$

Solution: Let A = $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$. The characteristic equation of the given matrix is $\lambda^2 - S_1\lambda + S_2 = 0$ where $S_1 = Sum\ of\ the\ main\ diagonal\ elements =$ 5 and $S_2 = |A| = -6$

Therefore, the characteristic equation is $\lambda^2 - 5\lambda - 6 = 0 \Rightarrow \lambda = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-6)}}{2(1)} = \frac{5 \pm 7}{2} = 6, -1$

Therefore, the eigen values of A are 6, -1

Hence, the matrix whose eigen values are $\frac{1}{6}$ $and - 1$ is A^{-1}

$$A^{-1} = \frac{1}{|A|}\ adj\ A$$

|A| = 4 -10 = - 6; adj A = $\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

Therefore, $A^{-1} = \frac{1}{-6}\begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$

9. Find the eigen values of the inverse of the matrix A = $\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{4} \end{bmatrix}$

Solution:We know that A is an upper triangular matrix. Therefore, the eigen values of A are 2, 3, 4. Hence, by using the property "If the eigen values of A are $\lambda_1, \lambda_2, \lambda_3$, then the eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$", the eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

10. Find the eigen values of A^3 given A = $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{2} & \mathbf{-7} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 1, 2, 3

Therefore, the eigen values of A^3 are $1^3, 2^3, 3^3$ i.e., 1,8,27

11. If 1 and 2 are the eigen values of a 2 x 2 matrix A, what are the eigen values of $\boldsymbol{A^2}$ *and* $\boldsymbol{A^{-1}}$?

Solution:Given 1 and 2 are the eigen values of A.

Therefore, 1^2 *and* 2^2 i.e., 1 and 4 are the eigen values of A^2 and 1 and $\frac{1}{2}$ are the eigen values of A^{-1}

12. If 1,1,5 are the eigen values of A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find the eigen values of 5A

Solution:By the property "If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A, then $k\lambda_1, k\lambda_2, k\lambda_3$ are the eigen values of kA, the eigen values of 5A are 5(1), 5(1), 5(5) ie., 5,5,25

13. Find the eigen values of A, $\boldsymbol{A^2, A^3, A^4, 3A, A^{-1}, A - I, 3A^3 + 5A^2 - 6A + 2I}$ if A = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

Solution:Given A = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 2, 5

The eigen values of A^2 are $2^2, 5^2$ i.e., 4, 25

The eigen values of A^3 are $2^3, 5^3$ i.e., 8, 125

The eigen values of A^4 are $2^4, 5^4$ i.e., 16, 625

The eigen values of 3A are 3(2), 3(5) i.e., 6, 15

The eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{5}$

A – I = $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = $\begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$

Since A - I is an upper triangular matrix, the eigen values of A- I are its main diagonal elements i.e., 1,4

Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are $3\lambda_1^3 + 5\lambda_1^2 - 6\lambda_1 + 2$ and $3\lambda_2^3 + 5\lambda_2^2 - 6\lambda_2 + 2$ where $\lambda_1 = 2$ and $\lambda_2 = 5$

First eigen value = $3\lambda_1^3 + 5\lambda_1^2 - 6\lambda_1 + 2$

= $3(2)^3 + 5(2)^2 - 6(2) + 2 = 24 + 20 - 12 + 2 = 34$

Second eigen value = $3\lambda_2^3 + 5\lambda_2^2 - 6\lambda_2 + 2$

= $3(5)^3 + 5(5)^2 - 6(5) + 2$

= 375+ 125-30 + 2 = 472

14. Find the eigen values of adj A if A = $\begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{4} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

<u>Solution</u>:Given A = $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. A is an upper triangular matrix. Hence, the eigen values of A are 3, 4, 1

We know that $A^{-1} = \frac{\mathbf{1}}{|\boldsymbol{A}|}\ adj\ A$

Adj A = $|A|\, A^{-1}$

The eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{4}, 1$

$|A|$ =Product of the eigen values = 12

Therefore, the eigen values of adj A is equal to the eigen values of 12 A^{-1} i.e., $\frac{12}{3}, \frac{12}{4}, 12$ i.e., 4, 3, 12

<u>Note:</u> A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$. Here, A is an upper triangular matrix, B is a lower triangular matrix and C is a diagonal matrix. In all the cases, the elements in the main diagonal are the eigen values. Hence, the eigen values of A, B and C are 1, 4, 6

15. Two eigen values of A = $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal and they are $\frac{1}{5}$ times the third. Find them

Solution:Let the third eigen value be λ_3

We know that $\lambda_1 + \lambda_2 + \lambda_3$ =2+3+2 = 7

Given $\lambda_1 = \lambda_2 = \frac{\lambda_3}{5}$

$$\frac{\lambda_3}{5} + \frac{\lambda_3}{5} + \lambda_3 = 7$$

$$\left[\frac{1}{5} + \frac{1}{5} + 1\right]\lambda_3 = 7 \Rightarrow \frac{7}{5}\lambda_3 = 7 \Rightarrow \lambda_3 = 5$$

Therefore, $\lambda_1 = \lambda_2 = 1$ and hence the eigen values of A are 1,1, 5

16. If 2, 3 are the eigen values of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$, *find the value of a*

Solution:Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$. Let the eigen values of A be 2, 3, k

We know that the sum of the eigen values = sum of the main diagonal elements

Therefore, 2 +3 +k = 2+ 2+2 = 6 ⇒ $k = 1$

We know that product of the eigen values = |A|

⇒2(3)(k) = |A|

$$\Rightarrow 6 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix} \Rightarrow 6 = 2(4) - 0 + 1(-2a) \Rightarrow 6 = 8 - 2a \Rightarrow 2a = 2 \Rightarrow a = 1$$

17. Prove that the eigen vectors of the real symmetric matrix A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are orthogonal in pairs

Solution:The characteristic equation of A is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$where $S_1 = sum\ of\ the\ main\ diagonal\ elements = 7$;

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements\ = 4 + (-8) + 4 = 0$

$$S_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 1(4) - 1(-2) + 3(-14) = -36$$

The characteristic equation of A is $\lambda^3 - 7\lambda^2 + 36 = 0$

3	1-7	0	36	
	0	3	-12	-36
	1	-4	-12	0

Therefore, $\lambda = 3, \lambda^2 - 4\lambda - 12 = 0 \Rightarrow \lambda = 3, \lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-12)}}{2(1)} = \frac{4 \pm 8}{2} = 6, -2$

Therefore, the eigen values of A are -2, 3, 6

To find the eigen vectors:

$$(A - \lambda I)X = 0$$

Case 1: When $\lambda = -2$, $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x_1 + x_2 + 3x_3 = 0$ -------- (1)

$x_1 + 7x_2 + x_3 = 0$ ---------- (2)

$3x_1 + x_2 + 3x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$$x_1 x_2 x_3$$

1 3 3 1

7 1 1 7

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case 2: When $\lambda = 3$, $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 + 2x_2 + x_3 = 0$ --------- (2)

$3x_1 + x_2 - 2x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$x_1 x_2 x_3$

1 3 -2 1

2 1 1 2

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case 3: When $\lambda = 6$, $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-5x_1 + x_2 + 3x_3 = 0$ --------- (1)

$x_1 - x_2 + x_3 = 0$ --------- (2)

$3x_1 + x_2 - 5x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication, we get,

$x_1 x_2 x_3$

1 3 -5 1

-1 1 1 -1

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \Rightarrow X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Therefore, $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

To prove that: $X_1^T X_2 = 0, X_2^T X_3 = 0, X_3^T X_1 = 0$

$$X_1^T X_2 = [-1 \quad 0 \quad 1] \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

$$X_2^T X_3 = [1 - 1 \quad 1]\begin{bmatrix}1\\2\\1\end{bmatrix} = 1 - 2 + 1 = 0$$

$$X_3^T X_1 = [1 \quad 2 \quad 1]\begin{bmatrix}-1\\0\\1\end{bmatrix} = -1 + 0 + 1 = 0$$

Hence, the eigen vectors are orthogonal in pairs

18. Find the sum and product of all the eigen values of the matrix A = $\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{3}\\ \mathbf{2} & \mathbf{2} & \mathbf{4}\\ \mathbf{1} & \mathbf{2} & \mathbf{7}\end{bmatrix}$**.Is the matrix singular?**

<u>**Solution**</u>:Sum of the eigen values = Sum of the main diagonal elements =Trace of the matrix

Therefore, the sum of the eigen values = 1+2+7=10

Product of the eigen values = |A| = 1(14 - 8) -2(14 - 4) + 3(4 - 2) = 6-20+ 6= - 8

|A|≠0. Hence the matrix is non-singular.

19. Find the product of the eigen values of A = $\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-2}\\ \mathbf{1} & \mathbf{0} & \mathbf{3}\\ \mathbf{-2} & \mathbf{-1} & \mathbf{-3}\end{bmatrix}$

<u>**Solution**</u>:Product of the eigen values of A = |A| = $\begin{vmatrix}1 & 2 & -2\\ 1 & 0 & 3\\ -2 & -1 & -3\end{vmatrix} = 1(3) - 2(3) - 2(-1) =$

$3 - 6 + 2 = -1$

ORTHOGONAL TRANSFORMATION OF A SYMMETRIC MATRIX TODIAGONAL FORM:

Orthogonal matrices:

A square matrix A (with real elements) is said to be orthogonal if $AA^T = A^TA = I$ or $A^T = A^{-1}$

Problems:

1. **Check whether the matrix B is orthogonal. Justify. B =** $\begin{bmatrix}\boldsymbol{\cos\theta} & \boldsymbol{\sin\theta} & \mathbf{0}\\ \boldsymbol{-\sin\theta} & \boldsymbol{\cos\theta} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\end{bmatrix}$

<u>**Solution**</u>: Condition for orthogonality is $AA^T = A^TA = I$

To prove that: $\boldsymbol{BB^T = B^TB = I}$

B = $\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$; $B^T = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$

$$BB^T = \begin{bmatrix} cos\,\theta & sin\,\theta & 0 \\ -sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} cos^2\theta + sin^2\theta & -sin\theta cos\theta + sin\theta cos\theta & 0 \\ -sin\theta cos\theta + \sin\theta\cos\theta + 0 & sin^2\theta + cos^2\theta + 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly,

$$B^T B = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} cos\,\theta & sin\,\theta & 0 \\ -sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} cos^2\theta + sin^2\theta & \sin\theta cos\theta - \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta - \sin\theta\cos\theta & sin^2\theta + cos^2\theta + 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, B is an orthogonal matrix

2. **Show that the matrix P =** $\begin{bmatrix} \boldsymbol{cos\,\theta} & \boldsymbol{sin\,\theta} \\ \boldsymbol{-sin\,\theta} & \boldsymbol{cos\,\theta} \end{bmatrix}$ **is orthogonal**

Solution:To prove that: $PP^T = P^T P = I$

$P = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$; $P^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$PP^T = \begin{bmatrix} cos^2\theta + sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & sin^2\theta + cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}$$

$$Similarly, P^T P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} cos^2\theta + sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & sin^2\theta + cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore, P is an orthogonal matrix

WORKING RULE FOR DIAGONALIZATION

[ORTHOGONAL TRANSFORMATION]:

Step 1: To find the characteristic equation

Step 2: To solve the characteristic equation

Step 3:To find the eigen vectors

Step 4: If the eigen vectors are orthogonal, then form a normalized matrix N

Step 5: Find N^T

Step 6: Calculate AN

Step 7: Calculate D = $N^T AN$

Problems:

1. **Diagonalize the matrix** $\begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{5} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = \text{Sum of the main diagonal elements} = 3 + 5 + 3 = 11$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonalelements = (15-1) + (9-1) + (15-1)$
$= 14 + 8 + 14 = 36$

$S_3 = |A| = 3(15-1) + 1(-3+1) + 1(1-5) = 3(14) - 2 - 4 = 42 - 6 = 36$

Therefore, the characteristic equation is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

2	1	-11	36	-36
	0	2	-18	36
	1	-9	18	0

$\lambda = 2, \lambda^2 - 9\lambda + 18 = 0 \Rightarrow \lambda = 2, \lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(18)}}{2(1)} = \frac{9 \pm \sqrt{81-72}}{2} = \frac{9 \pm 3}{2} = 6,3$

Hence, the eigen values of A are 2, 3, 6

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\boldsymbol{\lambda} = \mathbf{2}$, $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 + 3x_2 - x_3 = 0$ -------- (2)

$x_1 - x_2 + x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$x_1 x_2 x_3$

-1 1 1 -1

3 -1 -1 3

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1} \Rightarrow \frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case 2: When $\boldsymbol{\lambda = 3}$, $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 + 2x_2 - x_3 = 0$ -------- (2)

$x_1 - x_2 + 0x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$x_1 x_2 x_3$

-1 1 0 -1

2 -1 -1 2

$$\frac{x_1}{1-2} = \frac{x_2}{-1-0} = \frac{x_3}{0-1} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case 3: When $\boldsymbol{\lambda = 6}$, $\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1 - x_2 + x_3 = 0$ ---------- (1)

$-x_1 - x_2 - x_3 = 0$ -------- (2)

$x_1 - x_2 - 3x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

$x_1 x_2 x_3$

-1 1 -3 -1

-1 -1 -1 -1

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$X_1^T X_2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

$$X_2^T X_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 1 - 2 + 1 = 0$$

$$X_3^T X_1 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

Hence, the eigen vectors are orthogonal to each other

The Normalized matrix N = $\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$; $N^T = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

AN = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}} \\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \end{bmatrix}$

$$N^T AN = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}} \\ \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{4}{2} & \frac{0}{\sqrt{6}} & \frac{0}{\sqrt{12}} \\ \frac{0}{\sqrt{6}} & \frac{9}{3} & \frac{0}{\sqrt{18}} \\ \frac{0}{\sqrt{12}} & \frac{0}{\sqrt{18}} & \frac{36}{6} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

The diagonal elements are the eigen values of A

2. Diagonalize the matrix $\begin{bmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{bmatrix}$

<u>**Solution**</u>: Let A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$\mathrm{S}_1 = \text{Sum of the main diagonal elements} = 8 + 7 + 3 = 18$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonalelements = (21 - 16) + (24 - 4) + (56 - 36)$
$= 5 + 20 + 20 = 45$

$S_3 = |A| = 8(21 - 16) + 6(-18 + 8) + 2(24 - 14) = 8(5) - 60 + 20 = 0$

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$ i.e., $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0 \Rightarrow \lambda = 0, \lambda = \frac{18 \pm \sqrt{(-18)^2 - 4(1)(45)}}{2(1)} = \frac{18 \pm \sqrt{324 - 180}}{2} = \frac{18 \pm 12}{2}$
$= 15, 3$

Hence, the eigen values of A are 0, 3, 15

To find the eigen vectors:

$(A - \lambda I)X = 0$

$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case 1: When $\boldsymbol{\lambda = 0}$, $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$8x_1 - 6x_2 + 2x_3 = 0$ ---------- (1)

$-6x_1 + 7x_2 - 4x_3 = 0$ -------- (2)

$2x_1 - 4x_2 + 3x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

x_1 x_2 x_3

-6 2 8 -6

7 -4 -6 7

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36} \Rightarrow \frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case 2: When $\lambda = 3$, $\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$5x_1 - 6x_2 + 2x_3 = 0$ -------- (1)

$-6x_1 + 4x_2 - 4x_3 = 0$ ------- (2)

$2x_1 - 4x_2 + 0x_3 = 0$ --------- (3)

Solving (1) and (2) by rule of cross-multiplication,

x_1 x_2 x_3

-6 2 5 -6

4 -4 -6 4

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36} \Rightarrow \frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16} \Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Case 3: When $\lambda = 15$, $\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-7x_1 - 6x_2 + 2x_3 = 0$ ---------- (1)

$-6x_1 - 8x_2 - 4x_3 = 0$ ---------- (2)

$2x_1 - 4x_2 - 12x_3 = 0$ ---------- (3)

Solving (1) and (2) by rule of cross-multiplication,

x_1 x_2 x_3

-6 2 -7 -6

-8 -4 -6 -8

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36} \Rightarrow \frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$X_1^T X_2 = [1 \quad 2 \quad 2] \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 2 + 2 - 4 = 0$

$X_2^T X_3 = [2 \quad 1 \quad -2] \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ =4-2-2 = 0

$X_3^T X_1 = [2 \quad -2 \quad 1] \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 2 - 4 + 2 = 0$

Hence, the eigen vectors are orthogonal to each other

The Normalized matrix N = $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$N^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

AN =

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} =$$

$$\frac{1}{3} \begin{bmatrix} 8 - 12 + 4 & 16 - 6 - 4 & 16 + 12 + 2 \\ -6 + 14 - 8 & -12 + 7 + 8 & -12 - 14 - 4 \\ 2 - 8 + 6 & 4 - 4 - 6 & 4 + 8 + 3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

$$N^T AN = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 + 0 + 0 & 2 + 2 - 4 & 10 - 20 + 10 \\ 0 + 0 + 0 & 4 + 1 + 4 & 20 - 10 - 10 \\ 0 + 0 + 0 & 4 - 2 - 2 & 20 + 20 + 5 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 45 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

The diagonal elements are the eigen values of A

QUADRATIC FORM- REDUCTION OF QUADRATIC FORM TO CANONICAL FORM BY ORTHOGONAL TRANSFORMATION:

Quadratic form:

A homogeneous polynomial of second degree in any number of variables is called a quadratic form

Example: $2x_1^2 + 3x_2^2 - x_3^2 + 4x_1x_2 + 5x_1x_3 - 6x_2x_3$ is a quadratic form in three variables

Note:

The matrix corresponding to the quadratic form is

$$\begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$$

Problems:

1. **Write the matrix of the quadratic form $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$**

Solution:Q = $\begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$

Here $x_2x_1 = x_1x_2$; $x_3x_1 = x_1x_3$; $x_2x_3 = x_3x_2$

$$Q = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$$

2. **Write the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$**

Solution: Q = $\begin{bmatrix} coeff.of\ x^2 & \frac{1}{2}\ coeff.of\ xy & \frac{1}{2}\ coeff.of\ xz \\ \frac{1}{2}\ coeff.of\ yx & coeff.of\ y^2 & \frac{1}{2}\ coeff.of\ yz \\ \frac{1}{2}\ coeff.of\ zx & \frac{1}{2}\ coeff.of\ zy & coeff.of\ z^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$

3. **Write down the quadratic form corresponding to the following symmetric matrix** $\begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{4} & \mathbf{3} \end{bmatrix}$

Solution:Let $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$

The required quadratic form is

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2(a_{12})x_1x_2 + 2(a_{23})x_2x_3 + 2(a_{13})x_1x_3$$

$= 0x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 + 8x_2x_3$

NATURE OF THE QUADRATIC FORM:

Rank of the quadratic form: The number of square terms in the canonical form is the rank (r) of the quadratic form

Index of the quadratic form: The number of positive square terms in the canonical form is called the index (s) of the quadratic form

Signature of the quadratic form: The difference between the number of positive and negative square terms = s – (r-s) = 2s-r, is called the signature of the quadratic form

The quadratic form is said to be

(1) **Positive definite** if all the eigen values are positive numbers
(2) **Negative definite** if all the eigen values are negative numbers
(3) **Positive Semi-definite** if all the eigen values are greater than or equal to zero and at least one eigen value is zero
(4) **Negative Semi-definite** if all the eigen values are less than or equal to zero and at least one eigen value is zero
(5) **Indefinite** if A has both positive and negative eigen values

Problems:

1. **Determine the nature of the following quadratic form f(x_1, x_2, x_3) $= x_1^2 + 2x_2^2$**

Solution:The matrix of the quadratic form is Q = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The eigen values of the matrix are 1, 2, 0

Therefore, the quadratic form is Positive Semi-definite

2. **Discuss the nature of the quadratic form $2x^2 + 3y^2 + 2z^2 + 2xy$ without reducing it to canonical form**

Solution:The matrix of the quadratic form is Q = $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$D_1 = 2(+ve)$

$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5(+ve)$

$D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2(6-0) - 1(2-0) + 0 = 12 - 2 = 10(+ve)$

Therefore, the quadratic form is positive definite

REDUCTION OF QUADRATIC FORM TO CANONICAL FORM THROUGH ORTHOGONAL TRANSFORMATION [OR SUM OF SQUARES FORM OR PRINCIPAL AXES FORM]

Working rule:

Step 1: Write the matrix of the given quadratic form

Step 2: To find the characteristic equation

Step 3: To solve the characteristic equation

Step 4: To find the eigen vectors orthogonal to each other

Step 5: Form the Normalized matrix N

Step 6: Find N^T

Step 7: Find AN

Step 8: Find D = $N^T AN$

Step 9: The canonical form is $[y_1 y_2 y_3][D]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Problems:

1. **Reduce the given quadratic form Q to its canonical form using orthogonal transformation Q = $x^2 + 3y^2 + 3z^2 - 2yz$**

Solution: The matrix of the Q.F is A = $\begin{bmatrix} coeff.of\ x^2 & \frac{1}{2}\ coeff.of\ xy & \frac{1}{2}\ coeff.of\ xz \\ \frac{1}{2}\ coeff.of\ yx & coeff.of\ y^2 & \frac{1}{2}\ coeff.of\ yz \\ \frac{1}{2}\ coeff.of\ zx & \frac{1}{2}\ coeff.of\ zy & coeff.of\ z^2 \end{bmatrix}$

i.e., A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 1 + 3 + 3 = 7$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elemeents = (9-1)+(3-0)+(3-0)$
$= 8+3+3 = 14$

$S_3 = |A| = 1(9-1)+0+0 = 8$

The characteristic equation of A is $\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$

1	1	-7	14	-8
	0	1	-6	8
	1	-6	8	0

$$\lambda = 1, \lambda^2 - 6\lambda + 8 = 0 \Rightarrow \lambda = 1, \lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(8)}}{2(1)} = \frac{6 \pm \sqrt{4}}{2} = \frac{6 \pm 2}{2} = 4, 2$$

The eigen values are 1, 2, 4

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\boldsymbol{\lambda = 1}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0x_1 + 0x_2 + 0x_3$= 0 -------- (1)

$0x_1 + 2x_2 - x_3 = 0$ -------- (2)

$0x_1 - x_2 + 2x_3 = 0$ -------- (3)

Solving (2) and (3) by rule of cross multiplication, we get,

$x_1 x_2 x_3$

2 -1 0 2

-1 2 0 -1

$$\frac{x_1}{4-1} = \frac{x_2}{0-0} = \frac{x_3}{0-0} \Rightarrow \frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Case 2: When $\boldsymbol{\lambda = 2}$, $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-x_1 + 0x_2 + 0x_3$= 0 -------- (1)

$0x_1 + x_2 - x_3 = 0$ -------- (2)

$0x_1 - x_2 + x_3 = 0$ -------- (3)

Solving (1) and (2) by rule of cross multiplication, we get,

$x_1 x_2 x_3$

0 0 -1 0

1 -1 0 1

$$\frac{x_1}{0-0} = \frac{x_2}{0-1} = \frac{x_3}{-1-0} \Rightarrow \frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Case 3: When $\boldsymbol{\lambda = 4}$, $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1 + 0x_2 + 0x_3$= 0 -------- (1)

$0x_1 - x_2 - x_3 = 0$ -------- (2)

$0x_1 - x_2 - x_3 = 0$ -------- (3)

Solving (1) and (2) by rule of cross multiplication, we get,

$x_1 x_2 x_3$

0 0 -3 0

-1 -1 0 -1

$$\frac{x_1}{0-0} = \frac{x_2}{0-3} = \frac{x_3}{3-0} \Rightarrow \frac{x_1}{0} = \frac{x_2}{-3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

The Normalized matrix N = $\begin{bmatrix} \frac{1}{\sqrt{1}} & \frac{0}{\sqrt{2}} & \frac{0}{\sqrt{2}} \\ \frac{0}{\sqrt{1}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{0}{\sqrt{1}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$; $N^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

AN = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1}} & \frac{0}{\sqrt{2}} & \frac{0}{\sqrt{2}} \\ \frac{0}{\sqrt{1}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{0}{\sqrt{1}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & \frac{-4}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix}$

$$N^T AN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & \frac{-4}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

i.e., $D = N^T AN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

The canonical form is $[y_1 y_2 y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + 2y_2^2 + 4y_3^2$

canonical form is $[y_1 \quad y_2 \quad y_3] \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 8y_1^2 + 2y_2^2 + 2y_3^2$

1. **Reduce the quadratic form to a canonical form by an orthogonal reduction $2x_1x_2 + 2x_1x_3 - 2x_2x_3$.Also find its nature.**

Solution: The matrix of the Q.F is A = $\begin{bmatrix} coeff.of\ x_1^2 & \frac{1}{2}coeff.of\ x_1x_2 & \frac{1}{2}coeff.of\ x_1x_3 \\ \frac{1}{2}coeff.of\ x_2x_1 & coeff.of\ x_2^2 & \frac{1}{2}coeff.of\ x_2x_3 \\ \frac{1}{2}coeff.of\ x_3x_1 & \frac{1}{2}coeff.of\ x_3x_2 & coeff.of\ x_3^2 \end{bmatrix}$

i.e., A = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = Sum\ of\ the\ main\ diagonal\ elements = 0$

$S_2 = Sum\ of\ the\ minors\ of\ the\ main\ diagonal\ elements = -1 - 1 - 1 = -3$

$S_3 = |A| = 0(0-1) - 1(0+1) + 1(-1-0) = 0 - 1 - 1 = -2$

The characteristic equation of A is $\lambda^3 - 0\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda^3 - 3\lambda + 2 = 0$

1	1	0	-3	2
	0	1	1	-2
	1	1	-2	0

$$\lambda = 1, \lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda = 1, \lambda = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = -2, 1$$

The eigen values are 1, 1, -2

To find the eigen vectors:

$(A - \lambda I)X = 0$

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: When $\boldsymbol{\lambda = -2}$, $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$2x_1 + x_2 + x_3 = 0$ ------- (1)

$x_1 + 2x_2 - x_3 = 0$ ------- (2)

$x_1 - x_2 + 2x_3 = 0$ ------- (3)

$x_1 \quad x_2 \quad x_3$

1 ╳ 1 ╳ 2 ╳ 1

2   -1   1   2

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1} \Rightarrow \frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Case 2: When $\boldsymbol{\lambda = 1}$, $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-x_1 + x_2 + x_3 = 0$ ------- (1)

$x_1 - x_2 - x_3 = 0$ ------ (2)

$x_1 - x_2 - x_3 = 0$ ------ (3)

All three equations are one and the same.

Put $x_1 = 0$, $x_2 = -x_3$. Let $x_3 = 1$. Then $x_2 = -1$

$$X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$. Since X_3 is orthogonal to X_1 and X_2, $X_1^T X_3 = 0$ and $X_2^T X_3 = 0$

$$\begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \; and \; \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$-l + m + n = 0$ -------- (1)

$0l - m + n = 0$ --------- (2)

l m n

1 1 -1 1

-1 1 0 -1

$$\frac{l}{1+1} = \frac{m}{0+1} = \frac{n}{1-0} \Rightarrow \frac{l}{2} = \frac{m}{1} = \frac{n}{1}$$

$$X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

The Normalized matrix N = $\begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$; $N^T = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

AN = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{-2}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

$$N^T A N = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{-2}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{-6}{3} & \frac{0}{\sqrt{6}} & \frac{0}{\sqrt{18}} \\ \frac{0}{\sqrt{6}} & \frac{2}{2} & \frac{0}{\sqrt{12}} \\ \frac{0}{\sqrt{18}} & \frac{0}{\sqrt{12}} & \frac{6}{6} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e., $D = N^T A N = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The canonical form is $[y_1 \quad y_2 \quad y_3] \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -2y_1^2 + y_2^2 + y_3^2$

Nature: The eigen values are -2, 1, 1. Therefore, it is indefinite in nature.

Reference Links

1. Introduction to matrices." From *Math Insight*. http://mathinsight.org/matrix_introduction *mathinsight.org/**matrix_introduction***
2. *www.mathresource.iitb.ac.in/linear%20algebra/chapter2.0.html*
3. https://en.wikipedia.org/wiki/**Matrix**_(mathematics)
4. www.slideshare.net/moneebakhtar50/**application-of-matrices**-in-real-life
5. www.youtube.com/watch?v=jzHb1R5wWYU
6. *www.clarkson.edu/~pmarzocc/AE430/Matlab_Eig.pdf*