UNIT 3 SPRINGS

Introduction

A spring is defined as an elastic body, whose function is to distort when loaded and to recover its original shape when the load is removed. The various important applications of springs are as follows:

1. To cushion, absorb or control energy due to either shock or vibration as in car springs, railway buffers, air-craft landing gears, shock absorbers and vibration dampers.
2. To apply forces, as in brakes, clutches and spring-loaded valves.
3. To control motion by maintaining contact between two elements as in cams and followers.
4. To measure forces, as in spring balances and engine indicators.
5. To store energy, as in watches, toys, etc.

Types of Springs

Though there are many types of the springs, yet the following, according to their shape, are important from the subject point of view.

1. **Helical springs.** The helical springs are made up of a wire coiled in the form of a helix and are primarily intended for compressive or tensile loads. The cross-section of the wire from which the spring is made may be circular, square or rectangular. The two forms of helical springs are **compression helical spring** as shown in Fig (a) and **tension helical spring** as shown in Fig (b).

![Helical springs](image)

Helical springs.

(a) Compression helical spring.  
(b) Tension helical spring.

The helical springs are said to be **closely coiled** when the spring wire is coiled so close that the plane containing each turn is nearly at right angles to the axis of the helix and the wire is subjected to torsion. In other words, in a closely coiled helical spring, the helix angle is very small, it is usually less than 10°. The major stresses produced in helical springs are shear stresses due to twisting. The load applied is parallel to or along the axis of the spring.

In **open coiled helical springs**, the spring wire is coiled in such a way that there is a gap between the two consecutive turns, as a result of which the helix angle is large. Since the application of open coiled helical springs are limited, therefore our discussion shall confine to closely coiled helical springs only.

The helical springs have the following advantages:

(a) These are easy to manufacture.
(b) These are available in wide range.
(c) These are reliable.
(d) These have constant spring rate.
(e) Their performance can be predicted more accurately.
(f) Their characteristics can be varied by changing dimensions.

2. **Conical and volute springs.** The conical and volute springs, as shown in Fig. 23.2, are used in special applications where a telescoping spring or a spring with a spring rate that increases with the load is desired. The conical spring, as shown in Fig (a), is wound with a uniform pitch whereas the volute springs, as shown in Fig. (b), are wound in the form of paraboloid with constant pitch.

![Conical and volute springs](image)

(a) Conical spring.  
(b) Volute spring.
and lead angles. The springs may be made either partially or completely telescoping. In either case, the number of active coils gradually decreases. The decreasing number of coils results in an increasing spring rate. This characteristic is sometimes utilised in vibration problems where springs are used to support a body that has a varying mass.

The major stresses produced in conical and volute springs are also shear stresses due to twisting.

3. **Torsion springs.** These springs may be of *helical* or *spiral* type as shown in Fig. The *helical type* may be used only in applications where the load tends to wind up the spring and are used in various electrical mechanisms. The *spiral type* is also used where the load tends to increase the number of coils and when made of flat strip are used in watches and clocks.

The major stresses produced in torsion springs are tensile and compressive due to bending.

![Helical Torsion Spring](image1)

![Spiral Torsion Spring](image2)

4. **Laminated or leaf springs.** The laminated or leaf spring (also known as *flat spring* or *carriage spring*) consists of a number of flat plates (known as leaves) of varying lengths held together by means of clamps and bolts, as shown in Fig. These are mostly used in automobiles.

The major stresses produced in leaf springs are tensile and compressive stresses.

![Laminated or Leaf Springs](image3)

![Disc or Belleville Springs](image4)

5. **Disc or bellevile springs.** These springs consist of a number of conical discs held together against slipping by a central bolt or tube as shown in Fig. These springs are used in applications where high spring rates and compact spring units are required.

The major stresses produced in disc or bellevile springs are tensile and compressive stresses.

6. **Special purpose springs.** These springs are air or liquid springs, rubber springs, ring springs etc. The fluids (air or liquid) can behave as a compression spring. These springs are used for special types of application only.

**Uses of springs:**

(a) To apply forces and to control motions as in brakes and clutches.

(b) To measure forces as in spring balance.

(c) To store energy as in clock springs.
(d) To reduce the effect of shock or impact loading as in carriage springs.

(e) To change the vibrating characteristics of a member as inflexible mounting of motors.

**Derivation of the Formula :**

In order to derive a necessary formula which governs the behaviour of springs, consider a closed coiled spring subjected to an axial load $W$.

![Diagram of a spring](image.png)

Let

$W = \text{axial load}$

$D = \text{mean coil diameter}$

$d = \text{diameter of spring wire}$

$n = \text{number of active coils}$

$C = \text{spring index} = \frac{D}{d} \text{ for circular wires}$

$l = \text{length of spring wire}$

$G = \text{modulus of rigidity}$

$x = \text{deflection of spring}$

$q = \text{Angle of twist}$

When the spring is being subjected to an axial load to the wire of the spring gets be twisted like a shaft.

If $q$ is the total angle of twist along the wire and $x$ is the deflection of spring under the action of load $W$ along the axis of the coil, so that

$x = \frac{D}{2} \cdot q$

Again $l = \pi D n$ [consider, one half turn of a close coiled helical spring]
Assumptions: (1) The Bending & shear effects may be neglected

(2) For the purpose of derivation of formula, the helix angle is considered to be so small that it may be neglected.

Any one coil of a spring will be assumed to lie in a plane which is nearly \( \perp \) to the axis of the spring. This requires that adjoining coils be close together. With this limitation, a section taken perpendicular to the axis the spring rod becomes nearly vertical. Hence to maintain equilibrium of a segment of the spring, only a shearing force \( V = F \) and Torque \( T = F \cdot r \) are required at any \( X \) section. In the analysis of springs it is customary to assume that the shearing stresses caused by the direct shear force is uniformly distributed and is negligible.

So applying the torsion formula.

Using the torsion formula i.e

\[
\frac{T}{J} = \frac{\tau}{r} = \frac{G \cdot \theta}{l}
\]

and substituting

\[
J = \frac{\pi d^4}{32}, \quad T = \frac{w \cdot d}{2}, \quad \theta = \frac{2x}{D}, \quad l = \pi D \cdot x
\]

**SPRING DEFLECTION**

\[
\frac{w \cdot d/2}{\frac{\pi d^4}{32}} = \frac{G \cdot 2x/D}{\pi D \cdot n}
\]

Thus,

\[
\chi = \frac{8wD^3 \cdot n}{G \cdot d^4}
\]

**Spring stiffness:** The stiffness is defined as the load per unit deflection therefore

\[
k = \frac{w}{\chi} = \frac{w}{\frac{8wD^3 \cdot n}{G \cdot d^4}}
\]

Therefore

\[
k = \frac{G \cdot d^4}{8D^3 \cdot n}
\]

**Shear stress**
WAHL'S FACTOR:

In order to take into account the effect of direct shear and change in coil curvature a stress factor is defined, which is known as Wahl's factor

\[ K = \text{Wahl's factor and is defined as} \]

\[ K = \frac{4c - 1}{4c - 4} \cdot \frac{0.615}{c} \]

Where \( C \) = spring index = \( D/d \)

If we take into account the Wahl's factor than the formula for the shear stress becomes

\[ \tau_{\text{max}} = \frac{16.T.k}{\pi d^3} \]

**Strain Energy:** The strain energy is defined as the energy which is stored within a material when the work has been done on the material.

In the case of a spring the strain energy would be due to bending and the strain energy due to bending is given by the expansion

\[ U = \frac{t^2L}{2Et} \]

\[ L = \pi Dn \]

\[ I = \frac{\pi d^4}{64} \]

so after substitution we get

\[ U = \frac{32T^2Dn}{Ed^4} \]

**Example:** A close coiled helical spring is to carry a load of 5000N with a deflection of 50 mm and a maximum shearing stress of 400 N/mm². If the number of active turns or active coils is 8. Estimate the following:

(i) Wire diameter

(ii) Mean coil diameter

(iii) Weight of the spring.

Assume \( G = 83,000 \text{ N/mm}^2 \); \( r = 7700 \text{ kg/m}^3 \)

**Solution:**

(i) For wire diameter if \( W \) is the axial load, then
Further, deflection is given as

$$x = \frac{6wD^3 n}{GJd^4}$$

On substituting the relevant parameters we get

$$50 = \frac{6 \times 5000 \times (0.0314 \times 13.32)^3 \times 8}{83000 \times 13.32^4}$$

$$d = 13.32\text{mm}$$

Therefore,

$$D = 0.0314 \times (13.317)^3\text{mm}$$

$$= 74.15\text{mm}$$

$$D = 74.15\text{mm}$$

**Weight**

Mass or weight = Volume density

= area length of the spring density of spring material

$$= \frac{wD^2 \pi d}{4} \cdot \pi Dn \cdot \rho$$

On substituting the relevant parameters we get

Weight = 1.996 kg

= 2.0 kg

**Close – coiled helical spring subjected to axial torque T or axial couple.**

In this case the material of the spring is subjected to pure bending which tends to reduce Radius R of the coils. In this case the bending moment is constant throughout the spring and is equal to the applied axial Torque T. The stresses i.e. maximum bending stress may thus be determined from the bending theory.
Deflection or wind - up angle:

Under the action of an axial torque the deflection of the spring becomes the “wind – up” angle of the spring which is the angle through which one end turns relative to the other. This will be equal to the total change of slope along the wire, according to area – moment theorem.

\[ \theta = \int_0^L \frac{MdL}{EI} \]

But \( M = T \)

\[ = \int_0^L \frac{T \cdot dL}{EI} = \frac{T}{E} \int_0^L dL \]

Thus, as \( T \) remains constant

\[ \theta = \frac{TL}{EI} \]

Further

\[ L = \pi d^2 n \]

\[ l = \frac{\pi d^4}{64} \]

Therefore, on substitution the value of \( \theta \) obtained is

\[ \theta = \frac{64 T D_n}{Ed^4} \]

Springs in Series: If two springs of different stiffness are joined end on end and carry a common load \( W \), they are said to be connected in series and the combined stiffness and deflection are given by the following equation.

\[ \frac{W}{k} = \frac{W}{k_1} + \frac{W}{k_2} \]

or

\[ \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \]

Springs in parallel: If the two springs are joined in such a way that they have a common deflection ‘x’; then they are said to be connected in parallel. In this case the load carried is shared between the two springs and total load \( W = W_1 + W_2 \)
Terms used in Compression Springs

The following terms used in connection with compression springs are important from the subject point of view.

1. **Solid length.** When the compression spring is compressed until the coils come in contact with each other, then the spring is said to be **solid.** The solid length of a spring is the product of total number of coils and the diameter of the wire. Mathematically,

   \[ L_S = n'd \]

   where
   \[ n' = \text{Total number of coils}, \]
   \[ d = \text{of the wire}. \]

2. **Free length.** The free length of a compression spring, as shown in Fig. is the length of the spring in the free or unloaded condition. It is equal to the solid length plus the maximum deflection or compression of the spring and the clearance between the adjacent coils (when fully compressed). Mathematically,

   \[ L_F = L_S + \delta_{max} + 0.15 \delta_{max} \]

   where
   \[ \delta_{max} = \text{Maximum compression}. \]

   The following relation may also be used to find the free length of the spring, *i.e.*

   \[ L_F = n'd + \delta_{max} + (n' - 1) \times 1 \text{ mm} \]

   In this expression, the clearance between the two adjacent coils is taken as 1 mm.

3. **Spring index.** The spring index is defined as the ratio of the mean diameter of the coil to the diameter of the wire. Mathematically,

   \[ C = \frac{D}{d} \]

   where
   \[ D = \text{Mean diameter of the coil}, \]
   \[ d = \text{Diameter of the wire}. \]

4. **Spring rate.** The spring rate (or stiffness or spring constant) is defined as the load required per unit deflection of the spring. Mathematically,

   \[ k = \frac{W}{\delta} \]

   where
   \[ W = \text{Load}, \]
   \[ \delta = \text{Deflection of the spring}. \]

5. **Pitch.** The pitch of the coil is defined as the axial distance between adjacent coils in uncompressed state. Mathematically,

   \[ p = \frac{\text{Free length}}{n' - 1} \]

   The pitch of the coil may also be obtained by using the following relation, *i.e.*
Pitch of the coil,  
\[ p = \frac{L_F - L_S}{n'} + d \]

where  
\( L_F \) = Free length of the spring,  
\( L_S \) = Solid length of the spring,  
\( n' \) = Total number of coils, and  
\( d \) = Diameter of the wire.

In choosing the pitch of the coils, the following points should be noted:
(a) The pitch of the coils should be such that if the spring is accidently or carelessly compressed, the stress does not increase the yield point stress in torsion.
(b) The spring should not close up before the maximum service load is reached.

Note: In designing a tension spring (See Example 23.8), the minimum gap between two coils when the spring is in the free state is taken as 1 mm. Thus the free length of the spring,  
\[ L_F = n.d + (n - 1) \]

and pitch of the coil,  
\[ p = \frac{L_F}{n - 1} \]

Example 1. Design a helical compression spring for a maximum load of 1000 N for a deflection of 25 mm using the value of spring index as 5.

The maximum permissible shear stress for spring wire is 420 MPa and modulus of rigidity is 84 kN/mm\(^2\).

Take Wahl's factor,  
\[ K = \frac{4C - 1 + 0.615}{4C - 4} \]

Solution. Given:  
\( W = 1000 \) N;  
\( \delta = 25 \) mm;  
\( C = D/d = 5 \);  
\( \tau = 420 \) MPa = 420 N/mm\(^2\);  
\( G = 84 \) kN/mm\(^2\) = 84 \( \times 10^3 \) N/mm\(^2\).

1. Mean diameter of the spring coil
   Let  
\[ D = \text{Mean diameter of the spring coil, and} \]
\[ d = \text{Diameter of the spring wire.} \]

We know that Wahl's stress factor,  
\[ K = \frac{4C - 1 + 0.615}{4C - 4} \]

and maximum shear stress (\( \tau \)),  
\[ 420 = K \cdot \frac{8W.C}{\pi d^3} \cdot 1.31 \cdot \frac{8 \cdot 1000 \cdot 5}{\pi d^3} \cdot \frac{16677}{d^3} \]

\[ d^3 = 16677 / 420 = 39.7 \text{ mm} \]

or  
\[ d = 6.3 \] mm.

We shall take a standard wire of size SWG 3 having diameter (\( d \)) = 6.401 mm.

\[ D = C.d = 5 \times 6.401 = 32.005 \text{ mm Ans.} \]

and outer diameter of the spring coil,  
\[ D_o = D + d = 32.005 + 6.401 = 38.406 \text{ mm Ans.} \]

2. Number of turns of the coils
   Let  
\[ n = \text{Number of active turns of the coils.} \]

we know that compression of the spring (\( \delta \)),  
\[ \delta = \frac{8W.C^3}{G.d} = \frac{8 \cdot 1000 \cdot (5)^3 n}{84 \cdot 10^3 \cdot 6.401} = 1.86 n \]

\[ n = 25 / 1.86 = 13.44 \text{ say 14 Ans.} \]

For squared and ground ends, the total number of turns,  
\[ n' = n + 2 = 14 + 2 = 16 \text{ Ans.} \]

3. Free length of the spring
   We know that free length of the spring
4. Pitch of the coil

We know that pitch of the coil

\[
\text{Free length} = n' - 1 = 16 - 1 = 8.75 \text{ mm Ans.}
\]

Example 2. Design a close coiled helical compression spring for a service load ranging from 2250 N to 2750 N. The axial deflection of the spring for the load range is 6 mm. Assume a spring index of 5. The permissible shear stress intensity is 420 MPa and modulus of rigidity, \( G = 84 \text{ kN/mm}^2 \).

Neglect the effect of stress concentration. Draw a fully dimensioned sketch of the spring, showing details of the finish of the end coils.

Solution. Given:

\[
W_1 = 2250 \text{ N}; \quad W_2 = 2750 \text{ N; } \delta = 6 \text{ mm}; \quad C = D/d = 5; \quad \tau = 420 \text{ MPa} = 420 \text{ N/mm}^2; \quad G = 84 \text{ kN/mm}^2 = 84 \times 10^3 \text{ N/mm}^2
\]

1. Mean diameter of the spring coil

Let \( D \) = Mean diameter of the spring coil for a maximum load of \( W_2 = 2750 \text{ N} \), and 
\( d \) = Diameter of the spring wire.

We know that twisting moment on the spring,

\[
T = W_2 \cdot \frac{D}{2} = 2750 \cdot \frac{5d}{2} = 6875 \text{ d}
\]

We also know that twisting moment (\( T \)),

\[
6875 \text{ d} = \frac{\pi \cdot \tau \cdot d^3}{16} = \frac{\pi}{16} \cdot 420 \cdot d^3 = 82.48 d^3
\]

\[
\therefore \quad d^3 = 6875 / 82.48 = 83.35 \text{ or } d = 9.13 \text{ mm}
\]

From Table 23.2, we shall take a standard wire of size SWG 3/0 having diameter (\( d \)) = 9.49 mm. ∴ Mean diameter of the spring coil,

\[
D = 5d = 5 \times 9.49 = 47.45 \text{ mm Ans.}
\]

We know that outer diameter of the spring coil,

\[
D_o = D + d = 47.45 + 9.49 = 56.94 \text{ mm Ans.}
\]

and inner diameter of the spring coil,

\[
D_i = D - d = 47.45 - 9.49 = 37.96 \text{ mm Ans.}
\]

2. Number of turns of the spring coil

Let \( n \) = Number of active turns.

It is given that the axial deflection (\( \delta \)) for the load range from 2250 N to 2750 N (i.e. for \( W = 500 \text{ N} \)) is 6 mm.

We know that the deflection of the spring (\( \delta \)),

\[
6 = \frac{8 \cdot W \cdot C \cdot n^3}{G \cdot d} = \frac{8 \cdot 500 (5)^3 \cdot n}{84 \cdot 10^3 \cdot 9.49} = 0.63 n
\]

\[
\therefore \quad n = 6 / 0.63 = 9.5 \text{ say } 10 \text{ Ans.}
\]
For squared and ground ends, the total number of turns, \( n' = 10 + 2 = 12 \) \( \text{Ans.} \).

3. **Free length of the spring**
   
   Since the compression produced under 500 N is 6 mm, therefore maximum compression produced under the maximum load of 2750 N is
   
   \[ \delta_{\text{max}} = \frac{6}{500} \times 2750 = 33 \text{ mm} \]
   
   We know that free length of the spring,
   
   \[ L_F = n'd + \delta_{\text{max}} + 0.15 \delta_{\text{max}} \]
   
   \[ = 12 \times 9.49 + 33 + 0.15 \times 33 \]
   
   \[ = 151.83 \text{ say 152 mm} \text{ Ans.} \]

4. **Pitch of the coil**
   
   We know that pitch of the coil
   
   \[
   \text{Free length} = \frac{152}{n' - 1} = \frac{152}{12 - 1} = 13.73 \text{ say 13.8 mm} \text{ Ans.}
   \]
**Problem 16.39.** A closely coiled helical spring of mean diameter 20 cm is made of 3 cm diameter rod and has 16 turns. A weight of 3 kN is dropped on this spring. Find the height by which the weight should be dropped before striking the spring so that the spring may be compressed by 18 cm. Take \( C = 8 \times 10^4 \text{ N/mm}^2 \).

**Sol.** Given:
- Mean dia. of coil, \( D = 20 \text{ cm} = 200 \text{ mm} \)
- Mean radius of coil, \( R = \frac{200}{2} = 100 \text{ mm} \)
- Dia. of spring rod, \( d = 3 \text{ cm} = 30 \text{ mm} \)
- Number of turns, \( n = 16 \)
- Weight dropped, \( W = 3 \text{ kN} = 3000 \text{ N} \)
- Compression of the spring, \( \delta = 18 \text{ cm} = 180 \text{ mm} \)
- Modulus of rigidity, \( C = 8 \times 10^4 \text{ N/mm}^2 \)

Let \( h = \) Height through which the weight \( W \) is dropped

\( W = \) Gradually applied load which produces the compression of spring equal to 180 mm.

Now using equation (16.26),

\[
\delta = \frac{64W.R^3.n}{Cd^4}
\]

or

\[
180 = \frac{64 \times 3000 \times 100^3 \times 16}{8 \times 10^4 \times 30^4}
\]

or

\[
W = \frac{180 \times 8 \times 10^4 \times 30^4}{64 \times 100^3 \times 16} = 11390 \text{ N}
\]

Work done by the falling weight on spring

\[
= \text{Weight falling} \ (h + \delta) = 3000 \ (h + 180) \text{ N-mm}
\]

Energy stored in the spring = \( \frac{1}{2} W \times \delta \)

\[
\frac{1}{2} \times 11390 \times 180 = 1025100 \text{ N-mm}.
\]

Equating the work done by the falling weight on the spring to the energy stored in the spring, we get

\[
3000(h + 180) = 1025100
\]

or

\[
h + 180 = \frac{1025100}{3000} = 341.7 \text{ mm}
\]

\[
\therefore \ h = 341.7 - 180 = 161.7 \text{ mm. \ Ans.}
\]
Problem 16.43. A closely coiled helical spring made of 10 mm diameter steel wire has 15 coils of 100 mm mean diameter. The spring is subjected to an axial load of 100 N. Calculate:

(i) The maximum shear stress induced,
(ii) The deflection, and
(iii) Stiffness of the spring.

Take modulus of rigidity, \( C = 8.16 \times 10^4 \text{ N/mm}^2 \).

(AMIE, Winter 1990; Converted to S.I. units)

Sol. Given:
Dia. of wire, \( d = 10 \text{ mm} \)
Number of coils, \( n = 15 \)
Mean dia. of coil, \( D = 100 \text{ mm} \)

\[ \therefore \text{Mean radius of coil, } R = \frac{100}{2} = 50 \text{ mm} \]

Axial load, \( W = 100 \text{ N} \)
Modulus of rigidity, \( C = 8.16 \times 10^4 \text{ N/mm}^2 \).

(i) Maximum shear stress induced

Using equation (16.24), \( \tau = \frac{16WR}{\pi d^3} = \frac{16 \times 100 \times 50}{\pi \times 10^3} = 24.46 \text{ N/mm}^2 \). Ans.

(ii) The deflection \( \delta \)

Using equation (16.26),

\[ \delta = \frac{64W \times R^3 \times n}{C \times d^4} = \frac{64 \times 100 \times 50^3 \times 15}{8.16 \times 10^4 \times 10^4} \]

\[ = 14.7 \text{ mm. Ans.} \]

(iii) Stiffness of the spring

Stiffness = \( \frac{\text{Load on spring}}{\text{Deflection of spring}} \)

\[ = \frac{100}{14.7} = 6.802 \text{ N/mm. Ans.} \]
UNIT 4 - TORSION OF SHAFTS

Torsion occurs when any shaft is subjected to a torque. This is true whether the shaft is rotating (such as drive shafts on engines, motors and turbines) or stationary (such as with a bolt or screw). The torque makes the shaft twist and one end rotates relative to the other inducing shear stress on any cross section. Failure might occur due to shear alone or because the shear is accompanied by stretching or bending.

1.1. TORSION EQUATION

The diagram shows a shaft fixed at one end and twisted at the other end due to the action of a torque T.

![Figure 1](image)

The radius of the shaft is $R$ and the length is $L$.

Imagine a horizontal radial line drawn on the end face. When the end is twisted, the line rotates through an angle $\theta$. The length of the arc produced is $R\theta$.

Now consider a line drawn along the length of the shaft. When twisted, the line moves through an angle $\gamma$. The length of the arc produced is $L\gamma$.

If we assume that the two arcs are the same it follows that $R\theta = L\gamma$

Hence by equating $L\gamma = R\theta$ we get

$$\gamma = \frac{R\theta}{L}$$

.........................(1A)

If you refer to basic stress and strain theory, you will appreciate that $\gamma$ is the shear
strain on the outer surface of the shaft. The relationship between shear strain and shear stress is

\[ G = \frac{T}{\gamma} \] ..........................(1B)

\( \tau \) is the shear stress and \( G \) the modulus of rigidity.

\( G \) is one of the elastic constants of a material. The equation is only true so long as the material remains elastic.

\[ G\theta/L = T/R \] ..........................(1C)

Since the derivation could be applied to any radius, it follows that shear stress is directly proportional to radius 'r' and is a maximum on the surface. Equation (1C) could be written as

\[ \frac{G\theta}{L} = \frac{\tau}{r} \] ..........................(1D)

Now let's consider how the applied torque 'T' is balanced by the internal stresses of the material.

Consider an elementary ring of material with a shear stress \( \tau \) acting on it at radius \( r \).

The area of the ring is \( dA = 2\pi r \, dr \)

The shear force acting on it tangential is \( dF = \tau \, dA = \tau \, 2\pi r \, dr \)

This force acts at radius \( r \) so the torque produced is \( dT = \tau \, 2\pi r^2 \, dr \)

Since \( \tau = \frac{G\theta}{L} \) from equation (1D) then \( dT = \frac{G\theta}{L} \, 2\pi r^3 \, dr \)

The torque on the whole cross section resulting from the shear stress is \( T = \frac{G\theta}{L} \int_0^R r^3 \, dr \)

The expression \( 2\pi \int_0^R r^3 \, dr \) is called the polar second moment of area and denoted as 'J'. The Torque equation reduces to \( T = \frac{G\theta}{L} J \) and this is usually written as \( T = \frac{G\theta}{L} J \) ..........................(1E)

Combining (1D) and (1E) we get the torsion equation \( \frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{r} \) ..........................(1F)

1.2 POLAR SECOND MOMENTS OF AREA

This tutorial only covers circular sections. The formula for J is found by carrying out the integration or may be found in standard tables.
For a shaft of diameter \(D\) the formula is
\[
J = \frac{\pi D^4}{32}
\]
This is not to be confused with the second moment of area about a diameter, used in bending of beams \((I)\) but it should be noted that \(J = 2I\).

### 1.3 HOLLOW SHAFTS

Since the shear stress is small near the middle, then if there is no other stress considerations other than torsion, a hollow shaft may be used to reduce the weight.

The formula for the polar second moment of area is
\[
J = \frac{\pi}{32} \left( D_o^4 - D_i^4 \right)
\]

\(D_o\) is the outside diameter and \(D_i\) the inside diameter.

### 1.4 MECHANICAL POWER TRANSMISSION BY A SHAFT

In this section you will derive the formula for the power transmitted by a shaft and combine it with torsion theory.

Mechanical power is defined as work done per second. Work done is defined as force times distance moved. Hence

\[
P = Fx/t
\]

where

- \(P\) is the Power
- \(F\) is the force
- \(x\) is distance moved.
- \(t\) is the time taken.

Since distance moved/time taken is the velocity of the force we may write

\[
P = F \, v \quad \text{(2A)}
\]

where \(v\) is the velocity.

When a force rotates at radius \(R\) it travels distance moved in one revolution is one circumference in the time of one revolution. Hence the \(x = 2\pi R\)

If the speed is \(N\) rev/second then the time of one revolution is \(1/N\) seconds. The mechanical power is hence

\[
P = F \times \frac{2\pi R}{1/N} = 2\pi NFS\]

Since \(FR\) is the torque produced by the force this reduces to

\[
P = 2\pi NT \quad \text{(2B)}
\]

Since \(2\pi N\) is the angular velocity \(\omega\) radians/s it further reduces to

\[
P = \omega T \quad \text{(2C)}
\]

Note that equations (2C) is the angular equivalent of equation (2A) and all three equations should be remembered.
WORKED EXAMPLE No.1

A shaft 50 mm diameter and 0.7 m long is subjected to a torque of 1200 Nm. Calculate the shear stress and the angle of twist. Take $G = 90$ GPa.

**SOLUTION**

Important values to use are $D = 0.05$ m, $L = 0.7$ m, $T = 1200$ Nm, $G = 90 \times 10^9$ Pa

\[
J = \frac{\pi D^4}{32} = \frac{\pi \times 0.05^4}{32} = 613.59 \times 10^{-9} \text{ m}^4
\]

\[
\tau_{\text{max}} = \frac{TR}{J} = \frac{1200 \times 0.025}{613.59 \times 10^{-9}} = 48.89 \times 10^6 \text{ Pa or } 48.89 \text{ MPa}
\]

\[
\theta = \frac{TL}{J} = \frac{1200 \times 0.7}{90 \times 10^9 \times 613.59 \times 10^{-9}} = 0.0152 \text{ radian}
\]

Alternately \[
\theta = \frac{\tau L}{G R} = \frac{48.89 \times 10^6 \times 0.7}{90 \times 10^9 \times 0.025} = 0.0152 \text{ radian}
\]

Converting to degrees $\theta = 0.0152 \times \frac{180}{\pi} = 0.871^\circ$

WORKED EXAMPLE No.2

Repeat the previous problem but this time the shaft is hollow with an internal diameter of 30 mm.

\[
J = \frac{\pi (D^4 - d^4)}{32} = \frac{\pi \times (0.05^4 - 0.03^4)}{32} = 534.07 \times 10^{-9} \text{ m}^4
\]

\[
\tau_{\text{max}} = \frac{TR}{J} = \frac{1200 \times 0.025}{534.07 \times 10^{-9}} = 56.17 \times 10^6 \text{ Pa or } 56.17 \text{ MPa}
\]

\[
\theta = \frac{TL}{J} = \frac{1200 \times 0.7}{90 \times 10^9 \times 534.07 \times 10^{-9}} = 0.0175 \text{ radian}
\]

Alternately \[
\theta = \frac{\tau L}{G R} = \frac{56.17 \times 10^6 \times 0.7}{90 \times 10^9 \times 0.025} = 0.0175 \text{ radian}
\]

Converting to degrees $\theta = 0.0152 \times \frac{180}{\pi} = 1^\circ$

Note that the answers are nearly the same even though there is much less material in the shaft.
WORKED EXAMPLE No.3

A shaft 40 mm diameter is made from steel and the maximum allowable shear stress for the material is 50 MPa. Calculate the maximum torque that can be safely transmitted. Take G = 90 GPa.

SOLUTION

Important values to use are:

\[ D = 0.04 \text{ m}, \quad R = 0.02 \text{ m}, \quad \tau = 50 \times 10^6 \text{ Pa} \quad \text{and} \quad G = 90 \times 10^9 \text{ Pa} \]

\[ T = \frac{G \theta}{J \cdot L} = \frac{\tau}{r} \]

\[ J = \frac{\pi D^4}{32} = \frac{\pi \times 0.04^4}{32} = 251.32 \times 10^{-9} \text{ m}^4 \]

The complete torsion equation is \( T = \frac{G \theta}{J \cdot L} = \frac{\tau}{R} \). Rearrange and ignore the middle term.

\[ T = \frac{\tau_{\text{max}} J}{R} = \frac{50 \times 10^6 \times 251.32 \times 10^{-9}}{0.02} = 628.3 \text{ Nm} \]

WORKED EXAMPLE No.4

A shaft is made from tube. The ratio of the inside diameter to the outside diameter is 0.6. The material must not experience a shear stress greater than 500 kPa. The shaft must transmit 1.5 MW of mechanical power at 1500 rev/min. Calculate the shaft diameters.

SOLUTION

The important quantities are \( P = 1.5 \times 10^6 \) Watts, \( \tau = 500 \times 10^3 \) Pa, \( N = 1500 \text{ rev/min} \) and \( d = 0.6D \).

\( N = 1500 \text{ rev/min} = 1500/60 = 25 \text{ rev/s} \quad \Rightarrow \quad P = 2 \pi N T \)

\[ \text{hence} \quad T = \frac{P}{2 \pi N} = \frac{1.5 \times 10^6}{2 \pi \times 25} = 9549.3 \text{ Nm} \]

\[ J = \frac{\pi \left( D^4 - d^4 \right)}{32} = \frac{\pi \left( D^4 - (0.6D)^4 \right)}{32} = \frac{\pi \left( D^4 - 0.36D^4 \right)}{32} = 0.08545D^4 \]

\[ T = \frac{\tau}{J \cdot R} = \frac{2\tau}{D} \quad \text{hence} \quad \frac{9549.3}{0.08545D^4} = \frac{2 \times 500 \times 10^3}{D} \quad \Rightarrow \quad \frac{9549.3}{0.08545 \times 2 \times 500 \times 10^3} = \frac{D^4}{D} = D^3 \]

\[ D^3 = 0.11175 \quad D = \sqrt[3]{0.11175} = 0.4816 \text{ m} = 481.6 \text{ mm} \quad \Rightarrow \quad d = 0.6D = 289 \text{ mm} \]
Columns and Struts

Introduction

A machine part subjected to an axial compressive force is called a strut. A strut may be horizontal, inclined or even vertical. But a vertical strut is known as a column, pillar or stanchion. The machine members that must be investigated for column action are piston rods, valve push rods, connecting rods, screw jack, side links of toggle jack etc. In this chapter, we shall discuss the design of piston rods, valve push rods and connecting rods.

Failure of a Column or Strut

It has been observed that when a column or a strut is subjected to a compressive load and the load is gradually increased, a stage will reach when the column will be subjected to ultimate load. Beyond this, the column will fail by crushing and the load will be known as crushing load.

Types of End Conditions of Columns

In actual practice, there are a number of end conditions for columns. But we shall study the Euler’s column theory on the following four types of end conditions which are important from the subject point of view:

1.3 Both the ends hinged or pin jointed as shown in Fig (a),
1.4 Both the ends fixed as shown in Fig (b),
1.5 One end is fixed and the other hinged as shown in Fig. (c), and
1.6 One end is fixed and the other free as shown in Fig. (d).

Euler’s Column Theory

The first rational attempt, to study the stability of long columns, was made by Mr. Euler. He derived an equation, for the buckling load of long columns based on the bending stress. While deriving this equation, the effect of direct stress is neglected. This may be justified with the statement, that the direct stress induced in a long column is negligible as compared to the bending stress. It may be noted that Euler’s formula cannot be used in the case of short columns, because the direct stress is considerable, and hence cannot be neglected.

16.5 Assumptions in Euler's Column Theory

The following simplifying assumptions are made in Euler’s column theory:

1. Initially the column is perfectly straight, and the load applied is truly axial.
2. The cross-section of the column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic, and thus obeys Hooke’s law.
4. The length of column is very large as compared to its cross-sectional dimensions.
5. The shortening of column, due to direct compression (being very small) is neglected.
6. The failure of column occurs due to buckling alone.
7. The weight of the column itself is neglected.

16.6 Euler’s Formula

According to Euler’s theory, the crippling or buckling load \(W_{cr}\) under various end conditions is represented by a general equation,

\[
W_{cr} = \frac{C \pi \frac{E I}{(l/k)^2}}{l^2} = C \frac{\pi \frac{E A k^2}{l^2}}{l^2} \quad \ldots \quad \text{(Q} = A k^2\text{)}
\]

where

- \(E\) = Modulus of elasticity or Young’s modulus for the material of the column,
- \(A\) = Area of cross-section,
- \(k\) = Least radius of gyration of the cross-section,
- \(l\) = Length of the column, and
- \(C\) = Constant, representing the end conditions of the column or end fixity coefficient.

The following table shows the values of end fixity coefficient \((C)\) for various end conditions.

<table>
<thead>
<tr>
<th>S. No.</th>
<th>End conditions</th>
<th>End fixity coefficient ((C))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Both ends hinged</td>
<td>1</td>
</tr>
<tr>
<td>2.</td>
<td>Both ends fixed</td>
<td>4</td>
</tr>
<tr>
<td>3.</td>
<td>One end fixed and other hinged</td>
<td>2</td>
</tr>
<tr>
<td>4.</td>
<td>One end fixed and other end free</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Slenderness Ratio

In Euler’s formula, the ratio \(l/k\) is known as slenderness ratio. It may be defined as the ratio of the effective length of the column to the least radius of gyration of the section.

Limitations of Euler’s Formula

A little consideration will show that the crippling stress will be high, when the slenderness ratio is small. We know that the crippling stress for a column cannot be more than the crushing stress of the column material. It is thus obvious that the Euler’s formula will give the value of crippling stress of the column (equal to the crushing stress of the column material) corresponding to the slenderness ratio. Now consider a mild steel column. We know that the crushing stress for mild steel is 330 N/mm\(^2\) and Young’s modulus for mild steel is \(0.21 \times 10^6\) N/mm\(^2\).

Now equating the crippling stress to the crushing stress, we have

\[
\frac{C \pi \frac{E}{(l/k)^2}}{l^2} = 330
\]

\[
\frac{1 \cdot 9.87 \cdot 0.21 \cdot 10^6}{(l/k)^2} = 330
\]

\[\ldots\text{(Taking } C = 1\text{)}\]
\[ \frac{l}{k} = \sqrt{\frac{\pi^2 EL}{W_{cr}}} \]

Hence if the slenderness ratio is less than 80, Euler’s formula for a mild steel column is not valid.

Sometimes, the columns whose slenderness ratio is more than 80, are known as long columns, and those whose slenderness ratio is less than 80 are known as short columns. It is thus obvious that the Euler’s formula holds good only for long columns.

### 16.9 Equivalent Length of a Column

Sometimes, the crippling load according to Euler’s formula may be written as

\[ W_{cr} = \frac{\pi^2 EL}{L^2} \]

where \( L \) is the equivalent length or effective length of the column. The equivalent length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends to that of the given column. The relation between the equivalent length and actual length for the given end conditions is shown in the following table.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>End Conditions</th>
<th>Relation between equivalent length (L) and actual length (l)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Both ends hinged</td>
<td>( L = l )</td>
</tr>
<tr>
<td>2.</td>
<td>Both ends fixed</td>
<td>( L = \frac{l}{2} )</td>
</tr>
<tr>
<td>3.</td>
<td>One end fixed and other end hinged</td>
<td>( L = \frac{l}{2} )</td>
</tr>
<tr>
<td>4.</td>
<td>One end fixed and other end free</td>
<td>( L = 2l )</td>
</tr>
</tbody>
</table>

### Example 16.1

A T-section 150 mm × 120 mm × 20 mm is used as a strut of 4 m long hinged at both ends. Calculate the crippling load, if Young’s modulus for the material of the section is 200 kN/mm².

**Solution.**

Given: \( l = 4 \) m = 4000 mm; \( E = 200 \) kN/mm² = \( 200 \times 10^3 \) N/mm²

First of all, let us find the centre of gravity (G) of the T-section as shown in Fig. 16.2.

Let \( y \) be the distance between the centre of gravity (G) and top of the flange.

We know that the area of flange,
\[ a_1 = 150 \times 20 = 3000 \text{ mm}^2 \]

Its distance of centre of gravity from top of the flange,
\[ y_1 = \frac{20}{2} = 10 \text{ mm} \]

Area of web, \( a_2 = (120 - 20) \times 20 = 2000 \text{ mm}^2 \)

Its distance of centre of gravity from top of the flange,
\[ y_2 = \frac{20 + 100}{2} = 70 \text{ mm} \]

\[ y = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{3000 \times 10 + 2000 \times 70}{3000 + 2000} = 34 \text{ mm} \]

![Fig. 16.2](image)
We know that the moment of inertia of the section about X-X,

\[ I_{XX} = \left[ \frac{150}{12} (20)^3 + 3000 (34 - 10)^2 + \frac{20}{12} (100)^3 + 2000 (70 - 34)^2 \right] \]

\[ = 6.1 \times 10^6 \text{ mm}^4 \]

and

\[ I_{YY} = \frac{20}{12} (150)^3 + \frac{100}{12} (20)^3 = 5.7 \times 10^6 \text{ mm}^4 \]

Since \( I_{YY} \) is less than \( I_{XX} \), therefore the column will tend to buckle in Y-Y direction. Thus we shall take the value of \( I \) as \( I_{YY} = 5.7 \times 10^6 \text{ mm}^4 \).

Moreover as the column is hinged at its both ends, therefore equivalent length,

\[ L = l = 4000 \text{ mm} \]

We know that the crippling load,

\[ W_{cr} = \frac{\pi^2 EI}{L^2} = \frac{9.87 \times 200 \times 10^3 \times 5.7 \times 10^6}{(4000)^2} = 703 \times 10^3 \text{ N} = 703 \text{ kN} \text{ Ans.} \]
RANKINE'S FORMULA:

We have already discussed that Euler's formula gives correct results only for very long columns. Though this formula is applicable for columns, ranging from very long to short ones, yet it does not give reliable results. Prof. Rankine, after a number of experiments, gave the following empirical formula for columns.

\[
\frac{1}{W_{cr}} = \frac{1}{W_C} + \frac{1}{W_E} \tag{i}
\]

where

\[W_{cr} = \text{Crippling load by Rankine's formula},\]
\[W_C = \text{Ultimate crushing load for the column} = \sigma_c \times A,\]
\[W_E = \text{Crippling load, obtained by Euler's formula} = \frac{\pi^2 E I}{L^2}.\]

A little consideration will show that the value of \(W_C\) will remain constant irrespective of the fact whether the column is a long one or short one. Moreover, in the case of short columns, the value of \(W_E\) will be very high, therefore the value of \(1/W_E\) will be quite negligible as compared to \(1/W_C\). It is thus obvious, that the Rankine's formula will give the value of its crippling load (i.e. \(W_{cr}\)) approximately equal to the ultimate crushing load (i.e. \(W_C\)). In case of long columns, the value of \(W_E\) will be very small, therefore the value of \(1/W_E\) will be quite considerable as compared to \(1/W_C\). It is thus obvious, that the Rankine's formula will give the value of its crippling load (i.e. \(W_{cr}\)) approximately equal to the crippling load by Euler's formula (i.e. \(W_E\)). Thus, we see that Rankine's formula gives a fairly correct result for all cases of columns, ranging from short to long columns.

From equation \((i)\), we know that

\[
\frac{1}{W_{cr}} = \frac{1}{W_C} + \frac{1}{W_E} = \frac{W_C \times W_E}{W_C \times W_E}
\]

\[
\therefore \quad W_{cr} = \frac{W_C \times W_E}{1 + \frac{W_C}{W_E}}
\]

Now substituting the value of \(W_C\) and \(W_E\) in the above equation, we have

\[
W_{cr} = \frac{\sigma_c \times A}{1 + \frac{\sigma_c \times A \times L^2}{\pi^2 E I}} = \frac{\sigma_c \times A}{1 + \frac{\sigma_c \times A L^2}{\pi^2 E A k^2}}
\]

\[
= \frac{\sigma_c \times A}{1 + a \left(\frac{L}{k}\right)^2} = \text{Crushing load}
\]

where

\[\sigma_c = \text{Crushing stress or yield stress in compression},\]
\[A = \text{Cross-sectional area of the column},\]
\[a = \text{Rankine's constant} = \frac{\sigma_c}{\pi^2 E}.\]
\[ L = \text{Equivalent length of the column, and} \]
\[ k = \text{Least radius of gyration.} \]

The following table gives the values of crushing stress and Rankine’s constant for various materials.

**Table 16.3. Values of crushing stress (}\sigma_c\text{) and Rankine’s constant (}\alpha\text{) for various materials.**

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Material</th>
<th>(\sigma_c) in MPa</th>
<th>(\alpha = \frac{\sigma_c}{\pi^2 E})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Wrought iron</td>
<td>250</td>
<td>(\frac{1}{9000})</td>
</tr>
<tr>
<td>2.</td>
<td>Cast iron</td>
<td>550</td>
<td>(\frac{1}{1600})</td>
</tr>
<tr>
<td>3.</td>
<td>Mild steel</td>
<td>320</td>
<td>(\frac{1}{750})</td>
</tr>
<tr>
<td>4.</td>
<td>Timber</td>
<td>50</td>
<td>(\frac{1}{750})</td>
</tr>
</tbody>
</table>

**Problem 19.4 (a).** A simply supported beam of length 4 metre is subjected to a uniformly distributed load of 30 kN/m over the whole span and deflects 15 mm at the centre. Determine the crippling loads when this beam is used as a column with the following conditions:

(i) one end fixed and other end hinged

(ii) both the ends pin jointed.  

(Annamalai University, 1990)

**Sol.** Given:

Length, \(L = 4\) m = 4000 mm

Uniformly distributed load, \(w = 30\) kN/m = 30,000 N/m

\[
= \frac{30,000}{1000} \text{ N/mm} = 30 \text{ N/mm}
\]

Deflection at the centre, \(\delta = 15\) mm.

For a simply supported beam, carrying U.D.L. over the whole span, the deflection at the centre is given by,

\[
\delta = \frac{5}{384} \times \frac{w \times L^4}{EI}
\]

or

\[
15 = \frac{5}{384} \times \frac{30 \times 4000^4}{EI}
\]

\[
\therefore \quad EI = \frac{5}{384} \times \frac{30 \times 4000^4}{15} = \frac{5}{384} \times \frac{3 \times 256 \times 10^{13}}{3} \times 10^{13} = \frac{2}{3} \times 10^{13} \text{ N mm}^2.
\]

(i) Crippling load when the beam is used as a column with one end fixed and other end hinged.

The crippling load \(P\) for this case in terms of actual length is given by equation (19.4) as

\[
P = \frac{2\pi^2 \times EI}{L^2}, \text{ where } L = \text{actual length} = 4000 \text{ mm}
\]

\[
= \frac{2 \times \pi^2 \times \frac{2}{3} \times 10^{13}}{4000^2} = 8224.5 \text{ kN}. \quad \text{Ans.}
\]

(ii) Crippling load when both the ends are pin-jointed

This is given by equation (19.1) in terms of actual length as

\[
P = \frac{3\pi^2 \times EI}{l^3}, \text{ where } l = \text{actual length} = 4000 \text{ mm}
\]

\[
= \frac{\pi^2 \times \frac{2}{3} \times 10^{13}}{4000^3} = 4112.25 \text{ kN}. \quad \text{Ans.}
\]
**Problem 19.5.** A solid round bar 4 m long and 5 cm in diameter was found to extend 4.6 mm under a tensile load of 50 kN. This bar is used as a strut with both ends hinged. Determine the buckling load for the bar and also the safe load taking factor of safety as 4.0.

**Sol.** Given:
- Actual length of bar, $L = 4 \text{ m} = 4000 \text{ mm}$
- Dia. of bar, $d = 5 \text{ cm}$

\[ A = \frac{\pi}{4} \times 5^2 = 6.25\pi \text{ cm}^2 = 6.25\pi \times 10^2 \text{ mm}^2 = 625\pi \text{ mm}^2 \]

- Extension of bar, $\delta L = 4.6 \text{ mm}$
- Tensile load, $W = 50 \text{ kN} = 50000 \text{ N}$.

In this problem, the values of Young's modulus ($E$) is not given. But it can be calculated from the given data.

\[ E = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{\left( \frac{\text{Tensile load}}{\text{Area}} \right)}{\left( \frac{\text{Extension of bar}}{\text{Length of bar}} \right)} \]

\[ = \frac{W}{A} \times \frac{L}{\delta L} = \frac{50000}{625\pi} \times \frac{4000}{4.6} = 2.214 \times 10^4 \text{ N/mm}^2. \]

Since the strut is hinged at its both ends,

\[ L_e = \frac{L}{2} = \frac{4000}{2} = 2000 \text{ mm} \]

Let $P = $Crippling or buckling load.

Using equation (19.5), we get

\[ P = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 \times 2.214 \times 10^4 \times \pi \times 5^4 \times 10^4}{4000 \times 4000} = 4189.99 \text{ say } 4190 \text{ N. Ans.} \]

And safe load

\[ \frac{\text{Crippling load}}{\text{Factor of safety}} = \frac{4190}{4} = 1047.5 \text{ N. Ans.} \]

**Problem 19.13.** The external and internal diameter of a hollow cast iron column are 5 cm and 4 cm respectively. If the length of this column is 3 m and both of its ends are fixed, determine the crippling load using Rankine's formula. Take the values of $\sigma_c = 550 \text{ N/mm}^2$ and $\alpha = \frac{1}{1600}$ in Rankine's formula.

**Sol.** Given:
- External dia., $D = 5 \text{ cm}$
- Internal dia., $d = 4 \text{ cm}$

\[ A = \frac{\pi}{4} (5^2 - 4^2) = 2.25\pi \text{ cm}^2 = 2.25\pi \times 10^2 \text{ mm}^2 = 225\pi \text{ mm}^2 \]

- Moment of Inertia, $I = \frac{\pi}{64} [5^4 - 4^4] = 5.7656 \pi \text{ cm}^4 = 5.7656\pi \times 10^4 \text{ mm}^4$
Least radius of gyration,
\[ k = \sqrt{\frac{I}{A}} = \sqrt{\frac{57656\pi}{225\pi}} = 25.625 \text{ mm} \]

Length of column, \( l = 3 \text{ m} = 3000 \text{ mm} \)
As both the ends are fixed,

\[ \text{Effective length, } L_e = \frac{l}{2} = \frac{3000}{2} = 1500 \text{ mm} \]

Crushing stress, \( \sigma_c = 550 \text{ N/mm}^2 \)

Rankine's constant, \( a = \frac{1}{1600} \)

Let \( P = \text{Crippling load} \) by Rankine's formula

Using equation (19.9), we have

\[ P = \frac{\sigma_c \cdot A}{1 + \left( \frac{L_e}{k} \right)^2} = \frac{550 \times 225\pi}{1 + \left( \frac{1500}{25.625} \right)^2} \]

\[ = \frac{550 \times 225\pi}{3.1415} = 123750 \text{ N. } \text{ Ans.} \]

**Problem 19.9.** Determine the crippling load for a T-section of dimensions 10 cm \( \times \) 10 cm \( \times \) 2 cm and of length 5 m when it is used as strut with both of its ends hinged. Take Young's modulus, \( E = 2.0 \times 10^8 \text{ N/mm}^2 \).

**Sol. Given:**

Dimensions of T-section = 10 cm \( \times \) 10 cm \( \times \) 2 cm

Length actual, \( l = 5 \text{ m} = 5000 \text{ mm} \)

Young's modulus, \( E = 2.0 \times 10^8 \text{ N/mm}^2 \).

First of all, calculate the C.G. of the section. The given section is symmetrical about the axis Y-Y, hence the C.G. of the section will lie on Y-Y axis.

Let \( \bar{y} \) = Distance of C.G. of the section from bottom end.

For the flange, we have \( a_1 = 10 \times 2 = 20 \text{ cm}^2 \)

\( y_1 = \text{Distance of C.G. of area } a_1 \text{ from the bottom end} \)

\[ = 8 + 1 = 9 \text{ cm} \]

For the web, we have \( a_2 = 8 \times 2 = 16 \text{ cm}^2 \)

\( y_2 = \text{Distance of C.G. of area } a_2 \text{ from bottom end} \)

\[ = \frac{8}{2} = 4 \text{ cm} \]

Using the relation, \( \bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} \)

\[ = \frac{20 \times 9 + 16 \times 4}{20 + 16} = \frac{180 + 64}{36} = 6.777 \text{ cm} \]

Moment of inertia of the section about the axis X-X,

\[ I_{XX} = \left( \frac{10 \times 8^3}{12} + 20 \times 2.223^3 \right) + \left( \frac{2 \times 8^3}{12} + 16 \times 2.777^2 \right) \]

\[ = (6.667 + 98.834) + (85.333 + 123.387) = 314.221 \text{ cm}^4 \]

Moment of inertia of the section about the axis Y-Y,

\[ I_{YY} = \frac{2 \times 10^3}{12} + \frac{8 \times 2^3}{12} = 166.67 + 5.33 = 172 \text{ cm}^4 \]

Least value of moment of inertia is about Y-Y axis

\[ I = 172 \text{ cm}^4 = 172 \times 10^4 \text{ mm}^4 \]

Since the strut is hinged at both of its ends

\[ \therefore \text{ Effective length, } L_e = l = 5000 \text{ mm} \]

Let \( P = \text{Crippling load} \)

Using equation (19.5), we get

\[ P = \frac{\pi^2 E I}{L_e^2} = \frac{\pi^2 \times 2.0 \times 10^5 \times 172 \times 10^4}{5000^3} = 135805.7 \text{ N. } \text{ Ans.} \]
Problem 19.17. Find the Euler crushing load for a hollow cylindrical cast iron column 20 cm external diameter and 25 mm thick if it is 6 m long and is hinged at both ends. Take $E = 1.2 \times 10^6$ N/mm$^2$.

Compare the load with the crushing load as given by the Rankine's formula, taking $\sigma_c = 550$ N/mm$^2$ and $a = \frac{L}{1600}$; for what length of the column would these two formulae give the same crushing load?

Sol. Given:
External dia., $D = 20$ cm
Thickness, $t = 25$ mm = 2.5 cm
\[ \therefore \text{ Internal dia., } \quad d = (D - 2 \times t) = 20 - 2 \times 2.5 = 15 \text{ cm.} \]

Area,
\[ A = \frac{\pi}{4} (20^2 - 15^2) = \frac{175\pi}{4} = 137.44 \text{ cm}^2 = 13744 \text{ mm}^2 \]

Moment of inertia,
\[ I = \frac{\pi}{64} (20^4 - 15^4) = \frac{\pi}{64} (160000 - 50625) = 5368.93 \text{ cm}^4 = 53689300 \text{ mm}^4 \]

\[ \therefore \text{ Least radius of gyration, } \quad k = \sqrt{\frac{I}{A}} = \sqrt{\frac{53689300}{15744}} = 62.5 \text{ mm} \]

Length of column, $l = 6 \text{ m} = 6000$ mm
End conditions = Both ends are hinged
\[ \therefore \text{ Effective length, } L_e = l = 6000 \text{ mm} \]
Value of $E = 1.2 \times 10^6$ N/mm$^2$.

Euler's crushing load is given by equation (19.5),
\[ P = \frac{\pi^2 EI}{L_e^2} \]
\[ = \frac{\pi^2 \times 1.2 \times 10^6 \times 53689300}{6000^2} = 1766307 \text{ N. Ans.} \]

**Crushing load by Rankine's formula**
The value of $\sigma_c = 550$ N/mm$^2$

Value of $a = \frac{1}{1600}$

Let $P = \text{Crushing load by Rankine's formula}$

Using equation (19.9),
\[ P = \frac{\sigma_c \cdot A}{1 + a \left( \frac{L_e}{k} \right)^2} = \frac{550 \times 13744}{1 + \frac{1}{1600} \left( \frac{6000}{625} \right)^2} = 1118224.8 \text{ N. Ans.} \]